



**Tombstone.** The tombstone of Ludolph van Ceulen in Leiden, the Netherlands, is engraved with his amazing 35-digit approximation to pi. Notice that, in keeping with the tradition started by Archimedes, the upper and lower limits are given as fractions rather than decimals. (Photo courtesy of Karen Aardal. ©Karen Aardal. All rights reserved.)

# Digits of Pi

Barry Cipra

THE NUMBER  $\pi$ —the ratio of any circle’s circumference to its diameter, or, if you like, the ratio of its area to the square of its radius—has fascinated mathematicians for millennia. Its decimal expansion, 3.14159265 . . . , has been studied for hundreds of years. More recently its binary expansion, 11.001001 . . . , has come under scrutiny. Over the last decade, number theorists have discovered some surprising new algorithms for computing digits of  $\pi$ , together with new theorems regarding the apparent randomness of those digits.

In 2002, Yasumasa Kanada and a team of computer scientists at the University of Tokyo computed a record 1.2411 *trillion* decimal digits of  $\pi$ , breaking their own previous record of 206 billion digits, set in 1999. The calculation was a shake-down cruise of sorts for a new supercomputer. That’s one of the main reasons for undertaking such massive computations: Calculating  $\pi$  to high accuracy requires a range of numerical methods, such as the fast Fourier transform, that test a computer’s ability to store, retrieve, and manipulate large amounts of data.

The first rigorous calculation of  $\pi$  was carried out by the Greek mathematician Archimedes in around 250 BC. Archimedes did not have the decimal system at his disposal. Instead he worked with fractions. In his treatise *The Measurement of the Circle*, only parts of which have survived, Archimedes showed that  $\pi$  is greater than  $3\frac{10}{71}$  but less than  $3\frac{1}{7}$ . (The latter value, also written as  $22/7$ , is one many people remember from school as “the” value of  $\pi$ .) In modern, decimal terms, these values are correct to two decimal digits:  $3\frac{10}{71} = 3.14084 \dots$  and  $3\frac{1}{7} = 3.14285 \dots$

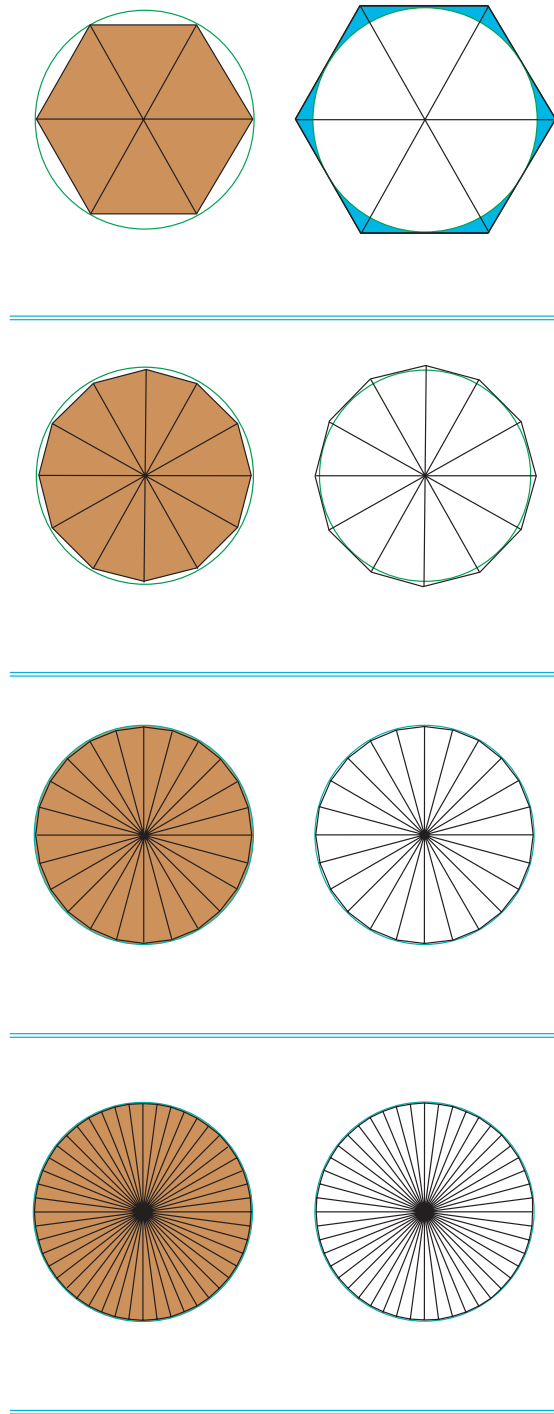
Archimedes’s method is based on polygonal approximations to the circle. The basic idea is to repeatedly double the number of sides in the polygon. Archimedes in effect worked out the relationship between the perimeter of the new polygon (with twice as many sides) and the perimeter of the old one (see Figure 1, next page). As the number of sides increases, the polygon becomes more and more indistinguishable from a circle, and the perimeter of the polygon becomes closer and closer to the circumference of the circle. If the polygon is inscribed in the circle, its perimeter will be slightly too short; if it is circumscribed around the circle, the perimeter will be slightly too long. Archimedes started with a hexagon and doubled the number of sides four times, so his under- and overestimates for  $\pi$  were based on a polygon with 96 sides.

