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by and for math graduate students

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Gauge Theory and Low-Dimensional Topology (Part I: Historical Context)
by Irving Dai, Princeton University

Hi! This month, I thought I would start a brief series of articles describing the uses of gauge theory in mathematics. Rather than discuss current research directions in gauge theory (of which there are many), I hope to give an overview of the sorts of mathematical questions that gauge theory was first used to answer and a general idea of what it is all about. Our goal for today will be to contextualize the initial advances in low-dimensional topology due to gauge theory by giving a picture of the state of affairs before its introduction. We will thus spend this post establishing some basic terms and ideas for the uninitiated; we will wait until later posts to discuss gauge theory itself and what it can tell us about topology.

I have attempted to make this article as introductory and motivational as possible, especially to readers who are less familiar with the finer historical developments in low-dimensional topology. No background is needed except for (at best) a passing recollection of basic algebraic and differential topology. In several places I have been a bit cavalier with precise definitions for the sake of the exposition. All errors are (of course) mine!

In the Beginning, There Were Four-Manifolds

In order to understand the development of (mathematical) gauge theory, we will first need to know a bit about the history of low-dimensional topology. In our case, the relevant history will be the story of four-dimensional manifolds (affectionately called "four-manifolds," for short). If you are less familiar with low-dimensional topology, then for the moment you can think of these objects simply as "spaces," much like the surface of a sphere or a torus, except (of course) in four dimensions. Four-manifolds include things like S^4 and T^4, which are the obvious higher-dimensional analogues of the two-sphere S^2 and the two-torus T^2, as well as things like the complex projective plane $\mathbb{C}P^2$. For those who are familiar with complex or algebraic geometry, slightly fancier constructions include taking complex hypersurfaces in $\mathbb{C}P^3$, and so on. It turns out, however, that mathematicians have come up with a tremendous number of different ways to construct four-manifolds beyond these simple cases, the result of which has been a veritable zoo of examples, each with its own properties and its own "flavor" of construction.

In any first course in topology, one is usually exposed to algebraic invariants such as the fundamental group and/or homology and cohomology. The basic idea behind these is to associate to a topological space an algebraic object—for example, a group (in the case of the fundamental group) or a sequence of abelian groups or vector spaces (in the case of homology or cohomology). One then asks precisely what topological information about a space these algebraic invariants capture. Indeed, one might even hope to construct a *complete* invariant—that is, an invariant capable of distinguishing two spaces whenever they are different. Failing this, one might at least hope to find an invariant which distinguishes spaces in some well-understood subclass of cases, or up to a known number of ambiguities.

It turns out that in four dimensions, the most interesting classical invariant to study is the *intersection form.* For those with a bit of background in algebraic topology, recall that this is defined as follows. Let M be a closed four-manifold, which for simplicity we assume to be simply-connected. We then have a unimodular bilinear pairing on the cohomology $H^2(M; \mathbb{Z})$ given by the cup product:

$$(\alpha, \beta) = \langle \alpha \smile \beta, [M] \rangle \in \mathbb{Z}$$

for any two classes $\alpha, \beta \in H^2(M;\mathbb{Z})$. If you are more familiar with de Rham cohomology, then the above is the same as taking the wedge product of two two-forms and integrating over M. If you are not at all familiar with algebraic topology, then the idea of the intersection form is (very roughly) as follows: each element of $H^2(M; \mathbb{Z})$ can be represented as (the dual of) an embedded two-dimensional surface Σ in M. Any pair of two-dimensional surfaces Σ_1 and Σ_2 in M generically intersect in a finite number of points, and counting the number of such points (with the proper sign) defines an integer-valued pairing on $H^2(M; \mathbb{Z})$.

We have thus associated to M an algebraic invariant which takes the form of an integer-valued bilinear pairing. Intuitively, this invariant captures the collection of all two-dimensional surfaces floating around inside M (up to the appropriate equivalence relation) and how they intersect each other. (This picture is very imprecise and should be used for motivational purposes only.) We can of course abstract away this topological intuition and simply study the space of all possible such integer pairings up to an appropriate notion of algebraic equivalence; this is identified with the set of all *unimodular lattices.* These may be thought of as corresponding to symmetric, integer-valued matrices with determinant ±1, where two matrices A_1 and A_2 are equivalent if(f) $A_1 = B^T A_2 B$ for some integer matrix B.

For those unfamiliar with intersection forms, we list here some simple examples. As an exercise, you can try to prove that the intersection forms are as I have stated them:

1) S^4: This has no second cohomology. Its intersection form is trivial (empty).

2) $S^2 \times S^2$: This has second cohomology isomorphic to $\mathbb{Z}^2$, with generators represented by the surfaces $S^2 \times$ (point) and (point) $\times S^2$. Its intersection form is given by the 2×2 matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

3) $\mathbb{C}P^2$: (Complex projective space.) This has second cohomology isomorphic to $\mathbb{Z}$, with a generator represented by the hyperplane class. Its intersection form is given by the singleton matrix [1].

4) $\overline{\mathbb{C}P^2}$ (Complex projective space with reversed orientation.) This has second cohomology isomorphic to $\mathbb{Z}$, with a generator represented by the hyperplane class. Its intersection form is given by the singleton matrix [-1].

