

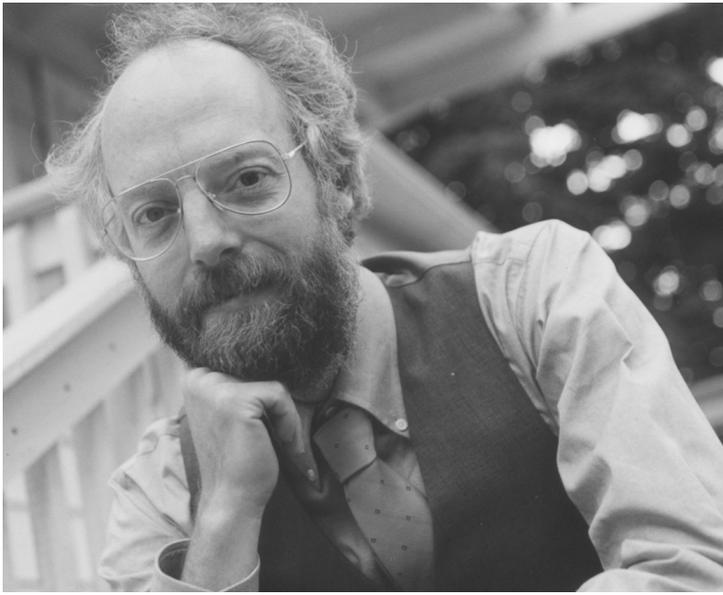
Hot Stuff. A map of the cosmic microwave background obtained by the Cosmic Background Explorer (top), and a computer simulation of what the Microwave Anisotropy Probe may find (bottom). (Figures courtesy of NASA/MAP Science team, <http://map.gsfc.nasa.gov>.)

Finite Math

The launch had been postponed again and again. Finally, on June 30, 2001, NASA's Microwave Anisotropy Probe (MAP) rocketed skyward from the Kennedy Space Center. Freelance geometer Jeff Weeks was delighted by the news. That's because the Canton, New York-based mathematician has a new-and-improved algorithm just waiting for data from the high-resolution satellite. Weeks has collaborated with scientists who will be analyzing MAP's map of the sky. And if all goes well, his algorithm may answer one of cosmology's most fundamental questions: What is the size and shape of the universe?

Theorists have pondered and debated the nature of space since antiquity. But only in the 20th century, with Einstein's general theory of relativity and astronomers' discovery of the expansion of the universe, has cosmology found its scientific footing. And only in recent years have cosmologists begun to think they may be able to tease precise answers out of astronomical observations.

Key to cosmologists' great expectations is the remnant whisper of primordial radiation known as the Cosmic Microwave Background. First detected by Bell Labs scientists Arno Penzias and Robert Wilson in the mid 1960s, the CMB is highly uniform:



Jeff Weeks. (Photo courtesy of the MacArthur Foundation.)

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No matter where you look in the sky, the spectrum of the microwave background is that of a theoretical black body radiating at a temperature of 2.73 degrees Kelvin, to within about a hundred thousandth of a degree. But those tiny departures are crucial. They echo the random fluctuations in density of the expanding early universe, which eventually spawned stars, galaxies, and astronomers.

Early observers couldn't see these fluctuations—it's like trying to eyeball bacteria—but data from the 1992 Cosmic Background Explorer (COBE) and more recent balloon-based measurements began to sharpen the picture. The CMB sky is now revealed as a patchwork of hot and cold spots (see Figure "Hot Stuff," page 32).

MAP will bring the picture into even sharper focus. "COBE had an angular resolution of 7 degrees. MAP has angular resolution of .2 degrees," says David Spergel, an astrophysicist at Princeton University and a member of MAP's science team. "So every place where COBE measured one number on the sky, MAP will measure a thousand."

In 1997, Spergel and colleagues Glenn Starkman at Case Western Reserve University and Neil Cornish, now at Montana State University, proposed testing whether the universe is finite by looking for "matched circles" in the fluctuations of the CMB. "When we look at the microwave background, we're basically looking out to the surface of a sphere," Spergel explains. If the universe is finite—and if it's small enough—then this "surface of last scattering," as it's called, must intersect itself (see Box, "Topology Makes the World Go Round," page 38). The intersections, if any, will show up as pairs of circles in different parts of the sky with precisely aligned temperature patterns.