Over the centuries, other mathematicians extended Archimedes’s calculation with more and more doublings. This approach peaked around 1600, when the Dutch mathematician Ludolph van Ceulen used 60 doublings of an inscribed square

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**Figure 1.** Archimedes computed one of the first good approximations to  $\pi$  by starting with a hexagon inscribed in a circle or circumscribed about it. The perimeter of the hexagon is easy to compute, and gives a rather inaccurate approximation to  $\pi$ . Each time the number of sides is doubled, the approximation gets better. Archimedes used basic trigonometry to figure out how to compute the perimeter of each “doubled” polygon from the perimeter of the one before. His final result was that  $\pi$  lies between  $3 \frac{1}{7}$  and  $3 \frac{10}{71}$ .

to obtain the first 35 digits of  $\pi$ . (See “Figure “Tombstone” on p. 28.) Incidentally, this is more than enough accuracy for any conceivable practical purpose. Even if a circle was perfect down to the atomic scale, the thermal vibrations of the molecules of ink would make most of those digits physically meaningless. Thus, for the last four centuries (and arguably since Archimedes), mathematicians’ attempts to compute  $\pi$  to ever greater precision have been driven primarily by curiosity about the number itself. In recent years, a second motivation has been the desire to improve the techniques of high-precision computing, for which computations of  $\pi$  have become a benchmark.

The seventeenth century brought a radically improved method of computing  $\pi$ , using the tools of calculus rather than geometry. Mathematicians discovered numerous infinite series associated with  $\pi$ . The most famous is known as Leibniz’s formula,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

Although it is one of the most striking formulas in calculus, Leibniz’s formula is completely impractical for approximating  $\pi$  to more than a couple of decimal places because it converges very slowly. Newton, who along with Leibniz was one of the co-discoverers of calculus, found a more efficient formula based on an arcsine integral:

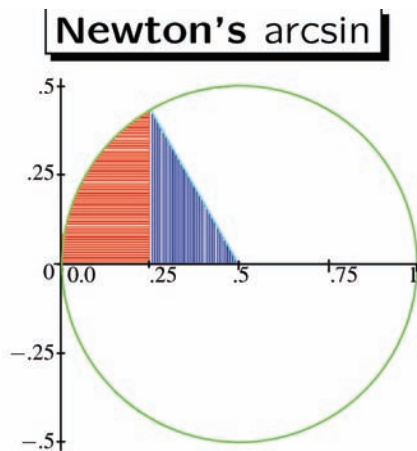
$$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{3 \cdot 8} - \frac{1}{5 \cdot 32} - \frac{1}{7 \cdot 128} - \frac{1}{9 \cdot 512} - \dots\right).$$