5) (Not so simple!) Let M be a nonsingular degree-four hypersurface in $\mathbb{C}P^3$. (That is, M is defined to be the zero locus of a homogenous degree-four polynomial in $\mathbb{C}P^3$; such a hypersurface is often called a K_3 surface.) Then the second cohomology of M has rank twenty-two, and its intersection form is given by $3H \oplus 2(-E_8)$. Here, H is the matrix of Example 2 and E_8 is given by

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

Freedom's Theorem

The amazing thing about the intersection form is that it turns out to be very nearly a complete invariant of (closed, simply-connected) topological four-manifolds. More precisely, in 1982, Michael Freedman proved the following:

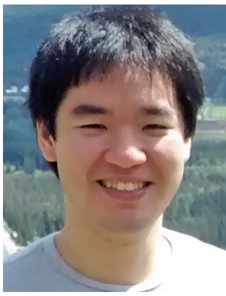
Let Q be any unimodular, symmetric bilinear pairing. Then there exists a closed, simply-connected topological four-manifold M whose intersection form is Q. Moreover, if Q is even, then M is unique up to homeomorphism. If Q is odd, then (up to homeomorphism) there are exactly two closed, simply-connected manifolds with intersection form Q.

Here, we say that an intersection form Q is even if $Q(x, x)$ is even for all x (or if it is empty), and is *odd* otherwise. Examples 1, 2, and 5 above are all even, while Examples 3 and 4 are odd.

It is worth pausing for a moment to appreciate this result. So far, we have encountered two distinct categories of objects in our discussion. On one hand, we have topological objects: simply-connected, topological four-manifolds. On the other, we have algebraic objects: unimodular lattices. At first glance, it might seem that these are two completely different sets of objects; and, indeed, the study of lattices has a long and storied history within number theory entirely independent of any topology (see for example previous posts on this blog about packing problems!). However, Freedman's theorem says that up to a factor of two, these two things *are the same*—for every possible intersection form, one can produce a corresponding four-manifold, and at most two such four-manifolds share any given intersection form! Results of this kind which link an algebraic invariant to the topology of a space are (in some sense) the holy grail of topology and are usually extremely hard to come by. For example, the three-dimensional Poincaré conjecture—which states that S_3 is the only three-manifold with trivial fundamental group—was only proven in 2006 after remaining open for nearly a century. (Indeed, the astute reader may have already realized that Freedman's theorem applied to Example 1 turns out to give a proof of the four-dimensional topological Poincaré conjecture!)

Roughly speaking, this was the state of affairs in four-manifold theory immediately before the introduction of gauge theory. Freedman's theorem provided a strong correspondence between four-manifolds and their intersection forms, and indeed was a powerful way to indirectly show that two four-manifolds were homeomorphic. However, a crucial distinction remained: this result held in the *topological* category, with the manifolds constructed in the theorem being topological manifolds and the notion of equivalence being homeomorphism. It was thus a natural question to ask whether a similar correspondence held for *smooth* four-manifolds (that is, four-manifolds with differentiable structure, up to diffeomorphism).

To the uninitiated, the distinction between smooth and topological manifolds might seem like a technical point. In fact, in three dimensions or less, every topological manifold admits a unique smooth structure—there is no distinction between continuous and smooth topology in dimension three! However, as we shall see, in dimension four the difference between continuous and smooth topology assumes an all-consuming importance. Indeed, the first major application of gauge theory was to show that smooth four-manifolds exhibit a radically different behavior with respect to their intersection forms—the relation between a four-manifold and its intersection form in the smooth category is very different from the tidy correspondence due to Freedman's theorem in the topological category. This opened up a chasm between continuous and smooth topology in four-dimensions that persists to the present day. (Indeed, while the topological four-dimensional Poincaré conjecture was solved by Freedman, the smooth four-dimensional conjecture remains a major open problem!) Next time, we will attempt to describe this difference and explain how gauge theory could possibly have something to say about it.



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How to Survive Grad School as a Woman in STEM

by Susannah Shoemaker,[1] Princeton University

There are certain moments from grad school that will always stick with me: the conference in Boston where my usually quiet lab mate opened up to me; the nights I spent drinking cheap beer with my closest friends in the grungy, student-run bar; the time (okay, times) I cried in my advisor's office, … my general exams, whose residual panic-inducing effects I can still feel, months later.

But I've also had experiences that extend beyond the normal ups and downs. I listened, trying not to cry, as a professor told me that I was too slow to do theoretical work. (A year later, I won an NSF grant to do just that.) I gritted my teeth as I, the only woman in the room, was asked to sort exams into piles, while my male colleagues graded them. These experiences didn't make me stronger, happier, more resilient, or more confident. They just wore away at my well-being.

Learning to survive graduate school as a woman in STEM—or any minority[1], for that matter—means finding ways to manage the effects of constant, subtle antagonism, because that antagonism won't make you a better scientist, mathematician, or engineer.

Here are seven things that will:

1. Reevaluate your definition of a mathematician… . It took some reexamining of unconscious beliefs before I realized that if I wanted to become a mathematician, I'd have to start thinking of myself as one.
2. Stop using the word "genius."[2] …
3. Start your own alt-boys' club. …
4. Find an involved advisor…
5. … and a good therapist. …
6. Accomplish something. Something outside of your project and coursework, that is. …
7. Know why you're in it… . Whatever your career goals are, communicate them—especially if they are unconventional… .

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Susannah Shoemaker recently earned her MA in applied math from Princeton University. She is interested in the mathematical foundations of structural techniques in biophysical chemistry. Her email address is `scs5@princeton.edu`.

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EDITOR'S NOTE. Also, see the helpful post by Jacob Gross on "Some funding opportunities for graduate students in mathematics" `blogs.ams.org/mathgradblog/2017/07/19/funding-opportunities-graduate-students-mathematics/`

[1] *See video "Neil Degrasse Tyson on being Black, and Women in Science" at* `www.youtube.com/watch?v=z7ihNLEDiuM`

[2] *See "Belief that some fields require 'brilliance' may keep women out" by Rachel Bernstein in* Science Magazine *at* `www.sciencemag.org/news/2015/01/belief-some-fields-require-brilliance-may-keep-women-out`