Nonlinear Fourier Analysis of Ocean Waves: Extremely Fast Numerical Integration of Surface Wave Dynamics

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LONG-TERM GOALS

1. Application of the ***inverse scattering transform*** **(IST)** to the ***time series analysis of laboratory and oceanic wave data***. The approach may be viewed as a **generalization of linear Fourier analysis** and is loosely referred to as "Nonlinear Fourier Analysis or **Generalized Fourier Transform**" (GFT).

2. A major focus has been the application of IST to the study of *"**rogue, freak or giant" ocean waves***. The emphasis has been on the study of the ***physical mechanisms leading to the generation of rogue waves in random sea states***.

3. A third long term goal **added this year** is the ***development of fast algorithms for numerically integrating the space/time dynamics of deep-water wave trains***. While IST is limited to the numerical/analytic integration of the so-called "soliton equations," I have discovered how the GFT can be used to solved higher order equations for which study of the dynamics have previously been limited to numerical (as opposed to analytical) approaches. Indeed I discuss herein how the GFT can be used ***both*** for the ***analytical study and extremely fast numerical integration*** of the ***extended nonlinear Schroedinger equation for fully three dimensional wave motion***. I discuss a number of considerations for the development of ***extremely fast GFT algorithms***, which I refer to as the ***xGFT***.

OBJECTIVES

1. The long-term goal of the present research program is the development of ***fast numerical inverse scattering*** techniques (FIST) to enable the nonlinear Fourier analysis of nonlinear ocean wave data and to consequently improve our understanding and predictive capability of ocean surface waves.

2. A major break through has occurred during this year's research: the need for fast ***numerical integration of nonlinear partial differential equations offers new opportunities for application of the GFT*** approach to Navy-relevant problems. Such ***integrations can be drastically accelerated*** by exploiting the ability of the GFT to solve, or approximately solve, nonlinear partial differential equations using the fully new numerical methods discussed here.

APPROACH

There exist large classes of nonlinear partial differential wave equations that are completely solvable by the inverse scattering transform. One of the most important of these equations is the nonlinear Schroedinger equation in one space (x) and one time (t) (1+1) dimensions:

$$i(\psi_t + C_g \psi_x) + \mu \psi_{xx} + \nu |\psi|^2 \psi = 0 \qquad (1)$$

Extending NLS to 2+1 dimensions (x,y,t) is a well-known result

$$i(\psi_t + C_g \psi_x) + \mu \psi_{xx} - 2\mu \psi_{yy} + \nu |\psi|^2 \psi = 0 \qquad (2)$$

And, finally, extending this result to higher order yields

$$i(\psi_t + C_g\psi_x) + \mu\psi_{xx} - 2\mu\psi_{yy} + \nu|\psi|^2\psi + i\rho|\psi|^2\psi_x + i\xi\psi^2\psi_x^* + i\zeta\psi\bar{\phi}_x|_{z=0} = 0 \qquad (3)$$

Equation (1) is integrable by IST, but (2) and (3) are not. Thus seemingly (2) and (3) cannot be studied by the powerful methods of IST.

Is there a way to study (2) and (3) analytically and numerically by methods analogous to IST? Can IST be extended to study these classes of nonlinear equations? I have found an approximate approach does indeed exist for studying these wave equations. This recipe supplies an important step in the extension of IST to the study of ***asymptotic solutions of (2) and (3)***. Here is the recipe:

(1) Begin with the two partial differential equations known as the Lax pair of the NLS equation, whose ***compatibility condition*** yields (1).
(2) Suitably ***perturb the Lax pair***. When the compatibility condition is applied to the resultant Lax pair, either (2) or (3) is obtained *to* ***the order of these latter equations***. This implies that the perturbed Lax pairs ***asymptotically integrate*** (2) and (3).
(3) Extend the analysis to ***perturb algebraic geometric loop integrals*** that determine the parameters in the GFT (Riemann theta function) for the asymptotic solutions of (2) and (3). An important ingredient of the present research is that by virtue of the Lax pair representation the wave dynamics has an ***underlying Riemann surface*** on which the Riemann theta functions exist. It is impossible to emphasize enough this important aspect of the approach for solving/approximately solving nonlinear partial differential equations.
(4) Indeed

$$\psi(x,y,t) = A_o \frac{\theta(x,y,t\,|\,\mathbf{B},\delta^-)}{\theta(x,y,t\,|\,\mathbf{B},\delta^+)} e^{2iA_o^2 t}$$

is the ***leading order approximation*** which asymptotically solves (2) and (3) for suitable choices for the Riemann matrix, **B**, and phases, $\delta^{\pm}$. The GFT is given by

$$\theta(x,y,t) = \sum_{\mathbf{m}=-\infty}^{\infty} e^{i\mathbf{m}\cdot\mathbf{X} + \frac{1}{2}i\mathbf{m}^T\mathbf{B}\mathbf{m}}$$

for $\mathbf{X} = \mathbf{K}x - \mathbf{\Omega}t + \boldsymbol{\delta}$, where the wave number, frequency and phase vectors are all are defined in terms of the appropriate loop integrals.
(5) Use the results of (4) to approximately numerically integrate either (2) or (3).

