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# Representations of Finite Groups as Permutation Groups

MICHAEL ASCHBACHER

In 1860 the Paris Academy offered its Grand Prix des Mathematiques for a contribution to the solution of the following problem:[1]

*For given n, what are the possible indices m of subgroups of the symmetric group of degree n, and given m, what are the subgroups of index m?*

Three manuscripts were submitted to the Academy in the prize competition; the contributers were Kirkman, Jordan, and Mathieu. None of the contributions were judged worthy of the prize.

I believe it is fair to say that there was little significant progress on this problem until about 1955, when dramatic developments in the study of finite simple groups began to make the possiblity of a solution more realistic. The classification of the finite simple groups in 1981 and the continued expansion of our knowledge of the finite simple groups themselves have now brought at least a weak solution to the problem within reach.

The effort to solve the problem is one of the current active areas of research in finite group theory and touches most of the other active areas of the subject. I propose to discuss this effort and to use that discussion as a focus for a more general discussion of the major developments in finite group theory of the last few decades and for speculation on the future of the subject.

Let us begin by restating our problem in modern language. A *representation* of a group $G$ on an object $X$ is a group homomorphism $\pi : G \to \mathrm{Aut}(X)$ of $G$ into the group of automorphisms or symmetries of $X$. Most mathematicians are familar with *linear representations,* where $X$ is a vector space over a field. But for finite groups a more basic class of representations are the *permutation representations,* where $X$ is a set. Thus a permutation representation of $G$ is a group homomorphism $\pi : G \to \mathrm{Sym}(X)$ of $G$ into the symmetric group on a set $X$.

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[1]My (limited) knowledge of the early history of finite groups comes from a set of lectures given by Peter Neumann at Oxford in 1983 and from an expository article by Walter Feit **[6]**.

In the mid-nineteenth century the term "group" meant "permutation group" or "group of transformations". The notion of an abstract group did not yet exist and hence one could not speak of group representations. Each group came equipped with a permutation representation. However today we can restate our problem in the following form:

*Describe up to equivalence all permutation representations of finite groups.*

When stated in this form we see that we have not really posed the right question. For one thing the problem is not realistic: We cannot hope to completely describe all finite groups, much less their permutation representations. What we can do is decompose our representation and our group into indecomposables and irreducibles and attempt to describe the irreducibles.

A permutation representation $\pi : G \to \mathrm{Sym}(X)$ is indecomposable if it is *transitive*; that is, for all $x$, $y$ in $X$ there is a permutation in $G\pi$ mapping $x$ to $y$. It is a fact that any transitive permutation representation of $G$ is equivalent to a representation of $G$ by right multiplication on the set $G/H$ of cosets of the subgroup $H$ of $G$ fixing $x$ in $X$. Thus the study of permutation representations is equivalent to the study of subgroup structure.

A transitive representation is irreducible if it is *primitive*; that is, $G$ preserves no nontrivial equivalence relation on $X$. This is equivalent to requiring that $H$ be a maximal subgroup of $G$. Many problems on finite permutation groups can be reduced to a problem about primitive groups. Thus we are lead to reformulate our problem as follows:

*Determine up to equivalence all injective primitive permutation representations of finite groups.*

I contend this is the right formulation of our problem. It is right because the hypotheses are on the one hand sufficient for most applications and on the other hand restrictive enough to admit a solution, at least in a weak sense, which is strong enough for our applications. For example the theory of groups began in the nineteeth century, where permutation groups were used to study the solutions to polynomial equations. Today the classification of the finite simple groups and our knowledge of the subgroup structure of the simple groups has made possible the solution to problems in areas of mathematics as diverse as model theory, number theory, topology, and combinatorics. Most of these applications arise by reducing the model-theoretic or number-theoretic problem to a problem on primitive permutation groups. A more complete description of primitive permutation representations of the finite groups should lead to even more applications.

The reason that our new problem admits a solution is that most finite groups do not admit an injective primitive permutation representation. Indeed in [2] it is shown that each such group has one of five general structures. The most interesting structure occurs when $G$ is *almost simple*; that is, $G$ has a unique minimal normal subgroup $L$ and $L$ is a nonabelian simple group. Equivalently, $\mathrm{Inn}(L) \leq G \leq \mathrm{Aut}(L)$. Thus we have our final formulation of our problem:

*Determine up to conjugation the maximal subgroups of each almost simple finite group.*

This formulation focuses attention on the simple groups and their subgroups. Thus I will interrupt our discussion of primitive groups to recall the statement of the Classification, to discuss the simple groups, and to make a few brief remarks about the history of the subject.

CLASSIFICATION THEOREM. *Each finite simple group is isomorphic to one of the following*:

(1) *A group of prime order.*
(2) *An alternating group.*
(3) *A group of Lie type.*
(4) *One of* 26 *sporadic simple groups.*

Of course there is a unique group of order $p$ for each prime $p$. The alternating group $\mathscr{A}_n$ of degree $n$ is the normal subgroup of all even permutations in the symmetric group of degree $n$. The groups of Lie type are analogues of the simple Lie groups. Finally we have the twenty-six sporadic groups, which fall into no known naturally defined infinite family.

