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Working and Playing with the 2-Disk

MICHAEL H. FREEDMAN

This article is simply a written lecture and what philosophy it contains should not necessarily be taken seriously. However, it is much easier to learn a whole story than a single theorem, so many of the latter are woven into the former. Our hero, for fun, is the two-dimensional disk which seems to intrude at many important junctures of geometric topology. Also, there is the theme that ideas of great importance can be enormously simple. As the Centennial recalls to each of us our small mortal places and seems to threaten even mathematics with a certain loss of youth—computer proofs, proofs too long to write (or think), the joint power and vacuity of abstration—I enjoy recalling a few forceful but simple ideas in the subject I know best. I have no prediction for the next century but am content to express the hope that mathematics will still, from time to time, be extraordinarily easy—that the last simple idea is still far off.

By now, topologists have learned to watch developments in analysis with an opportunistic eye. In 1913, I do not know how much attention was given to the topological implications of:

THEOREM (C. Carathéodory **[Car]** and, independently, W. F. Osgood and E. M. Taylor **[OT]**). *If $\mathcal{D}$ is a Jordan domain, then any Riemann mapping of the unit disk $U \to \mathcal{D}$ extends to a homeomorphism of the closures $\overline{U} \to \overline{\mathcal{D}}$.*

It follows that every imbedding of the circle S^1 into the plane extends to an imbedding of the disk:

$$\begin{array}{ccc} S^1 & \overset{i}{\hookrightarrow} & R^2 \\ {\scriptstyle\partial}\big\downarrow & \nearrow{\scriptstyle j} & \\ D^2 & & \end{array}$$

The hooked arrows are 1-1 maps—in general not supposed to be more than continuous. The dotted arrow is the conclusion, whereas the solid arrows are hypotheses. (The diagram is commutative.)

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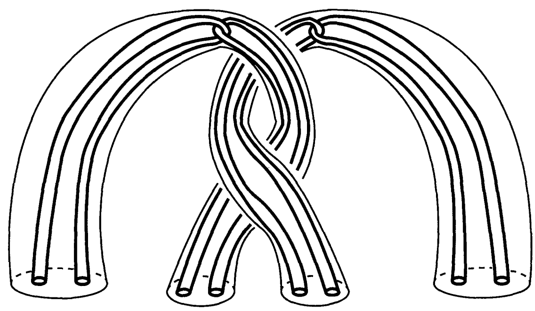


FIGURE 1. (S^3, B^3) /Cantor set of arcs $\cong$ (S^3 , horned ball).

Let $\mathscr{D}$ be the interior domain of $i(S^1)$. The theorem finds a continuous (and, in fact, conformal on the interior) extension j' of some other parametrization i' of $i(S^1)$. The one-dimensional problem of isotoping i' is i is not hard and this leads to j.

Carathéodory's proof was an application of his recently developed theory of prime ends—a subject which is still a source of topological arguments (e.g., Sullivan's solution [S] of the Wandering Domain Problem). The other proof, while of less long-term importance, was discovered after W. F. Osgood had served (1905–1906) as President of the American Mathematical Society and is a striking example of life after bureaucratic service.

In 1922, J. Alexander, one of the founders of homology theory, announced (unpublished) a similar result regarding imbeddings of the two-dimensional spheres S^2 in R^3. The argument was short-lived, for in 1924 Alexander published [A] the seminal counterexample, the *Alexander Horned Sphere.* Here we described it in a possibly unfamiliar way—but the usual image of infinitely interlocking horns can be retrieved with some scrutiny.

Imagine $S^3 = R^3 \cup \infty$. Attached to the horizontal plane P are a nested collection of solid cylinders as pictured in Figure 1.

At the " nth level" there are 2^n solid cylinders and these are arranged so that the intersection of all levels is a Cantor set's worth of arcs which braid as they move upward. (The components of the intersection are arranged to be arcs by making each intersect horizontal planes in at most one point.) The braiding is increasingly rapid toward the upper end points and they are not topologically tame but wild.