Matched circles would be one signature of a finite universe. Researchers, including Spergel's group in the U.S. and a group in France headed by Jean-Pierre Luminet at Observatoire de Paris, are developing circle-spotting algorithms to crunch the data from MAP, hoping to find John Hancock's in the sky. That in itself is a major undertaking—in essence, their algorithms must examine all possible pairs of circles, to see if there are any matches. Only then does the question arise, what do the matches mean?

That's where topology takes over.

The astronomers asked Weeks, an expert in topology, for help interpreting the matches they hope to find. They got more than they expected. "Jeff was the one who realized how you would use the circles to actually determine what was the underlying topology," Spergel says. Matched circles, Weeks saw, could be more than a signature; they could be a mathematical blueprint of the universe.

That's because each pair of matched circles provides a mathematical clue to the way space is stitched together.

The easiest possibility to visualize is a "toroidal" universe slightly smaller than the surface of last scattering (see

Figure 1). Such a universe can be thought of mathematically as a kind of magical box in which, when you exit through any wall, you immediately reenter through the opposite wall. There is, of course, no literal, physical wall, much as there's no edge to the Earth just because maps always show one. And much as U.S. map-makers conveniently place North America in the center, it's convenient to imagine Earth as being at the center of the box universe.

In this universe, the surface of last scattering intersects itself in circles on the walls. Since points on apparently opposite walls are in fact identical, this means that an observer on Earth would see the same circular patterns of cosmic radiation when looking in these three pairs of opposite directions. If MAP were to see three such pairs of matched circles, that would establish a toroidal geometry for the universe. (If the universe is toroidal but considerably smaller than the surface of last scattering,

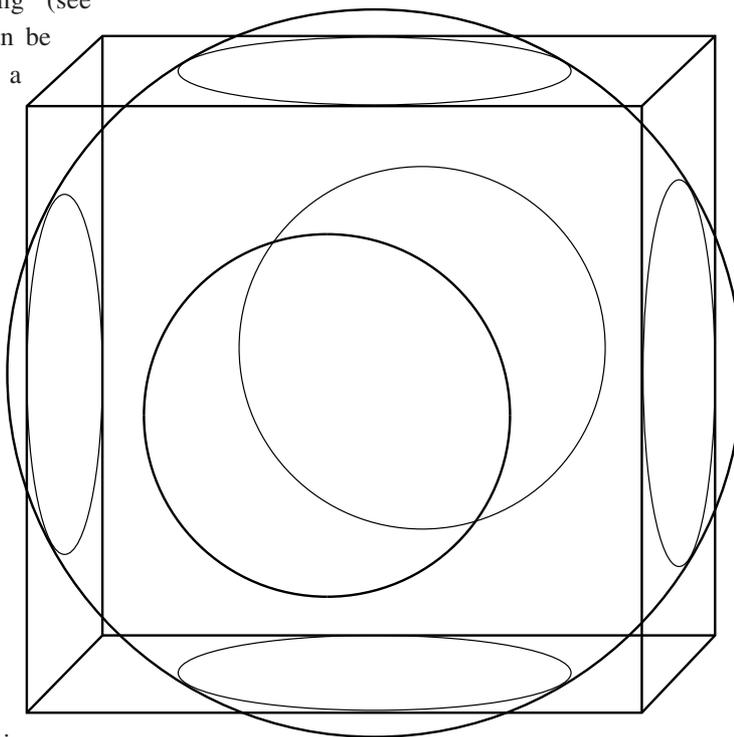


Figure 1. *The topology of a toroidal universe slightly smaller than the surface of last scattering might reveal itself in three pairs of matched circles.*