The first term to the right of the parentheses comes from finding the area of the blue triangle in Figure 2; the remaining terms come from setting up an integral to evaluate the area of the red region on the left and evaluating it with an infinite series. In this way, Newton computed  $\pi$  to 15 digits. (In a famous letter to a friend, written in 1666, Newton confessed “I am ashamed to tell you to how many figures I carried these computations, having no other business at the time.”)

Most of the infinite series for  $\pi$  are based on an infinite series for the inverse tangent function,  $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$  and the fact that  $\tan(\pi/4) = 1$  (i.e.,  $\arctan(1) = \pi/4$ ). Leibniz’s formula simply lets  $x = 1$ . To get more rapid convergence, it is necessary to use smaller values of  $x$ . One pretty method uses the geometric diagram in Figure 3 (see next page), from which it is easy to show that

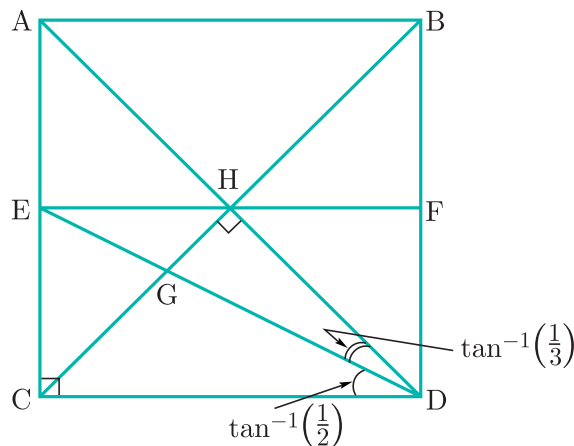
$$\pi = 4\left(\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)\right).$$

[To see that this is true, notice that  $\angle CDH = \pi/4$  and that  $\angle CDE = \arctan(\frac{1}{2})$ , from the two corresponding right triangles. Meanwhile,  $\angle GDH = \arctan(GH/DH) = \arctan(GH/CH) = \arctan(1/3)$ , because of the classic result from Euclidean geometry that the medians of a triangle (in this case  $\triangle ACD$ ) divide each other in a 1:3 ratio.] Starting with the above trigonometric identity, which was discovered by Euler in 1738, one can substitute  $\frac{1}{2}$  and  $\frac{1}{3}$  for  $x$  into the arctangent series. Even the first five terms of the series (or ten terms



**Figure 2.** The geometric reasoning behind Newton’s calculus-based 15-digit approximation of  $\pi$ . The circular sector shown has area  $\pi/24$ . The area of the blue triangle is easily computed by elementary geometry. The area of the red triangle can be computed by an integral, which Newton rewrote as an infinite series. By stopping the summation after a finite number of terms, one can compute any desired number of decimal digits of  $\pi$  (with enough patience). (Figure courtesy of Jonathan Borwein.)

total—five for  $x = \frac{1}{2}$  and five for  $\frac{x-1}{3}$ ) are enough to obtain a pretty good approximation,  $\pi \approx 3.1417\dots$



**Figure 3.** Another geometrically inspired formula for  $\pi$ , discovered by Euler:  $\pi = \arctan 1 + \arctan 2 + \arctan 3$ , is equivalent to  $\pi/4 = \arctan \frac{1}{2} + \arctan \frac{1}{3}$ . Angle  $CDG$  has measure  $\arctan \frac{1}{2}$ , and angle  $GDH$  has measure  $\arctan \frac{1}{3}$ . The arctangents can be computed by using the well-known convergent series for  $\arctan x$  from calculus. (Based on a Figure by Michael Ecker, *College Mathematics Journal*, May 2006, p. 219. Courtesy of the Mathematical Association of America.)