Consider the quotient space (with the weak or *quotient* topology) $S^3/$ arcs wherein each of these arcs is declared to be a point. The Alexander horned sphere is $\pi(\overline{P})$ and the Alexander horned ball is π (upper half-space). By taking a limit of homeomorphisms, $S^3 \to S^3$, it is possible to find a map $\theta\colon S^3 \to S^3$ whose nonpoint (usually called nontrivial) point preimages are exactly these arcs. The composition $\pi \circ \theta^{-1}\colon S^3 \to S^3 \xrightarrow{\pi} S^3/$ arcs is a homeomorphism (in spite of the fact that θ^{-1} is a relation!). In this way it is seen

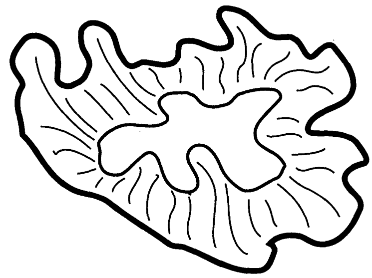


FIGURE 2

that these wild objects are actually subspaces of S^3. In fact, understanding this homeomorphism leads to the usual "horned" picture.

The horned sphere had intellectual descendants of two lineages. First it suggested two conjectures—repairs for the damage done by the counter-example—which became theorems during the following fifty-eight years. An imbedding of a space X is *collared* if it extends as $X \times \frac{1}{2}$ to an imbedding of $X \times [0, 1]$. The theorems are:

SCHOENFLIES THEOREM (Proved by B. Mazur with finishing touches by M. Morse and slightly later by M. Brown, 1959; see **[M]**, **[Br]**). *Any collared (topological) imbedding of $S^{n-1} \to S^n$ extends to an imbedding of B^n.*

ANNULUS THEOREM (R. Kirby + ϵ for $n > 4$, 1968, and F. Quinn for $n = 4$, 1982, see **[K]**, **[Q]**). *Any collared imbedding of $(S^{n-1} \coprod S^{n-1}) \to S^n$ extends to an imbedding of $S^{n-1} \times [0, 1]$.*

See Figure 2.

The other chain of descent attempted to explore rather than to define away the phenomenon. A key development came in 1952 when R. H. Bing **[Bi]** found that the *double* of the horned ball, DHB, is homeomorphic to the 3-sphere S^3. The double is defined by

$$DHB = HB \times \{0, 1\}/(x, 0) \sim (x', 1),$$

where $(x, 0) \sim (x', 1)$ iff $x = x'$ and $x \in$ frontier (HB). Bing's argument may be cast in the previous form by saying that he constructs a sequence of homeomorphisms $S^3 \xrightarrow{\theta_i} S^3$ whose limit $S^3 \xrightarrow{\theta} S^3$ has as its nontrivial point preimages the doubly wild Cantor set of arcs made by reflecting Figure 1 in the horizontal plane.

If $\pi: S^3 \to (S^3/\text{doubly wild Cantor set of arcs}) \cong$ DHB is the projection to the quotient space, the desired homeomorphism is $\pi \circ \theta^{-1}$. Unlike the earlier example, the shrinking homeomorphisms θ_i are of extraordinary subtlety. They are generated by successive shears defined near the boundaries of the pictured tori (Figure 3 on next page). To get a feel, notice that rotation by roughly 90° in the angular coordinate of the large solid tori reduces the diameters of the smaller solid tori contained within them. Such

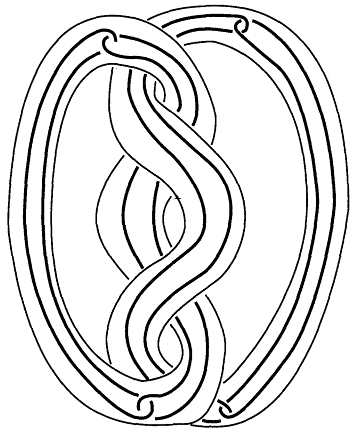


FIGURE 3

diameter reductions are painstakingly composed to reduce the diameter of each component of the ∞-stage—that is, each doubly wild arc—to zero.

The piéce de résistance of such shrinking arguments is the unpublished theorem of R. Edwards (1978; see **[Da]**). It gives a sharp criterion for when a quotient map of a high-dimensional topological manifold is approximable by homeomorphisms.