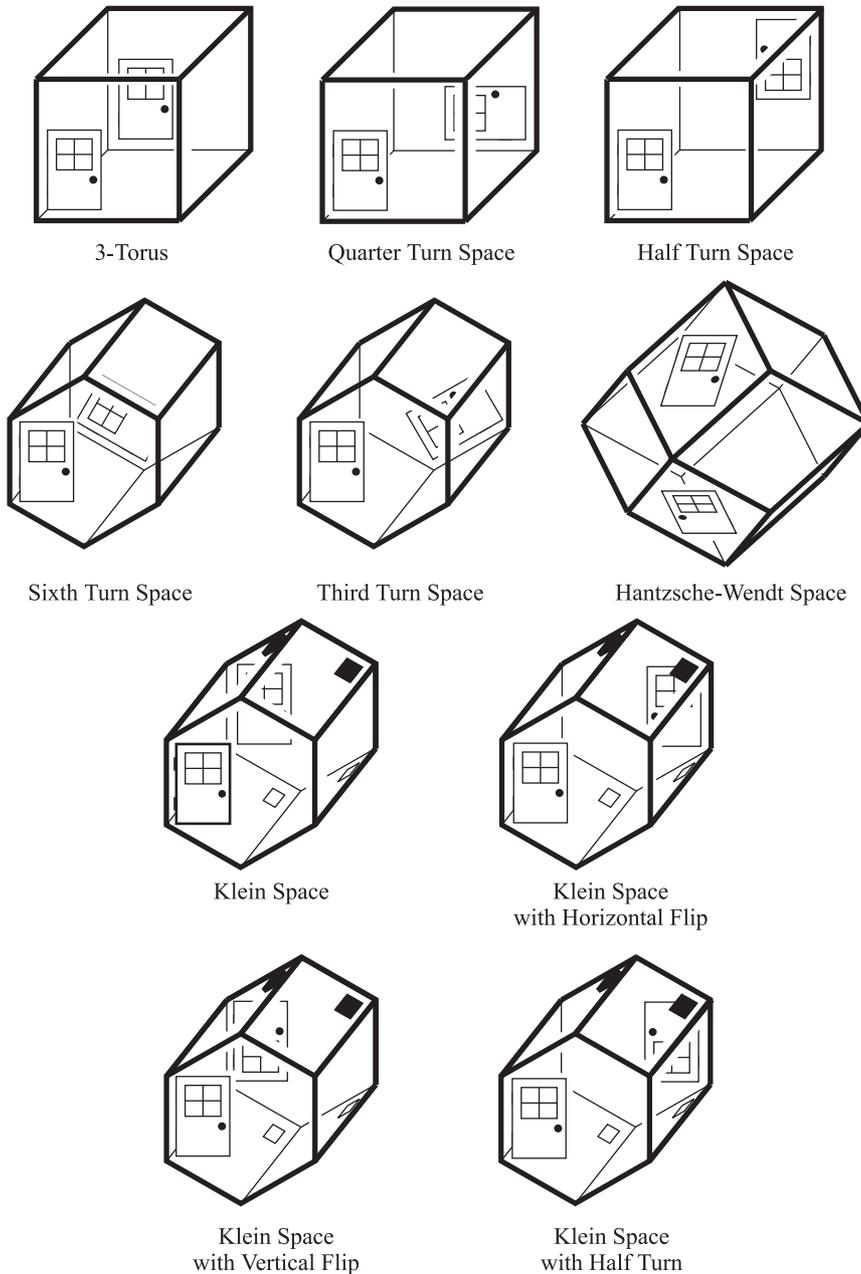


Figure 2. Ten flat topologies. If the universe is finite but flat, there are only six orientable and four non-orientable possibilities. The doors in each figure indicate how a person exiting through one side would appear entering through another. In the first five orientable cases, the other sides are identified as they are for the torus. In the four non-orientable cases, the identification is indicated by the placement of small “windows.” In the 12-sided Hantzsche–Wendt space, whose shape is a rhombicuboctahedron (see Figure 3 in “Nothing to Sphere But Sphere Itself,” page 25), there are six pairs of identified sides (only one pair is shown). They are symmetrically pegged to the six points where four sides meet. (Figure courtesy of Adam Weeks Marano.)

then the pattern of self-intersections becomes more complicated, but the math works out the same.) And if astronomers' best guess for the age of the universe is correct, that would establish the *size* of space as well.

Toroidality is only one possibility. It's one of ten different topologies for a "flat" universe (see Figure 2). And if the universe turns out to be curved—which is currently considered unlikely—then there are infinitely many more possibilities. Depending on whether the curvature is negative or positive, these alternative universes are what cosmologists traditionally call "open" or "closed" and mathematicians call "hyperbolic" or "spherical." (The near-uniformity of the CMB is evidence that the overall curvature of space is constant. There are, of course, local perturbations due to gravity, which are accounted for by the general theory of relativity.) Matched circles that correspond to a curved universe would contain an added bonus: The distance to the surface of last scattering (i.e., the age of the universe) and the radius of curvature of the universe would be related by a definite formula, giving a check on two otherwise independent numbers.