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Even better is a formula discovered by the English mathematician John Machin and first published in 1706:

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}.$$

Getting two digits of  $\pi$  requires only three terms:  $\pi \approx 16(\frac{1}{5} - \frac{1}{375}) - \frac{4}{239} \approx 3.14$ . Machin's formula became the basis for almost all computations of  $\pi$  for the next two centuries. Machin himself computed a hundred digits, and the zenith of his formula's popularity undoubtedly arrived in 1853, when William Shanks published a book in which he calculated 607 digits of  $\pi$ . Twenty years later, he added another hundred digits, giving  $\pi$  to 707 digits. (Shanks also calculated extensive tables of prime numbers, reciprocals, and digits of another famous number known as Euler's constant.)

Shanks's calculation stood unchallenged until 1945, when D.F. Ferguson, working with a different formula ( $\pi = 12 \arctan(1/4) + 4 \arctan(1/20) + 4 \arctan(1/1985)$ ), produced a new approximation—and discovered an error in Shanks's value. Not only that, Ferguson managed to find where Shanks made his mistake. Shanks had omitted a zero in the 531st decimal place of  $1/497 \cdot 5^{497}$ , so that the number  $\dots 80482897\dots$  was incorrectly replaced by  $\dots 8482897\dots$ . As a consequence (after multiplying the  $\arctan(1/5)$  by 16) Shanks's value of  $\pi$  was correct to only 527 digits.

Ferguson went on to calculate 710 digits of  $\pi$ , using a mechanical calculator. This was a first, as all earlier approximations had been done by hand, but an even bigger change was

coming. By 1949, ENIAC, one of the first electronic computers, set a new record of more than 2000 digits. As electronic computers improved, the number of known digits of  $\pi$  took off into the stratosphere: ten thousand digits in 1958, a hundred thousand in 1961, and a million in 1973.

Remarkably, one of the obstacles to high-precision computation of  $\pi$ , which surfaced as early as the 1950s, was the inefficiency of the “grade school” method of multiplying two numbers together. In many-digit computations, computer programmers typically store 32 bits (or 8 decimal digits) in each computer “word.” Computing the product of two  $N$ -digit numbers by the conventional method involves multiplying every pair of computer words, so that a product of two  $N$ -digit numbers requires a constant times  $N^2$  multiplications, plus a comparable number of additions. In the 1960s, computer scientists realized that there is a much faster method for large numbers, which involves first taking a fast Fourier transform (FFT) of each of the strings of words. This elegant scheme simplifies the computation so that it requires only a constant times  $N \log N$  steps, which for large  $N$  is much less than  $N^2$ .

Unfortunately, FFTs turn out to be difficult on today’s massively parallel supercomputers because of the need for massive exchanges of data between nodes of the system. It is better, if possible, to structure the computation so that FFTs are only performed on a single node, and other schemes are used between nodes. In short, a state-of-the-art computation of  $\pi$  turns out to be even more difficult than it appears at first glance; the computer programmer has to be very careful about how to implement the calculations efficiently. One might even say that the mathematics is the easiest part!

Nevertheless, Kanada and colleagues, who have set most of the records since the early 1980s, reached the billionth digit of  $\pi$  by 1989. And in 2002, they pushed the record to a staggering one trillion digits. The trillion-digit computation took some 600 hours. It was performed by a 64-processor parallel supercomputer built by Hitachi. Kanada’s team actually computed  $\pi$  twice, using two different formulas (see Figure 4 next page). The calculations were initially carried out in hexadecimal, or base-16, which is convenient for computers but also has theoretical significance (as explained below). In hexadecimal, the “digits” for ten through fifteen are usually denoted by the letters A through F, so that the hexadecimal expansion of  $\pi$  is 3.243F6A8..., which means  $3 + (2/16) + (4/16^2) + (3/16^3) + (15/16^4) + \dots$ . The computer calculated just over a trillion hexadecimal digits of  $\pi$  with each formula, and checked that the results agreed. It then ran an algorithm to convert the hexadecimal result into an ordinary, base-10 expansion. Because base 10 is smaller than base 16, the conversion produced 1.2411 trillion decimal digits. As one final check, Kanada’s team converted their decimal expansion *back* into hexadecimal and verified that it agreed with the original result.

