EUCLIDEAN GEOMETRY
— A Guided Inquiry Approach —

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INSTRUCTOR SUPPLEMENT

To the Instructor

In the Preface of *Euclidean Geometry: A Guided Inquiry Approach* you will read about a troublesome mathematical and pedagogical dilemma that developed around the teaching of Euclidean geometry during the twentieth century. Without a resolution to this dilemma the axiomatic development of the subject was virtually abandoned in our national curriculum, calling into question the rationale for teaching the subject at all.

This text offers a resolution to that dilemma which is carefully described in the last section of the Preface: "Resolution". More detailed guidelines as to how to use this book to implement this resolution are given in Appendix B: "Guidelines for the Instructor". A familiarity with both of those documents will be essential for anyone teaching from this text.

This supplement is primarily written to provide interpretations of the guidelines relative to specific definitions, problems and theorems. It also gives some mathematical background that goes beyond the level of this text. In some places it refers to Hilbert's Axioms, which are listed in Appendix C. Instructors are encouraged to consult appropriate topics in this supplement prior to presenting them in class.

1. Congruent Figures

1.1 Congruences and Isometries

congruent, p3.

Here we use this transformational definition of "congruence" that applies to all plane figures. Later it will be specialized to segments, angles, triangles and other specific geometric figures.

It is important, in this section, that students develop a sense of confidence with the material. Take as much time as is necessary to see that they all do each problem and write their solutions in a notebook.

Problem 1.

This problem helps students recognize the difference between solving a problem and articulating a justification of the solution. Students who see how to orient two figures to demonstrate their congruence can not convey this orientation to their entire class without effectively expressing it in words. Learning to express ideas correctly and precisely in words is a major goal of this class.

CPCFC, p3.

This argument is a very simple form of indirect proof. To see that two figures are *not* congruent we observe that, if they were congruent, then some part of one would have to appear in the other. Since it does not appear in the other, they can not be congruent. For example, suppose cyberfrogs **A** and **B** were congruent. Then the cross at the back of **A** would match some part of **B**. But there is no cross in **B**, a contradiction. Therefore **A** and **B** are not congruent.

It is recommended that "indirect proof" (or "proof by contradiction") not be mentioned here directly. This idea will first come up explicitly in the proof of Theorem 40, at which point a reference back to these early problems will be useful.

Problem 2.

At this point it would be helpful to give each student a sheet of tracing paper, as this would simplify the drawing process.

Problem 3.

As students do this, they will inevitably have to talk about how they intend to move a figure by sliding, turning and even flipping over. Try to draw out of their comments the three basic isometries that follow, and explain the terms "translation", "rotation" and "reflection" once they have generated the underlying ideas.

Problem 4.

As this problem is presented, it is useful to draw a 6-by-6 table with the letters **A** to **F** down the side and across the top. In the box of the row labeled **X** and column labeled **Y**, insert $|$ if **X** is congruent to **Y** and $\times$ if not. Then observe that there is a better way to convey this information using the idea of congruence classes.

Problem 7.

Time would be well spent letting students present different solutions for each part of Problem 7. Here are two relevant theorems that interested students may like to try proving.

Theorem. If two figures in the plane are congruent, then one can be made to coincide with the other either by a translation followed by a rotation, or by a translation followed by a rotation followed by a reflection.

Theorem. If two figures in the plane are congruent, then one can be made to coincide with the other by a sequence of reflections.

Problem 9.

See if your students can come up with more than one answer. Here are some options: three isolated points in a row; two twin pairs and no isolated corner point; two twin pairs that don't have an isolated point neighboring both.

Problem 10.

In Problem 9 students will check that $\mathbf{K} \cong \mathbf{B}$, $\mathbf{K} \cong \mathbf{E}$, $\mathbf{K} \cong \mathbf{W}$ and $\mathbf{K} \cong \mathbf{Y}$. These facts are indicated by the four $*$s in the table below. Students will want to conclude that $\{\mathbf{K}, \mathbf{B}, \mathbf{E}, \mathbf{W}, \mathbf{Y}\}$ is a congruence class. Although they will have established that none of these figures are congruent to any of the other twenty figures, it will remain to show that any two of these five figures are actually congruent to each other.