THEOREM (Edwards). *Let $\pi\colon M^n \to X$, $n \geq 5$, be a C.E. map from a topological manifold onto a finite-dimensional ANR. Then π is approximable by homeomorphisms iff any map of the two-dimensional disk into the quotient $f\colon D^2 \to X$ is approximable by an imbedding.*

A map is "C.E." if every point inverse is null homotopic within any neighborhood of itself. All the hypotheses are now known to be indispensible. Roughly, we think of the theorem as saying that a quotient which might be a manifold is a manifold, provided it has manifold-like general position with respect to maps of the two-dimensional disk.

The proof is beautiful but too long to summarize here except to say that the two disks enter as the important parts of the dual to the $(n-3)$-skeleton of M. Experts know that $(n-3)$ is the critical dimension for engulfing. I will not dwell on engulfing but later will spend a little time on its close cousin the h-cobordism theorem for which the 2-disk is also the key.

To redeem my promise that important results can be simple I now give a rather complete sketch of Brown's proof of B. Mazur's Schoenflies theorem **[M]**. It is that proof which, transfigured, reappears in the study of four-dimensional manifolds. First I should say that the stunning advances of algebraic, differential, and combinatorial topology in the forties and fifties together with the stunning stasis of the Schoenflies problem and its many relatives had led to a deep and well-informed pessimism on the prospects for naive geometric arguments in topology. It must have been a wonderful day

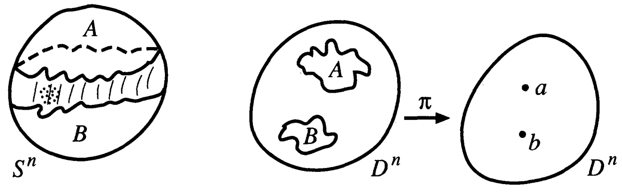


FIGURE 4

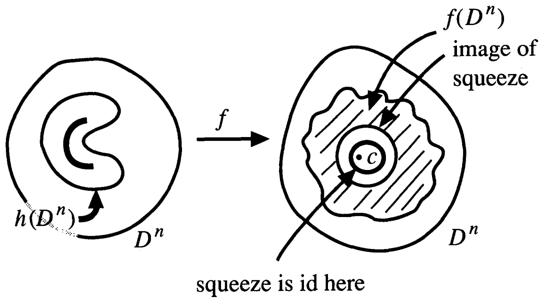


FIGURE 5

when Barry Mazur, then a graduate student at Princeton, cast the first bright light through the gloom.

PROOF-SKETCH OF THE SCHOENFLIES THEOREM ACCORDING TO BROWN. Let A and B be the (closed) complementary pieces of $S^n \backslash S^{n-1} \times (0, 1)$. A subset on S^n is *cellular* if it can be written as a nested intersection of balls $\bigcap_{i=1}^{\infty} B_i^n$, $B_{i+1}^n \subset \operatorname{int} B_i^n$. The key is to show that A (or B) is cellular, for then a simple limiting argument shrinks A to a point and in the process stretches the product collar lines of $S^{n-1} \times (0, 1)$ into the radial lines of some polar coordinate system on one of the closed complementary components of $S^{n-1} \times \frac{1}{2}$—identifying it as a disk.

Remove a small open disk from $S^{n-1} \times (0, 1)$ to obtain the picture shown in Figure 4.

The following lemma almost applies to the π in Figure 4.

LEMMA. *If $f \colon D^n \to D^n$ has a single nontrivial point preimage $f^{-1}(c) = C$ for $c \in \operatorname{int} D^n$, then C is cellular.* (Note that we do not assume f is onto.)