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### Back to the Future

In December 2002, Kanada computed  $\pi$  to over **1.24 trillion decimal digits**. His team first computed  $\pi$  in hexadecimal (base 16) to 1,030,700,000,000 places, using the following two arctangent relations:

$$\pi = 48 \tan^{-1} \frac{1}{49} + 128 \tan^{-1} \frac{1}{57} - 20 \tan^{-1} \frac{1}{239} + 48 \tan^{-1} \frac{1}{110443}$$
$$\pi = 176 \tan^{-1} \frac{1}{57} + 28 \tan^{-1} \frac{1}{239} - 48 \tan^{-1} \frac{1}{682} + 96 \tan^{-1} \frac{1}{12943}$$

due to Takano (1982) and Stöfmer (1896). Kanada verified the results of these two computations agreed, and then converted the hex digit sequence to decimal.

The resulting decimal expansion was checked by converting it back to hex.

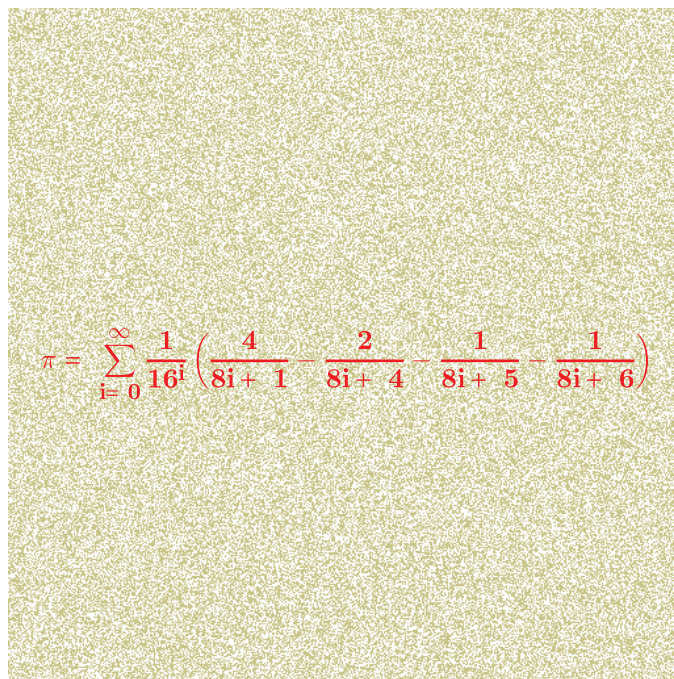
**Figure 4.** The formulas Kanada used for computing  $\pi$ . (Figure courtesy of Jonathan Borwein.)

The new calculation also matched up with the 206 billion digits of the 1999 computation. Kanada's team did one more spot check on the correctness of their computation: Using another algorithm, they recomputed 20 hexadecimal digits beginning with the trillion-and-first. The result (B4466E8D215388C4E014, if you must know) agreed with the main computation.

The algorithm for computing isolated digits of  $\pi$  is based on a remarkable formula discovered in 1996 by David Bailey, now at the Lawrence Berkeley Laboratory, Peter Borwein at Simon Fraser University in Burnaby, British Columbia, and Simon Plouffe, now at the University of Montreal. The BBP formula, as it's called, looks on the surface like many other infinite series approximations for  $\pi$  (see Figure 5), but it represents another major breakthrough in  $\pi$  computation. For the first time, a computer can compute the hexadecimal digits of  $\pi$  starting anywhere. That is, to obtain the trillion-and-first hexadecimal digit, it is not necessary to compute any of the first trillion digits. Until Bailey, Borwein and Plouffe discovered their formula, no one had even dreamed such a thing was possible.