For example, is $\mathbf{E} \cong \mathbf{Y}$? $\mathbf{B} \cong \mathbf{W}$? Rather than finding all of the other necessary isometries, you can – with assistance from the class – put $*$s in these other 21 entries of the table by making some simple observations. Certainly every figure is congruent to itself, giving us 5 more stars along the diagonal line.

Since $\mathbf{K} \cong \mathbf{B}$, it must be that $\mathbf{B} \cong \mathbf{K}$. And if $\mathbf{B} \cong \mathbf{K}$ and $\mathbf{K} \cong \mathbf{W}$, then it must be that $\mathbf{B} \cong \mathbf{W}$. Get the class to fill in the rest of the table with $*$s using these kind of arguments. (These arguments are actually using Hilbert's Axiom 9, though you need not cite it explicitly.)

≅	**B**	**E**	**K**	**W**	**Y**
B					
E					
K	*	*		*	*
W					
Y					

Problem 12.

You may want to do this in class together as a review of the concepts in this section, without asking students to write it up.

1.2 The Language of Geometry

Problem 15.

Because these equations require foundational principles of betweenness, we do not expect formal arguments to justify the answers. Doing this problem will help encourage students to recognize that a figure is a certain *sets of points.* Notice that we define rays, angles, circles, lines and triangles all as sets of points. In contrast, some texts will define an angle as a pair of rays and a triangle as a set of three segments. In their 1913 text Wentworth and Smith define an angle as “The opening between two straight lines drawn from the same point”. Like other older texts they use the confusing definition of a circle as “the locus of *a point*” that marches in some way around some central point.

sides of a line, p10.

This is Hilbert’s Axiom 7, an example of a foundational principle of intersections (Preface: “Resolution”). As with all definitions, draw a line on the board and ask the students when two points are on the same side of the line. Encourage them to pay attention to sides of lines, and to use what they intuitively know about them.

angle, p10.

It is important to emphasize the fact that we require the rays to be non-collinear, since some texts do not have this restriction. This means that we do not consider a “zero angle” (consisting of a single ray) to count as an angle. Also, we do not consider a “straight angle” (consisting of two opposite rays) to count as angle. Both concepts cause unnecessary difficulties that we would prefer to avoid.

Problem 17.

At this point students have no means to justify a positive response; they can only justify a negative response by producing a counterexample. For example, they may believe SSA. If they do, we suggest letting it pass. At some point some student will want to know why they are proving many things that seem so evident. That will be the time to ask them to review SSA, and learn a lesson about how things can seem evident without being true!

1.3 Construction Problems

construct, p13.

Up to now justifications have been informal, relying upon intuition and pictures. In this section Step 3 will require more formal justifications based on definitions and axioms. These particular justifications will draw on foundational principles about the existence of intersections (Preface: “Resolution”), which need not be stated explicitly.

Problems 19 - 22.

Each of the next four problems require the construction of a triangle. Each construction builds on the previous one, adding additional complexity. The last two also require the use of a new axiom.

equilateral triangle, p13.

Give students time to work this out, and to present and discuss it until they are all satisfied that they explain why it is correct.

Axiom 3, p14

The SAS axiom represents a first small step toward *indirect measurement* (discussed in the Introduction to the Student). It tells us that if we know one angle of a triangle and we know the lengths of its two sides, then there is only one possible value of the length of the third (opposite) side. It will not be until Chapter 6 when we will actually be able to find this length, thereby doing the indirect measurement.

construction problems, p14.

Remind your students that the problems in this text are not merely practice doing something they are supposed to already know. They need to take time to figure out each problem, which may require trying different approaches.

angle bisector, p15.