Roger Howe will be discussing Lie theory in more detail in a later talk in this series. Lie theory plays an important role in the study of finite simple groups. The simple Lie groups were classified by Killing and Cartan in the late nineteenth century; associated to each is a simple Lie algebra. In 1955, Chevalley [**3**] showed that each simple Lie algebra $X$ over $\mathbf{C}$ possesses a Chevalley basis with respect to which the structure constants of $X$ are integers. Then the basis elements can be exponentiated and reduced modulo $p$ for each prime $p$ to produce a *Chevalley group* $X(F)$ over any field $F$. When $F$ is finite $X(F)$ is finite and esssentially simple. Chevalley's work was extended to produce other groups of Lie type: the *twisted Chevalley groups* analogous to real forms of Lie groups. The Lie theory also gives important information about these groups such as their automorphism groups and certain subgroups. Borel, Ree, Springer, Steinberg, and Tits made important contributions here. The finite simple groups of Lie type are divided into two classes: the *classical groups* and the *exceptional groups*. The classical groups are the special linear group plus the isometry groups of nondegenerate bilinear and hermitian symmetric sesquilinear forms. The exceptional groups correspond to the exceptional simple Lie algebras.

The sporadic groups are fascinating discrete objects. Each group, by the nature of its existence, corresponds to a number of pathological group-theoretic, combinatorial, and number-theoretic phenomena. I will say more about the sporadic groups in a moment.

The appearance of the Chevalley groups and twisted Chevalley groups was one of the important group theoretic events occurring in the midfifties. The other was the beginning of modern local group theory. *Local group theory* studies a finite group $G$ via its $p$-subgroups and the normalizers in $G$ of these $p$-subgroups. Sylow's Theorem is perhaps the earliest result in local group theory. Philip Hall proved his extended Sylow theorem for solvable groups in 1937 and Brauer introduced his program for characterizing simple groups by the centralizers of involutions in the fifties. However the first spectacular success of the local theory was Thompson's verification of the Frobenius conjecture in his thesis in the late fifties, followed several years later by the verification by Feit and Thompson of the old conjecture of Burnside that groups of odd order are solvable. The local theory was the principal tool used to establish the Classification. While many mathematicians made major contributions to the local theory, I believe it is fair to say that Thompson had the largest role in its creation.

The next major event in finite group theory was the appearance of the sporadic groups. The first five sporadic groups were discovered by Mathieu (remember he made one of the contributions to the 1860 Paris Prize) in the nineteeth century as multiply transitive permutation groups. The next sporadic group was not discovered until 1965 by Janko, using the local theory. After that sporadic groups were discovered at the rate of about two or three a year until Janko also discovered the last of the groups in 1976.

The largest sporadic group (known as the *Monster*) was discovered independently by Fischer and Griess in 1974. There are a number of mysteries involving the Monster. For example it is conjectured [4] that there is a series $\chi_i$, $1 \leq i \leq \infty$, of characters (*Thompson series*) of the Monster such that $1/q + \sum_i \chi_i(1)q^i$ is the elliptic modular function and $1/q + \sum_i \chi_i(g_p)q^i$ is a generator for the function field of genus 0 of a congruence subgroup for the prime $p$, as $g_p$ ranges over elements of prime order $p$ in the Monster. Moreover Frankel, Lepowsky, and Meurman [7] have shown that the Monster is a symmetry group of a holomorphic two-dimensional quantum field theory.

In finite group theory, the seventies was the decade of the push toward the Classification. As a new Ph.D. entering the field at the beginning of the seventies, I can vouch for the excitement created by the regular appearance of sporadic groups and the stream of wonderful theorems that appeared at that time. Many finite group theorists participated in the effort, but the most influencial figure in the movement, both through his mathematical contributions and his orchestration of the program, was Danny Gorenstein.

I would like to say a few words about the complexity of the proof of the Classification and its implications for mathematics. The existing proof of the Classification is very long (Gorenstein estimates 10,000 pages), complicated, and messy. There are efforts to shorten and clean up the proof, but in the absence of some totally new idea, such efforts will still leave us with a complicated proof. I personally do not believe the proof will ever be simple. For one thing, the existence of the sporadic groups insures that the set of examples is rather complex. The groups of Lie type of small rank over small fields also exhibit sporadic behavior.

Many mathematicians seem to be uncomfortable with complicated proofs and pathological mathematical objects. I feel the sporadic groups are beautiful; without them, finite group theory would be less interesting. I also feel the Classification is a wonderful theorem. In discrete mathematics, assumptions of symmetry provide the structure which distinguishes interesting objects from the mundane and takes the place of the analytic or algebraic structure of classical mathematics. The Classification is a means for compactly encoding this structure. I believe it will come to be viewed as one of the most important results in discrete mathematics and as indispensible. If such a result requires a difficult proof, so be it.