The other unusual aspect of the BBP formula is the way it was discovered. Most formulas for  $\pi$ —indeed, most formulas of any sort—are derived by hand, starting from other known formulas. The BBP formula, though, was the product of a computer search. In 1995, Borwein and Plouffe realized that any infinite sum whose terms are of the form  $P(k)/b^k Q(k)$ , where  $P$  and

$Q$  are polynomials, permits the rapid determination of individual base- $b$  digits. Their starting point was a familiar formula for the natural logarithm of 2,  $\ln 2 = \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \dots$ , but they were soon able to find similar formulas in the mathematical literature for many other well and lesser known numbers. However, their literature search did not turn up anything for  $\pi$ . They had better luck looking in the computer.



$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left( \frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right)$$

**Figure 5.** *The Borwein-Bailey-Plouffe formula for  $\pi$ , which for the first time made it possible to compute any given digit of  $\pi$  without knowing all the previous digits. However, it works only in base 16 (or more generally, a power of 2). In the background is a computer image of the first 250,000 binary digits of  $\pi$ , with each 0 rendered as a dark pixel and each 1 rendered as a light pixel. The image is 500 pixels by 500 pixels. (Figure courtesy of Jonathan Borwein.)*

Borwein and Plouffe turned to an algorithm called PSLQ, developed by Bailey and Helaman Ferguson, who was then at the Supercomputing Research Center in Lanham, Maryland. PSLQ is a streamlined version of an algorithm discovered in the 1970s by Ferguson and Rodney Forcade of Brigham Young University for numerically detecting integer relations among real numbers. Borwein and Plouffe input  $\pi$  along with all the numbers for which they had found base-2 type infinite sums. After several weeks—they restarted the computation several times as they found more formulas in the literature—PSLQ discovered a remarkably simple relation involving the natural logarithm, the arctangent, and a special function known as the hypergeometric function, usually written  $F(a, b; c; z)$ . The number  $\pi$ , it turns out, is equal to  $4F(1/4, 5/4; 1; -1/4) + 2\arctan(1/2) - \ln(5)$ . Each of these three terms can be expanded as a sum of the type Borwein and Plouffe had been studying. When the computational dust



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settled, they had a striking new formula for  $\pi$  (see Figure 6), which can be used to compute individual digits of  $\pi$  in base 16 (hexadecimal notation). As often happens in mathematics, after the fact they discovered a more elementary proof of the formula that uses only techniques from a first-year calculus course (and no hypergeometric function).

Incidentally, it remains impossible to compute the trillion-and-first digit in the *decimal* expansion of  $\pi$  without computing all the preceding ones. The BBP formula only makes this trick possible in base 16 (as well as 2, 4, and 8). There are strong grounds for believing that no such formula exists in base 10, the base humans normally use for arithmetic. It seems as if an accident of our anatomy (the fact that we have 10 fingers) has stuck us with the “wrong” notation system for computing  $\pi$ .

The analysis of BBP-type formulas has led Bailey and Richard Crandall, a computational number theorist at Reed College in Portland, Oregon, to some intriguing new results in the study of so-called “normal” numbers. Roughly speaking, a number is normal (base 10) when its decimal expansion doesn’t favor any string of digits over any other—that is, each digit appears, on average, ten percent of the time, each *pair* of digits appears one percent of the time, and so forth. There is a similar meaning for being normal base- $b$ : Every string of  $k$  base- $b$  digits appears, in the limit, with frequency  $1/b^k$ . In other words, it seems as if the digits in base  $b$  have been generated by a random-number generator.