Weeks's original algorithm simply used matched circles to determine the "walls" of the universe. For example, if six pairs were found, the box would be a dodecahedron. Depending on how the pairs were oriented, the underlying geometry would be either spherical or hyperbolic. Whichever kind of box were found, purely mathematical considerations would then take over. In particular, if the geometry turned out to be hyperbolic, a computer program called SnapPea, which Weeks developed in the early 1990s, would finish the computation, determining the exact size and shape of space.

To work, though, this algorithm depended on having highly accurate data. But "real data are never as nice as mathematicians would like," Weeks notes. Not only will there be errors in the locations of observed circles, he notes, "but also there's the possibility of a few false matches creeping in, where two temperature patterns match to within some error tolerance just by chance. So you need some way of sifting out the false matches."

Weeks's new-and-improved algorithm does just that. It tolerates a considerable amount of error and uncertainty, and can even predict circles that the astronomers' circle-finding algorithms may have missed. The key new ingredient is group theory. Each pair of matched circles determines an element in a mathematical group.

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Much like a crystal, the group necessarily has a rigid structure. This requirement locks in the geometry and corrects errors. The matched circles are something like regularly spaced trees in an orchard: The observations must reflect the regular spacing, but can also tell whether the orchard is on level ground or rolling terrain.

Because the new algorithm can cope with uncertainty, “we’ll start taking a look as soon as any sort of data is available,” Weeks says. “If there are some big circles out there just screaming to be observed, then we can get them with incomplete data. But if the incomplete data isn’t good enough, then we’ll look again with more complete data later on.”

There is no guarantee, of course, that MAP will find any matched circles: The universe could in fact be infinite, or simply too large. Nevertheless, it’s an exciting time to be a cosmologist, Spergel says: “In two years we could know that we live in a finite universe.”

Topology Makes the World Go Round

The surface of last scattering is not a physical surface, like an eggshell or a balloon. It’s a mathematically defined surface, consisting of points in the universe that currently are 15 billion (give or take a billion) light years from Earth—in other words, points from which the remnant microwave photons are only now arriving to be seen by astronomers. The self-intersections of this “virtual” surface are simply those points from which there are two straightline paths to the Earth of exactly the same length. For each such point, the two paths correspond to our looking in different directions in the sky—say toward Vega for one and Arcturus for the other.

How can there be two paths of the same length between two points (the Earth being the second point)? The answer is easiest to explain with 1- and 2-dimensional analogies. The basic idea is kin to the joking notion that if you could see all the way around the world, you’d be looking at the back of your head. A 1-dimensional, circular “universe” can be viewed mathematically as a periodically repeating straight line, which mathematicians call, appropriately enough, the “universal cover” of the circle. In 2 dimensions the simplest finite “universe” is a torus (more commonly called a donut). The universal cover of the torus is an infinitely repeating grid. If the “circle of last scattering” for such a universe is large enough, its repeating images intersect, in patterns ranging from simple to baroque (see Figure 3).