After this long digression on simple groups, it is time to return to our problem. Recall we seek to describe the maximal subgroups of each finite simple group $G$. To do so we realize $G$ as the group of automorphisms of a suitable mathematical object $X(G)$. We then seek to prove:

STRUCTURE THEOREM FOR $G$. *A proper subgroup $H$ of $G$ either stabilizes some member of a set $\mathscr{C}(G)$ of natural structures on $X(G)$, or is almost simple and irreducible on $X(G)$.*

Such a structure theorem reduces our problem to the study of structures on $X(G)$ and to the irreducible representation theory of simple groups in the category of $X(G)$. Our structures include substructures, coproduct structures, and product structures; I will give an example soon.

If $G$ is a classical group of Lie type over a field $F$ then $X(G)$ is the pair $(V, f)$, where $V$ is an $FG$-module, $f$ is a bilinear or sesquilinear form on $V$, and $G$ is the isometry group of $f$. A Structure Theorem exists for $G$ **[1]** and it is conjectured that, with a short explicit list of exceptions, if $H$ is almost simple and absolutely irreducible on $V$ with the representation writable over no proper subfield of $F$, preserving no bilinear form other than $f$, and preserving no tensor product structure, then the normalizer in $G$ of $H$ is maximal in $G$. If this conjecture is established then in a weak sense we have determined the maximal subgroups of $G$. To do more would require an enumeration of the irreducible linear representations of finite simple groups over all finite fields.

Extending work of Dynkin on Lie groups **[5]**, Seitz **[10]** has established the conjecture for algebraic groups and used his theorem to establish the conjecture when $H$ is of Lie type with the same characteristic as $G$.

Clearly the study of the maximal subgroups of the classical groups impinges on another active area of finite group theory: the study of linear representations of finite groups. I do not have time to discuss this activity.

I believe the correct object $X(G)$ for an exceptional group $G$ over $F$ is a minimal dimensional $FG$-module together with a three- or four-linear form on $F$. This approach has been successful with groups of type $G_2$ and $E_6$, but much work remains to be done.

The maximal subgroups of twenty-three of the twenty-six sporadic groups have been enumerated. However the treatments are ad hoc and often involve extensive machine calculation, so the situation is not entirely satisfactory.

I will close by considering the alternating group $G = \mathscr{A}_n$ on a set $X$ of order $n$ as an example. We take $X(G)$ to be $X$. Let $n = |X|$ and $S = \mathrm{Sym}(X)$. Except when $n = 6$, $S$ is $\mathrm{Aut}(G)$. We have:

STRUCTURE THEOREM FOR $\mathscr{A}_n$ (O'Nan-Scott **[9]**). *Let $H$ be a proper subgroup of S. Then one of the following holds:*

(1) *$H$ preserves a proper nonempty subset of $X$. (Substructure)*
(2) *$H$ preserves a nontrivial partition of $X$. (Coproduct structure)*
(3) *$H$ preserves a nontrivial realization of $X$ as a set product. (Product structure)*
(4) *$H$ preserves an affine space structure on $X$.*
(5) *The socle of $H$ is the direct product of $k$ copies of some nonabelian simple group $L$ with $n = |L|^{k-1}$. (Diagonal structure)*
(6) *$H$ is almost simple and primitive on $X$.*

Moreover it has been shown that, with known exceptions, the stabilizers of the structures listed in (1)–(5) and the normalizers of primitive almost simple subgroups are indeed maximal **[8]**. Thus, in a weak sense, we know the maximal subgroups of the alternating and symmetric groups.

As I have tried to show, it seems possible that within this century we will be able to completely describe in a weak sense all primitive finite permutation groups. Our present knowledge of such groups has already been applied effectively in various areas of mathematics. As the theory becomes more complete and as mathematicians become aware of its potential, I believe many more applications will be discovered.

## REFERENCES

1. M. Aschbacher, *On the maximal subgroups of the classical groups*, Invent. Math. **76** (1984), 469–514.
2. M. Aschbacher and L. Scott, *Maximal subgroups of finite groups*, J. Algebra **92** (1985), 44–80.
3. C. Chevalley, *Sur certains groupes simples*, Tôhoku Math. J. **7** (1955), 14–66.
4. J. Conway and S. Norton, *Monstrous moonshine*, Bull. London Math. Soc. **11** (1979), 308–339.
5. E. Dynkin, *Maximal subgroups of the classical groups*, Amer. Math. Soc. Transl. **6** (1957), 245–378.
6. W. Feit, *Theory of finite groups in the twentieth century*, Amer. Math. Heritage: Algebra and Applied Math. **13** (1981), 37–60.
7. I. Frenkel, J. Lepowsky, and A. Meurman, *Vertex operator algebras and the Monster*, Academic Press, San Diego, 1988.
8. P. Kleidman and M. Liebeck, *The subgroup structure of the finite classical groups*, Cambridge Univ. Press, Cambridge, 1990.
9. L. Scott, *Representations in characteristic $p$*, Proc. Sympos. Pure Math., vol. 37, Amer. Math. Soc., Providence, RI, 1980, pp. 319–331.
10. G. Seitz, *The maximal subgroups of the classical algebraic groups*, Mem. Amer. Math. Soc., no. 365, Amer. Math. Soc., Providence, RI, 1987, pp. 1–286.

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CALIFORNIA 91125