It is an option here to say "...is *a* bisector of the angle..." and then to prove that an angle has a unique bisector. We have omitted this digression since it is not necessary for any of our subsequent work.

Problem 24 - 27.

The next four constructions are all very similar. Step 3 of each uses a foundational principle about intersections, namely, if two isosceles triangles have the same base and their third vertices are on opposite sides of the base, then the segment connecting the third vertices intersects the base between the two common vertices.

parallel, p16.

Before defining "parallel", draw two parallel lines and ask the students give a definition. A definition of "parallel" can't refer to "distance apart" because we do not yet have a way to measure the "distance" between two lines.

Problem 28.

Here "construct a line" means to construct two points on the line and then draw as much of the line as the straight edge will allow. At this stage students should be able to discover one of several methods to construct parallel lines, but they do not yet have the necessary axioms to prove lines are parallel. Let them present these constructions, and then ask them to keep them and see if later they can use new results to prove their constructions correct.

For example, they may guess the alternate interior angle theorem (or some variation). But they will not be able to prove that the lines are actually parallel until they prove Theorem 40 in Chapter 2. Guessing Theorem 40 now will make it exciting to prove when they reach it in Chapter 2.

trisquare, p16.

In Chapter 2 we will prove that there are no trisquares in a surface that satisfies our axioms. However, there are trisquares on a sphere, where there is no notion of betweenness that will satisfy Axiom 1(iii).

2. Axioms, Theorems and Proofs

Theorems 30 - 35.

In each of these theorems the existence of the points constructed along the way are foundational principles about intersections, and the necessary congruences come from the three axioms in Chapter 1.

Problem 23.

Note that we know this is true for *every* angle because the construction of Problem 22 was justified without any reference to what particular angle was being used. Consequently it could have been any angle.

isosceles triangle, p19.

We now define a series of standard notions of geometry. Like all definitions, these should be presented through H1, H2 and then H3 level discussions. After presenting a definition, make appropriate drawings on the board and get students to make a conjecture about what might always be true in the illustrated context. If they can guess a theorem before it is stated, then they will have the satisfaction of proving their own conjecture. This is particularly so if you name the theorem after the person who guessed it: "Kate's Theorem", "Keith's Theorem", ...

Theorem 36.

Encourage students to present different proofs of this theorem. There are several. Proofs like these can illustrate an important point that goes well beyond mathematics. The fact that a good idea is short and easy to understand does not at all mean that it was easy to find. A great deal is learned and achieved in the process of finding it, and the students who do this are to be commended.

Theorem 37.

Students may need some discussion to see how to restate this with named points so that they know what to prove.

Corollary 38.

Hilbert's Axiom 9 that you presented in solving Problem 10 is also useful here.

Weak Right Angle Theorem 39.

This is the first of four "Weak" theorems we will prove in this chapter. These theorems require only a few simple axioms to prove, and will be sufficient for the applications we need. Subsequently we will prove a "Strong" version of each one, requiring further axioms but giving us more powerful tools. For example, the Strong Right Angle Theorem will say that an angle is congruent to a right angle if and only if it is a right angle.

It is a common error to misquote this theorem by confusing it with its converse: if two angles are both right angles, then they are congruent. If a student does this, you will have an ideal opportunity to introduce the notion of the *converse* of a statement and the fact that it differs from the original statement. Exploiting student errors like this is one of the great advantages of guided inquiry learning.

As an alternative and short cut here, it would be an option to state Theorem 49 as an axiom immediately after Theorem 39. In doing so it will still be important to explain the difference. It would also be appropriate to tell students that this "axiom" can in fact be proven, and to challenge them to do so.

Axiom 4, p21.

This is a very weak version of our Axiom 7 in the sense that it follows easily from Axiom 7, but it requires that we assume much less. It provides us with the only angle comparison that we will need prior to Chapter 4.

Weak Alternate Interior Angle Theorem 40.

Ask your students if there is a "strong" version of this theorem. Try to get them to state it precisely, and to tell if they think that it is true. In fact, it is not possible to prove without Axiom 5 that comes much later.