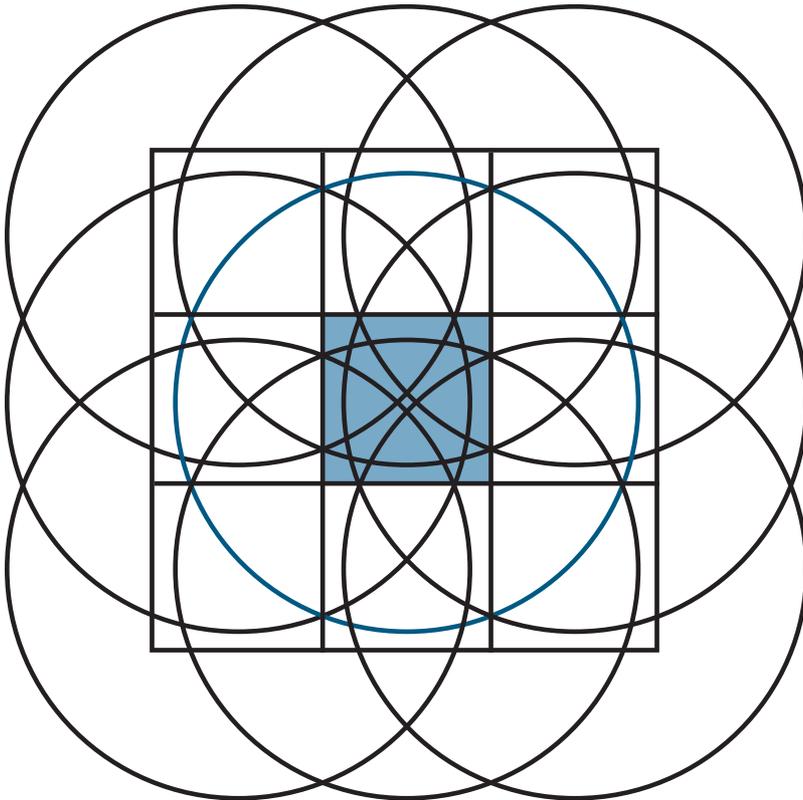
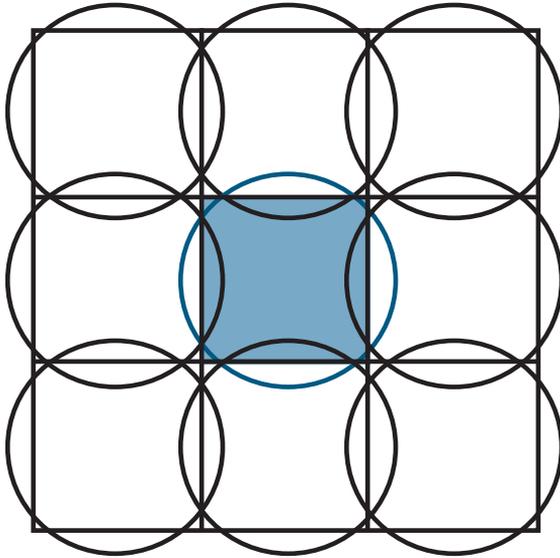


Figure 3. In a 2-dimensional toroidal space, a “circle of last scattering” just slightly larger than its universe intersects itself in a simple pattern (top); a larger circle has a more intricate pattern of self-intersections (bottom). Something similar occurs in 3 dimensions, except that the self-intersections are circles rather than points.

If two compact surfaces have the same number of holes, then they are topologically identical.

Although a torus may look curved (and a real donut, of course, *is* curved!), its geometry is actually “flat,” because its universal cover is simply the flat, Euclidean plane. But the torus is only one of many topological possibilities, most of which can only be realized using non-euclidean geometry. In 2 dimensions, every finite (technically called “compact”) surface can be represented by a polygon with various pairs of sides identified. The torus, for example is a square (or, more generally, a parallelogram) with opposite sides identified. As it turns out, the same surface may be represented in more than one way (see Figure 4). But topogists have a simple way to tell when that happens: they can compute a number, called the “genus,” which counts the number of holes in the surface. The torus, for example, has one hole, while the sphere has none. The genus is said to be an “invariant” of the surface—it stays the same no matter how the surface is deformed (picture a pliant balloon or underinflated inner tube).

A fundamental theorem in topology states that if two compact surfaces have the same number of holes, then they are topologically identical. In other words, each surface is characterized by a single invariant, namely its genus. (To be precise, this only applies to “orientable” surfaces. The theory also treats non-orientable surfaces, such as the Klein bottle, a near relative of the famous Möbius strip.)

Things are much more complicated in 3 dimensions (and even more so in 4-d). Finite spaces can still be represented geometrically, in this case by polyhedra with various pairs of faces identified. But mathematicians have no sure way of telling whether or not two spaces, represented by different polyhedra, are topologically the same. No single invariant, such as genus, is known to be different for different spaces.