It is suspected that  $\pi$  is normal in every base. Such numbers are called “absolutely normal.” Curiously, even though almost all real numbers are absolutely normal (“almost all” has a technical meaning in real analysis), there is not a single known example of such a number! In fact, researchers suspect that all of the irrational numbers they encounter in “normal” mathematics, including square roots and logarithms such as  $\sqrt{2}$  or  $\ln 2$ , are absolutely normal, but none of these numbers is yet known to be normal in any base, much less all bases. (Rational numbers, of course, have no hope of being normal, since they are characterized as having repeating—and therefore highly non-random—decimal expansions.)

Kanada’s latest calculation of  $\pi$  exhibits no sign of abnormality in either base 10 or base 16 (see Figure 6). But no calculation of digits can ever settle the matter one way or the other. It is always possible that  $\pi$  could go through a trillion digits without any sign of abnormality, and then suddenly start churning more 9’s than 0’s. Such behavior would seem too perverse to be credible—but nevertheless, in the absence of a solid theoretical proof it cannot be ruled out. No such theorem has been proved for  $\pi$ . However, Bailey and Crandall have proved a criterion for base- $b$  normality for numbers with a BBP-type formula, and they have used their theorem to prove normality for one class of such numbers.

Decimal Digit	Occurrences
0	99999485134
1	99999945664
2	100000480057
3	99999787805
4	100000357857
5	99999671008
6	99999807503
7	99999818723
8	100000791469
9	99999854780
Total	1000000000000

**Figure 6.** The number of occurrences of each decimal digit in the first trillion digits of  $\pi$ . The digits appear to occur equally often, with a few random fluctuations, as one would expect if  $\pi$  is a normal number. However, no one has proved yet that  $\pi$  is normal, and in fact no specific instance of a normal number is known. (Figure courtesy of Jonathan Borwein.)

Bailey and Crandall’s normality condition for a number  $\alpha$  given by a BBP-type formula  $\sum P(k)/b^k Q(k)$  involves looking at a sequence of fractions between 0 and 1 that starts  $x_0 = 0$  and, for  $k = 1, 2, \text{etc.}$ , defines  $x_k$  as the fractional part of  $b x_{k-1} + P(k)/Q(k)$ . For  $\ln 2 = \sum 1/2^k$ , the sequence thus defined begins 0, 0, 1/2, 1/3, 11/12, 1/30, 7/30, 64/105, 289/840, etc. Their theorem says that  $\alpha$  is base- $b$  normal if and only if the sequence of  $x_k$ s is equidistributed—that is, no subinterval of  $[0, 1]$  gets more (or less) than its fair share of  $x_k$ s (more precisely, the fraction of  $x_k$ s landing in any subinterval of length  $\ell$  tends to  $\ell$  as  $k$  tends to infinity).

Using this theorem, Bailey and Crandall have proved normality for a particular class of numbers with BBP-type formulas: Numbers of the form  $\sum 1/b^{c^k} c^k$  are base- $b$  normal whenever  $b$  and  $c$  are relatively prime (i.e., have no prime divisor in common). This had been proved for the case  $b = 2, c = 3$  by Richard Stoneham in 1970; the new proof is simultaneously simpler and more general. Unfortunately, these numbers that have been “cooked up” as examples of normality do not have any known independent meaning as roots of polynomials or values of significant functions.

Proving the normality of  $\pi$ , or any other familiar irrational number, therefore remains a daunting challenge. But it wasn’t so long ago that computing a trillion—or even a mere billion—digits of  $\pi$  seemed forever out of reach. Stay tuned.

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### Irrationalities

The discovery that the square root of 2 cannot be expressed as the ratio of two whole numbers was one of the turning points of ancient Greek mathematics. According to one legend, the fact was discovered by members of the Pythagorean brotherhood, who sought to keep it secret. When the secret was revealed by one of their members (the Deep Throat of his day), the brothers reacted by tossing him overboard during a sea voyage.