Use your judgment as to how much assistance your students require for this proof. The key is to observe that the theorem says that something is *not* true. Although the word "not" is missing, it is implicit. The statement really says that there does not exist a point that is on both lines. As we saw in Problem 2, negative statements like this are almost always proven indirectly. We *suppose* that such a point X does exist and show that this would be impossible since it would lead to a contradiction.

It is helpful here to discuss the notion of **indirect proof** or **proof by contradiction** explicitly. We like to reserve the word "suppose" for this purpose. When we reach a contradiction, we can then look back for that word to see what must be false.

Theorems 44 and 45.

These two theorems offer more examples of proof by contradiction. If your students really need help on them, you might point out that the two triangles certainly would be congruent if one of the other pairs of corresponding sides were congruent. So, suppose they are not and see what that tells you.

Theorem 46.

This is an opportunity to reiterate the meaning of 'converse', and the fact that an implication can be true without its converse being true. If two segments are congruent, then they have the same length. The converse is also true: if two segments have the same length, then they are congruent. Similarly, if two triangles are congruent, then they have congruent angles. But the converse is not true: two triangles can certainly have congruent angles without being congruent. Ask your students to think of other implications that are true but have converses which are false.

Strong Right Angle Theorem 49.

This proof uses a foundational principle about intersections: a ray emanating from a point inside a circle must intersect the circle. You will need to judge how much, if any, assistance they need.

Problem 57.

See the Introduction to the Student for a discussion of indirect measurement.

3. Area Measure

3.1 Closed Regions and Parallels

closed path, p29.

By a "closed path" we mean a simple closed curve, i.e., a homeomorphic image of a circle.

closed region, p29.

By a "closed region" we mean a homeomorphic image of a disc whose boundary is a "piecewise smooth" closed path. To say what this means, let $x, y : [0,1] \to \mathbb{R}$ be two continuous functions from the unit interval into the real numbers that are each differentiable at all but at most finitely many points. Let $H{:}[0,1] \to \mathbb{R}^2$ with $H(t) := (x(t), y(t))$ and assume that x and y are chosen so that H is one-to-one on $[0,1)$ and $H(0) = H(1)$. Then the range of H is a **piecewise smooth** closed path.

An informal definition of a "piecewise smooth" closed path would be a closed path with at most finitely many sharp turns. This idea can be easily conveyed to students. This requirement precludes exotic closed paths like space filling curves that have positive measure, leading to violations of Axiom 6(iii). (See the note on Axiom 6 below.)

Problem 58.

This problem turns out to be more problematic than the students will at first expect. Hopefully, among them, they will discover different promising ways to construct a square. For example, one way is to start with a segment and constructs two segments congruent to it and perpendicular to each of its endpoints. Then connect the far endpoints. Another way is to construct two congruent perpendicular segments with a common endpoint. Then construct perpendiculars to the two far endpoints with their own end points coinciding to form the last vertex of the square.

The difficulty arises when, in Step 3, they try to prove that their constructions actually do generate squares. In the first construction above, why is the last side congruent to the other three? Why are the last two angles right angles? In the second construction, why should the second two sides be congruent to the first two? Why is the last angle a right angle? How do we know that the two perpendiculars will intersect at all?

During the eighteenth and nineteenth centuries, mathematicians struggled with these questions. It wasn't until the advent of non-Euclidean geometry in the late nineteenth century that it was discovered that *none* of these statements follow from the axioms (and the foundational principles) that we have so far. In fact, no non-Euclidean geometry contains a square at all. (For example, a square can not be constructed on a sphere.) This becomes an H5 level topic which we will not further pursue here. In order to carry out Step 3, another axiom will be required.

When your students realize that they are unable to complete this problem, tell them that you suggest setting it aside but that they should let you know if they get any ideas. This will be a useful lesson in itself. Not every problem has an immediate solution, and we often have to set a problem aside and learn more about its context before we can solve it.