PROOF. Consider $h =$ "$f^{-1} \circ$ squeeze $\circ f$" where squeeze is a "reimbedding" of D^n into a small neighborhood of c, which is the identity on a still smaller neighborhood. The quotient marks mean the imbedding which can be easily fashioned out of the relation that the notation literally describes. These imbeddings, associated to progressively stronger squeezes, show C is cellular. See Figure 5. □

To conclude the Brown proof, observe that " $f^{-1} \circ \text{squeeze}_b \circ f$ " : $D^n \to D^n$ has A as its only nontrivial preimage. By the lemma, A is cellular. □

The Annulus Theorem could not be proved until manifold theory had reached maturity. It then required a brillant device—the *torus trick.* The Schoenflies theorem can be used to replace the annulus conjecture with the conjecture that all homeomorphisms are *stable.* A homeomorphism $h: R^n \to R^n$ is stable if it is a finite composition of homeomorphisms g_i, each of which is differentiable or piecewise linear on at least some open set $U_i \subset R^n$. This said, the pseudogroup of stable homeomorphisms and stable structures can be studied. It was known that all $h: R^n \to R^n$ stable implies the annulus conjecture in S^n.

By a marvelous device which I cannot describe here, Kirby showed that any germ of h determines a potentially exotic triangulation of the n-torus T^n_Γ. The problem became the: "Hauptvermutung for Tori." That is, given T^n_Γ find a P.L. homeomorphism $k: T^n_{\text{Standard}} \to T^n_\Gamma$. If this could be done "$\text{id} \circ k = T^n_{\text{Standard}} \circlearrowleft$ can be constructed. Any self-homeomorphism of T^n_{Standard} must be stable (by the controlled behavior of its lift to R^n) and since k is P.L. it is a formality that "id": $T^n_\Gamma \to T^n_{\text{Standard}}$ and, therefore, $h: R^n \to R^n$ are stable.

Finding k involves deep manifold theory and actually cannot be done before a (harmless) passage to a 2^n-fold covering space. The idea to pass to a cover was L. Siebenmann's; the construction of k (after covering) was carried out independently by T. Farrell, by W.-C. Hsiang and J. Shaneson, and by C.T.C. Wall.

It is in the depths of manifold theory that the 2-disk reenters the story. I began with the Riemann mapping theorem, skipped dimension $= 3$ permanently (the fundamental technical tool in three-manifold topology, Dehn's lemma—loop theorem—is a theorem for imbedding two-dimensional disks, however, an entire hour will be devoted to three-manifolds in a later lecture) and dimension $= 4$ temporarily, and we are now discussing the tools of high dimensional $(n \geq 5)$ smooth (or P.L.) manifold topology. This theory does more than help solve the annulus problem, but in this lecture we are oblivious to the rest.

We need to construct k. The method, rather odd at first glance, is to stick some P.L. manifold W in between T^n_Γ and T^n_{Standard} and then to try to simplify W so that the two inclusions of boundary components $T^n_\Gamma \to W$ and $T^n_{\text{Standard}} \to W$ are (simple) homotopy equivalences. Then one establishes a P.L. product structure on W. Following the product structure from bottom to top would give k. The process of simplification is called surgery. The construction of product structures is s-cobordism theorem. See Figure 6.

Surgery began with J. Milnor's discovery of new differentiable structures on the seven-sphere S^7 and was extensively developed by the mid-1960s through the work of J. Milnor, M. Kervaire, W. Browder, S. P. Novikov,

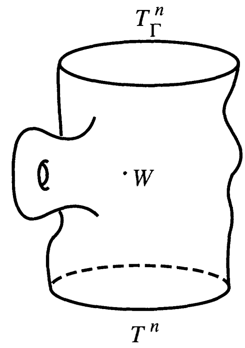


FIGURE 6

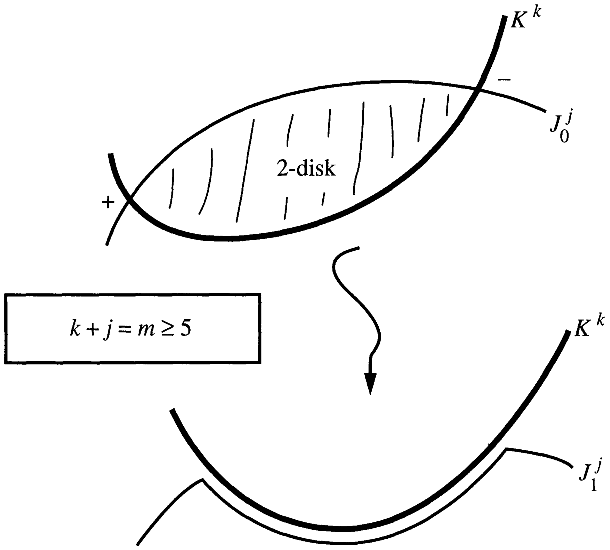
C.T.C. Wall, and others (for more details see the books of W. Browder and C.T.C. Wall [**Br**, **Wa**]). It is an obstruction to surgery on W that necessitates the passage to a finite cover.