There is, however, a conjectured way to classify 3-d spaces. Known as the Thurston Geometrization conjecture, this classification says, roughly speaking, that every 3-d space can be uniquely decomposed into pieces, each of which has a prescribed geometric structure. Moreover, in most cases (“most” having a precise mathematical meaning) the geometry is hyperbolic. The conjecture was formulated in the 1970s by William Thurston, now at the University of California at Davis, who proved that this approach accounts for a large class of spaces known as Haken manifolds. Mathematicians are confident that the Geometrization conjecture is correct—like the Taniyama–Shimura conjecture (see “New Heights for Number Theory,” page 2), it has been checked in a great many cases and is seen as a powerfully unifying theory—but a proof seems to be a long ways off. On the other hand, things that appear distant in one direction may actually be quite close.

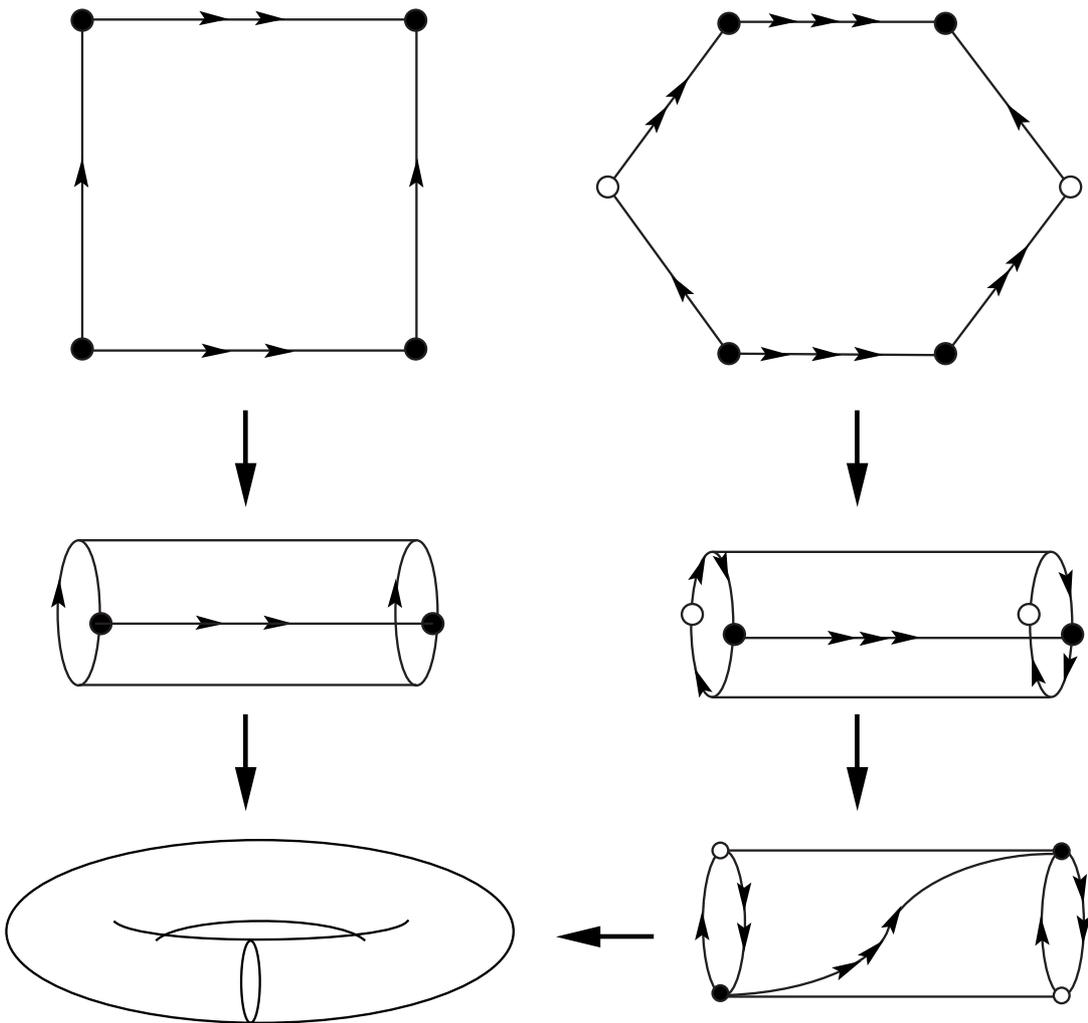


Figure 4. Two different polygons can produce the same topology.