Another way to appreciate this problem is to observe that we are studying the geometry of a flat plane. We could also study the geometry of other surfaces, but we would need different axioms. Consider, for example, a sphere. Is it possible to draw a square on a sphere? What about a trisquare?

Theorem 63.

This is not true in the hyperbolic plane, where the Parallel Axiom fails but our other axioms hold.

Theorem 65.

This theorem should be proven twice, showing that each of the two proposed constructions for a square are correct. This theorem is essential for measuring areas. Squares do not exist in either hyperbolic or elliptic geometry.

3.2 Measuring Areas

Axiom 6.

In the Preface: "Hilberts Geometry", we discussed the fact that Euclid was unable to adequately axiomatize area and volume measure because the necessary advances in mathematics did not come until the late nineteenth century. Our Axiom 6 on area measure builds on those advances, namely, the modern theories of measure developed by Riemann, Jordan and Lebesgue.

These theories provide three ways of measuring area of closed regions in the coordinate plane $\mathbb{R}^2$: the Riemann integral, the Jordan Measure and the Lebesgue Measure. Because we require a closed region $\mathbf{X} \subseteq \mathbb{R}^2$ to have a piecewise smooth boundary, $\mathbf{X}$ is a measurable set under all three measures and all three give the same value for its measure. This value can be expressed in familiar terms as

$$\mathcal{A}(\mathbf{X}) = \iint_{\mathbf{X}} 1 dx dy.$$

If we define area to be this value, then the four parts of Axiom 6 are true. The cover page illustration of Chapter 3 and Figure 3.9 illustrate the computation of the Jordan measure as a sum of areas of inscribed rectangles overlapping only on their boundaries.

Rectangle Area Theorem, p32.

For completeness, here is an outline of the proof of the Rectangle Theorem. First, finish this argument to prove by induction that it is true for positive integers $x = m$ and $y = n$. Then show that it is true for rational values $x = m/d$ and $y = n/d$ by applying the integer result with a $1/d \times 1/d$ square as unit. Now let x and y be any positive real numbers and use rational approximations of them to show that the area differs from xy by less than every positive number ϵ. This last part is an application of Axiom 6(iv) and Hilbert's Axiom 14, neither of which were available to Euclid.

Figure 3.5.

The fact that either $X = A$, or $X = C$, or is between A and C, or A is between X and C, or C is between A and X is a foundational principle of betweenness.

Problem 79.

Doing this would be tedious, but it is worth discussing the fact that, with patience, students can use this method to estimate the area of any closed region to as much accuracy as the drawing they have will allow by making a grid with small enough squares. This process amounts to approximating the measure of the closed region.

Theorem 80.

This problem is a good review, as it requires quoting a series of previous results: Problems 18 and 22, Corollary 61, Corollary 59, ASA and Theorem 72. It will be needed later in Problem 161.

4. Angle Measure

Angle Measure.

Although your students come to this chapter with previous experience in angle measurement, remind them (again) that what they "know" only counts in this course if they can explain why it is true. Lots of people "know" lots of things that just are not so. Perhaps your students can think of some examples of this.

Axiom 7.

Before using a protractor, it is important for students to see that Axiom 7 gives them enough information to determine the measure of any angle to as much accuracy as they like. Remember that, in this text, an angle $\angle X$ consists of two *non-collinear* rays. Consequently $\mathcal{D}(\angle X)$ is greater than 0 and less than 180.

protractor, p43.

Just as area approximations lead to a definition of area that satisfies Axiom 6, angle measure approximations lead to a definition of angle measure that satisfies Axiom 7. The angles we get from a right angle by successive bisections are called *standard angles*. Their measures are defined as in Problem 81. We can then define $\mathcal{D}(\angle ABC)$ to be the least upper bound of all measures $\mathcal{D}(\angle XBC)$ where $\angle XBC$ is a standard angle and X is in the interior of $\angle ABC$. Using this definition of $\mathcal{D}$, Axiom 7 becomes a theorem. The cover page of this chapter illustrates a protractor designed on this principle.