The s-cobordism theorem was developed in the simply connected setting (where the letter h replaces s) by S. Smale in 1959. It remains the most powerful method in topology for constructing isomorphisms between manifolds. D. Barden, B. Mazur, and J. Stallings worked out the obstructions which arise in the nonsimply connected setting. (These vanish for the fundamental group of an n-torus.)

In both of these major developments the 2-disk plays a key role in fitting geometry to algebra. The process is called the *Whitney trick* after H. Whitney's use of it [W] to construct imbeddings of n-manifolds in R^{2n}. See Figure 7 on next page.

In surgery theory manifolds are changed by manipulating spheres disjointly imbedded in them. The imbedding and disjointness information (when n is even and the spheres have dimension $= n/2$) arrives in algebraic form: a total number of crossing points sums to zero. It must be converted into geometric information (disjointness and imbeddedness) by pushing portions of spheres across *Whitney disks* which pair crossings of opposite sign. In the h-cobordism theorem, the bubbling bouncing flow of a "gradient-like" vector field must be shifted and simplified to the greatest extent consistent with homology. This is also accomplished by standard moves guided by two-dimensional Whitney disks.

Most (but not all) of standard high-dimensional topological theory can now be brought down to dimensions $n = 4$. Quinn's proof of the annulus conjecture is a prime example of this. A crucial step was finding an imbedding theorem for 2-disks—Whitney 2-disks—to aid in simplifying s-cobordisms. Technically imbedded disks are not enough. The Whitney disk guides an isotopy and transverse coordinates are needed to write it down. So what is



picture in M^m

FIGURE 7

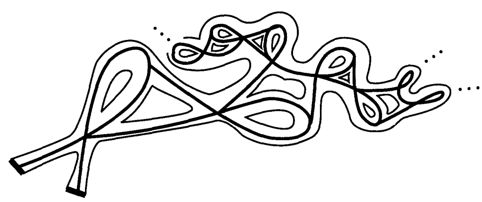
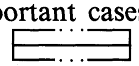


FIGURE 8

sought is a theorem for imbedding 2-handles $H = (D^2 \times R^2, \partial D^2 \times R^2)$ when presented with a Whitney problem arising from surgery or in a five-dimensional s-cobordism.

In 1974 A. Casson found these handles **[Cas]** (in many important cases where $\pi_1 \cong 0$) but they looked like Figure 8, and not like  . His infinite construction gave smooth manifolds which we denote CH for "Casson Handle" which plausibly might be homeomorphic or diffeomorphic to H.

My contribution (in 1981) was to recognize (any) CH as homeomorphic to H by finding a common quotient:

$$H \xrightarrow{\alpha} CH/\mathscr{D} \xleftarrow{\beta} CH.$$

The projection α is shown to be approximable by homeomorphisms by a difficult shrinking argument in the spirit of R. H. Bing with essential details supplied by R. Edwards, as explained in my paper **[F]**. The projection β is only known indirectly but, by the first step, its quotient is well in hand (i.e., homeomorphic to H). Following the spirit of M. Brown's Schoenflies argu-

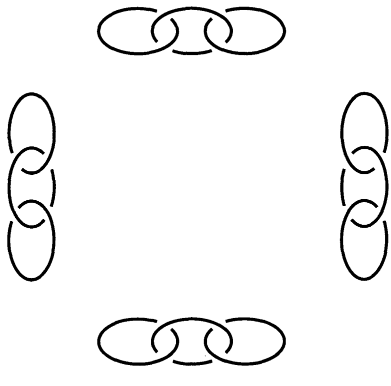


FIGURE 9. \$1 per opening-closing.

ment β is also shown to be approximable by homeomorphisms. At this point our story has come to a full circle, the two reactions to Alexander's horned sphere—one leading to α and the other to β—are united in establishing $H \cong CH$, with the outcome joining usefully with the waiting machinery of manifold theory. But neither of these was to be the most surprising confluence.