The irrationality of  $\sqrt{2}$  is easy to prove. For centuries mathematicians wondered about  $\pi$ . Finally, in the eighteenth century, they found a proof that  $\pi$  is irrational. Unlike  $\sqrt{2}$ , however, the proof is complicated. Later in the nineteenth century, it was proved that  $\pi$  is not only irrational but “transcendental,” meaning that it is not the root of any polynomial with integer coefficients. (Numbers that are roots of polynomials, such as  $\sqrt{2}$ , which is a root of  $x^2 - 2$ , are called algebraic.) In general, proofs of transcendental are very complicated.

Many other mathematical constants are known to be irrational, and quite a few are known to be transcendental. Others have remained stubbornly mysterious. Among them are certain special numbers associated with number theory’s most famous function, the Riemann zeta function,  $\zeta(s) = \sum 1/n^s$  (see “A Prime Case of Chaos,” *What’s Happening in the Mathematical Sciences*, Volume 4). For positive integers  $s = 2n$ , the value of the zeta function is a rational multiple of  $\pi^{2n}$ . For example,  $\zeta(2) = \pi^2/6$ ,  $\zeta(4) = \pi^4/90$ ,  $\zeta(6) = \pi^6/945$ , and so forth. All this was discovered by the Swiss mathematician Leonhard Euler in the eighteenth century, who proved the irrationality of the number  $e$ .

It may seem natural to suppose that the same is true for odd values of  $s$  as well. But it is almost certainly *not* true. The ratio  $\zeta(3)/\pi^3$  has been computed to millions of decimal places, with no sign of a repeating pattern. If the ratio is a fraction, the numerator and denominator are extremely large.

Because  $\pi$  is transcendental, its powers are all irrational, and hence the values of the zeta function at even values of  $s$  are all irrational (indeed, transcendental). Can anything be said about the irrationality of the zeta function at odd values of  $s$ ? Yes. In 1979, Roger Apéry at the University of Caen in France surprised number theorists with a proof that  $\zeta(3)$  is irrational. Apéry's proof, while complicated, did not use any modern methods; it was dubbed "the proof that Euler missed."

It was initially anticipated that Apéry's proof would open the floodgates for similar results on the zeta function at other odd values of  $s$ . It hasn't worked out that way. Recently, however, Tanguy Rivoal at the Université de Grenoble in France and Wadim Zudilin at Moscow State University in Russia have opened the sluices a bit.

Rivoal has shown that infinitely many of the values  $\zeta(3)$ ,  $\zeta(5)$ ,  $\zeta(7)$ ,  $\zeta(9)$ , etc. are irrational—and not only irrational, but *independently* irrational. That is, there are infinitely many odd numbers  $2n + 1$  for which  $\zeta(2n + 1)$  cannot be written as a rational combination of the form  $r_1 + r_3\zeta(3) + \dots + r_{2n-1}\zeta(2n - 1)$  with rational numbers  $r_1, r_3, \dots, r_{2n-1}$ . In fact, Rivoal showed that the number of independent irrationals among the numbers first  $n$  odd values of the zeta function grows at least with the logarithm of  $n$  (more precisely, it exceeds  $(\ln n)/3$ ). By carefully examining the estimates that underlie his proof, Rivoal was able to show that at least one of the nine numbers  $\zeta(5)$ ,  $\zeta(7)$ ,  $\dots$ ,  $\zeta(21)$  is irrational.

Zudilin found similar results independently, and has strengthened some of Rivoal's results. In particular, Zudilin has shown that there is an irrational number among the next four values of the zeta function,  $\zeta(5)$ ,  $\zeta(7)$ ,  $\zeta(9)$ ,  $\zeta(11)$ . He has also shown that there is an irrational value independent of  $\zeta(3)$  among the values  $\zeta(5)$  up to  $\zeta(145)$  (Rivoal had gotten an upper limit of 169).

Further research will undoubtedly continue to chip away at the irrational nature of the zeta function. At least today's mathematicians don't worry about getting thrown overboard for making their discoveries public.