Pythagorean Theorem 92, Part 1.

This construction provides a good review of compass and straight edge constructions and of some of our previous theorems.

5. Similar Figures

5.1 Similarity and Dilations

dilation, p51.

Given ray $\overrightarrow{OP}$ and $k > 0$ in the definition of "dilation", are we guaranteed that there is a point P' on $\overrightarrow{OP}$ at distance $k\mathfrak{L}(OP)$ from O? The foundational principle of Intersections says that we are, as the circle with center O and radius $k\mathfrak{L}(OP)$ will intersect the ray $\overrightarrow{OP}$ at some point P'. The existence of P' can be proven in Hilbert's system using his Axiom 15. (Appendix C).

similar, p52.

Students may want to suggest some facts about similarity now or as this work progresses. These facts may come from something they were taught before or (preferably) from their own observations. Either way, take time to discuss these ideas. Encourage them to formulate their beliefs as specific conjectures which you hope they will eventually either prove or refute.

Similarity Problems.

Problems 105, 106, 107 and 109 are not theorems, since they are based on pictures. They should be done like the isometry problems in Chapter 1, explaining the steps to make one figure coincide with the other.

Problem 108.

You will want books put away while students discuss ideas for (iv). Try to see if you can get your students conjecture some facts about dilations based on this diagram. For example, they might conjecture that a dilation takes a line to a line (Axiom 8(i)), that a dilation takes a segment to a segment (Axiom 8(ii)) or that a dilation from a point not on a line takes the line to a line parallel to it (Theorem 110).

Axiom 8.

Part (ii) of this axiom actually follows from part (i) using foundational principles about betweenness. Although part (ii) is in this sense redundant, we have chosen to state it here in order to avoid making this digression.

Axiom 8.

Although the facts stated immediately after Axiom 8 are really immediate consequences of (i) and (ii), we have refrained from introducing the formalism required to prove them. These restatements say that a dilation restricts to a one to one and onto function from a line to a line and a segment to a segment. You can mention this explanation to students with this background, or explain these claims with a drawing to students who do not.

Scaling Theorem, p57.

Here is an outline of the proof of the Scaling Theorem. First prove by induction that it is true when k is a positive integer n. For $k = 1$ this is just reflexivity of congruence. We have proven it for $k = 2$ and $k = 3$ in Theorems 112 and 114. Show that it is true for $k = 1/n$ by applying the inverse transformation with scaling factor n. Now combine these facts to show that it is true for every rational value of k. Finally, let k be a positive real number and use rational approximations of k to show that $\mathcal{L}(P'Q')$ differs from $k\mathcal{L}(PQ)$ by less than every positive number ϵ. Here you will again need Hilbert's Axiom 14.

5.2 Similar Triangles

Problem 124.

Once they answer this, you might ask the same question for polygons with more than four sides. Do the same for Problem 124 below. These observations point to some special properties of triangles.

Problem 131.

This problem can be made into a nice field trip, where students measure the heights of tall objects.

6. Trigonometric Ratios

trigonometry, p65.

Students who know trigonometry will be able to move quickly through this chapter. What they will learn that is new is exactly how trigonometry fits into and depends upon the logical development of geometry.

Figure 6.2.

The problems given in this chapter can all be done using this table without the need for a scientific calculator. Care has been taken to see that all angles have degree measure a multiple of 5. If students have access to scientific calculators, you can show them that they can solve right triangles with any angle measures. This would also require using inverse trig functions, which we achieve here by reading the table backwards. Do be sure that students can do these problems without calculators before they are introduced.

Problem 137

By the end of these problems, your students should be able to tell you how they decide which altitude to draw.

7. Circle Measure

Problem 138.

Your students of course "know" about π; that $c = 2\pi r$ and that $A = \pi r^2$. By now they also surely know that what they "know" only counts in this course if they can explain why it is true. In this case there is an even better reason to look again at the circumference and area formulas.