The next year (1982) it became evident from S. Donaldson's work **[Do]** that at least many CH, though homeomorphic, were not diffeomorphic to H. From this came exotic structures on H and then R^4 and a whole world of four-dimensional subtlety.

In the last few years, it has been necessary to look back to the most resistant settings where the starting material, Casson Handles, have not been found—and may not exist. An algebraic obstruction has been formulated in terms of Poincaré transversality [**F**′] and may be studied using the secondary theory of link invariants. This has been started by X.-S. Lin and me **[FL]** but talking about it is too much work. However, whether or not you can find disks where you want them, you can always play games on them. To set the mood, consider the necklace puzzle in Figure 9.

A topologist presents the above fragments to a jeweler and asks that they be hooked up into a necklace. He says: "I'll charge you one dollar for each link I must open and close so that will be four dollars, please." The topologist, of course, sees how to do it for three dollars.

In another game, suppose we consider a barber pole shear of infinite cylinder:

$$s_k\colon R \times S^1 \to R \times S^1$$
$$(t, \theta) \to (t, \theta + kt).$$

The map s_k is linear and given by the matrix $M = \left|\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right|$. The largest characteristic vale of M (eigenvalue of $\sqrt{M^{\mathrm{T}}M}$) measures the factor by which s_k can distort distance. For k large this eigenvalue is quite close to k.

Suppose (at some rather specialized place of business) that for a charge of \$1 any homeomorphism of distortion roughly 10 or less can be performed on $R \times S^1$. How much does it cost to make s_{10^4}?

Well,

$$\begin{vmatrix} 1 & 10 \\ 0 & 1 \end{vmatrix}^{10^3} = \begin{vmatrix} 1 & 10^4 \\ 0 & 1 \end{vmatrix},$$

so maybe \$1, 000. However,

$$\begin{vmatrix} 1 & 0 \\ 0 & 1/10 \end{vmatrix} \begin{vmatrix} 1 & 10 \\ 0 & 1 \end{vmatrix}^3 \begin{vmatrix} 1 & 0 \\ 0 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 10^4 \\ 0 & 1 \end{vmatrix},$$

so S_{10^4} can be realized for five dollars.

The analytic theory (the Beltrami equation) for conformal distortion finds an even more graceful version of this factoring trick. The distortion discretely follows a geodesic in the Poincaré upper half-plane from $(1, 10^4)$ to $(1, 0)$, the upper half-space being the Tiechmüller space for $R \times S^1$ relative to its ideal boundary.

There is a conformal isomorphism

$$R \times S^1 \overset{e^{-t}}{\to} U \backslash \{0\}$$
$$(t, \theta) \mapsto (e^{-t}, \theta)$$

and conjugating s_k by e^{-t} sends it to the logarithmic spiral $\bar{s}_k(\rho, \theta) = (\rho, \theta + k \log \rho)$. Thus we may see explicitly how logarithmic spirals can be quickly (in fact logarithmically) factored into compositions of quasiconformal maps of smaller conformal distortion. It is a joint result with Z.-X. He **[FH]** that no such rapid factoring exists for $\bar{s}_k$ on D^2 when small metrical distortion of the factors is required. Our result, in this example, says that if, s_{10^4} is to be written as a composition of n factors, each of which produces a metrical distortion of less than or equal to the distortion of s_{10}, then $n \geq 996$. One wonders if n must actually be $\geq 1{,}000$. The general problem, in which no real progress has yet been made, is to understand the behavior of metrical distortion on the 2-disk under composition and factoring. For example, it appears not to be known that a K-quasi-isometry of D^2 can be factored into a composition of L-quasi isometries for any constant L which is smaller than K.

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Department of Mathematics, University of California at San Diego, La Jolla, California 92093-0112