Ask your students what number "π" stands for. They will not be able to tell you. Numbers like 3.14 and 22/7 can not both be π, since they are not equal to each other. Is π the ratio of circumference to diameter? To say that we would need to know that this ratio is the same for all circles. Is the ratio of height to age the same for all people? The point is that we would like to know what number the symbol π stands for, and we would like to know exactly what it has to do with circles.

three possible facts, p74.

Mathematical and scientific discoveries arise in three stages. First we gather relevant information and data through experiments. Then we look at the information we have and make conjectures as to what might be true. Finally, we find ways to prove our conjectures are true. The previous two problems provide us with the first step. Try to get your students to help you formulate the second step. In the rest of this chapter, we will do the third step.

Figure 7.3.

Here is an opportunity for a useful discussion. Ask your students to look at Figure 7.3 and tell you what the dots are supposed to be. They will probably agree that the dots represent the centers of the regular polygons. Draw an example, like a regular octagon, on the board, and ask what you would get if you were to connect the center to each vertex. They will probably recognize that you should get congruent isosceles triangles.

Now ask them how they would prove these triangles congruent. They may have a number of suggestions. While they may think they understand what "the center" means, they should realize that they can't possibly prove anything about the center without a definition of "center".

Then ask them to give a definition of "center". If they suggest that it is the point where segments connecting opposite vertices meet, ask them to try this on a regular pentagon. Hopefully, without too much guidance, they will suggest defining it as the intersection of the angle bisectors.

You can then ask (if a student does not ask first, which would be better,) how we know that all the angle bisectors intersect at one point. Draw a diagram illustrating Theorem 140, and show them – before they prove it – how this theorem would imply that the bisectors do all intersect at one point.

Problems 145, 146, 147.

These construction problems will provide further review of compass and straight edge constructions and of some of our theorems. They are also mathematically significant since it is not possible to construct a regular n-gon for many values of n; for example, $n = 18$.

Problem 154.

Have your students do this problem in class with books put away. They will now need a scientific calculator to compute tangents. Plan in advance to have enough scientific calculators on hand that the can share in small groups. Hopefully they will identify the results they get with that of Problem 138. This problem will illustrate the concept of limit that is central to calculus.

Problem 156.

The value for $n = 2^{50}$ is a positive number. But a calculator will register 0 because this difference is too small for the processor to detect.

Problem 161.

This is an example of one of their "reasonable approximations" referenced in the Introduction. It corresponds to a value of π of $(4/3)^4$, or about 3.16.

8. Perspective Geometry

8.1 Solid Geometry

Each line contains . . . , p87.

This statement is an extension of the foundational Non-triviality Principle to space. While these facts can be easily axiomatized, formally developing them is an unnecessary digression in the present context. Students should assume that lines, planes and space have enough points to do the geometry they need to do.

Axiom 9.

A simpler but less precise form of these three statements is easy to remember.

(i) Two points determine a line.
(ii) Two non-parallel planes determine a line.
(iii) A line and a point not on it determine a plane.

Propositions 164 - 168.

These facts require only Axiom 5, Corollary 42 and Axiom 9. With the exception of Theorem 169 they should be easy to establish.

Axiom 10.

For part (i), ask your students if the stronger statement is true: "All pairs of parallel angles are congruent." Part (ii) is over stated, as it is not hard to prove that there can not be two such lines.

Figure 8.7.

To justify this description of m' we must explain why a point R is in m' if and only if R is *between* P_m and P_n, $R = P_m$ or P_m is *between* R and P_n. Since we are not using Hilbert's Axioms 4, 5, 6 and 7, and have no axioms about betweenness, this can not be formally proven in our system. But you can give a good explanation as to why it is true using the Foundational Principles of Betweenness.

Problems 193 - 197.

The drawing of the cabin in Problems 193 - 195 constitutes a bit of an extended project. You may choose to discuss this construction and then ask students not to carry it out themselves, but rather to use it to do either Problem 196 or Problem 197.