WORK COMPLETED

Numerical results, for the dynamics of a complex rogue wave train, have been completed using the procedure outlined in the previous Section. The main results are discussed in the next Section. One can ask at this juncture, why has the above normally computationally intensive GFT algorithm suddenly become extremely fast, yielding xGFT? There are several reasons why xGFT, as presently implemented, is fast:

(1) A preprocessor allows for the computation of a number of results at *t*=0 which never need be computed again for future times. Computation of the coefficients containing the Riemann matrix, $e^{\frac{1}{2}i\mathbf{m}^T\mathbf{B}\mathbf{m}}$, are stored for use at future times. Wave numbers, $\mathbf{m}\cdot\mathbf{X}$, are likewise stored and need to be computed only once. Indeed only those GFT components which significantly contribute to the nonlinear dynamics are retained for use in the time evolution.
(2) Coherent structures, or rogue waves, are coded as two-by-two submatrices along the diagonal of the Riemann matrix. This constitutes a simple and efficient way to encode very complex phenomena normally requiring many thousands of FFT components.

(3) Recursion relations can be used to determine the trigonometric functions.
(4) The FFT can be used to compute spatial variations in the dynamics.
(5) Time evolution of the GFT components can be computed with recursion and, in particular, the time steps do not need to be exceedingly small (as with traditional FFT approaches) but need be computed only at (much larger) graphical precision.

In consequence of these advances the numerical simulation discussed below in the RESULTS section is about 200 times faster than a conventional FFT numerical simulation of equation (3).

RESULTS

The numerical results are shown in the Figures. The initial condition consists of two ***small-amplitude modulations***, simultaneously in the dominant wave direction and perpendicular to it. Two nonlinear Fourier components are specified, one slightly to the right and the other slightly to the left of the dominant direction. Figure 1 shows the evolution at early time, after the small amplitude modulations have grown to be visible to the eye. Figure 2 shows the evolution as the rogue wave begins to grow and Figure 3 shows the evolution at the maximum rogue wave amplitude. Finally in Figure 4, somewhat after the maximum rogue wave amplitude, the wave train still has quite complex behavior.

IMPACT/APPLICATION

The impact of this research will occur in general for the extremely fast xGFT analysis of shallow and deep water wave trains, the analysis of internal wave trains and acoustic waves on the continental shelf, the design of floating surface and subsurface vessels, the fatigue life of tethered vehicles, etc.

TRANSITIONS

Collaborative work is now underway with researchers at William & Mary (aspects of inverse scattering theory) and at the U. S. Army Waterways Experiment Station (analysis of data).

RELATED PROJECTS

An intimate relationship between our results and other projects exists because the sea surface provides a major forcing input to many kinds of offshore activities, including the dynamics of floating and drilling vessels, barges, risers and tethered vehicles. The present work leads to a nonlinear representation of the sea surface forcing and vessel response functions.

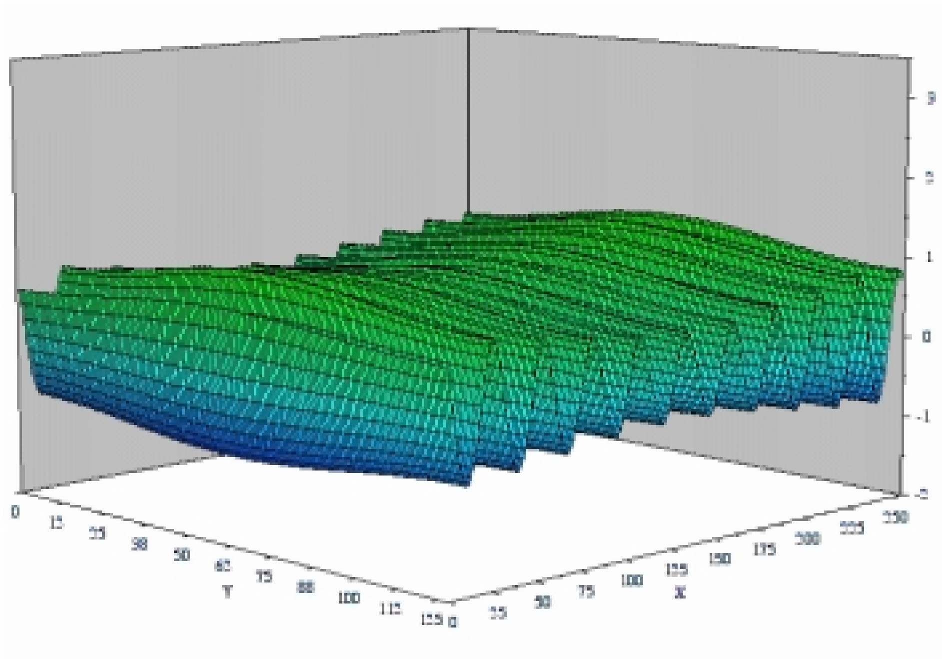

Figure 1 Extremely fast numerical simulation: Somewhat after the "small-amplitude" initial conditions used at the beginning of the numerical run, t = 50.

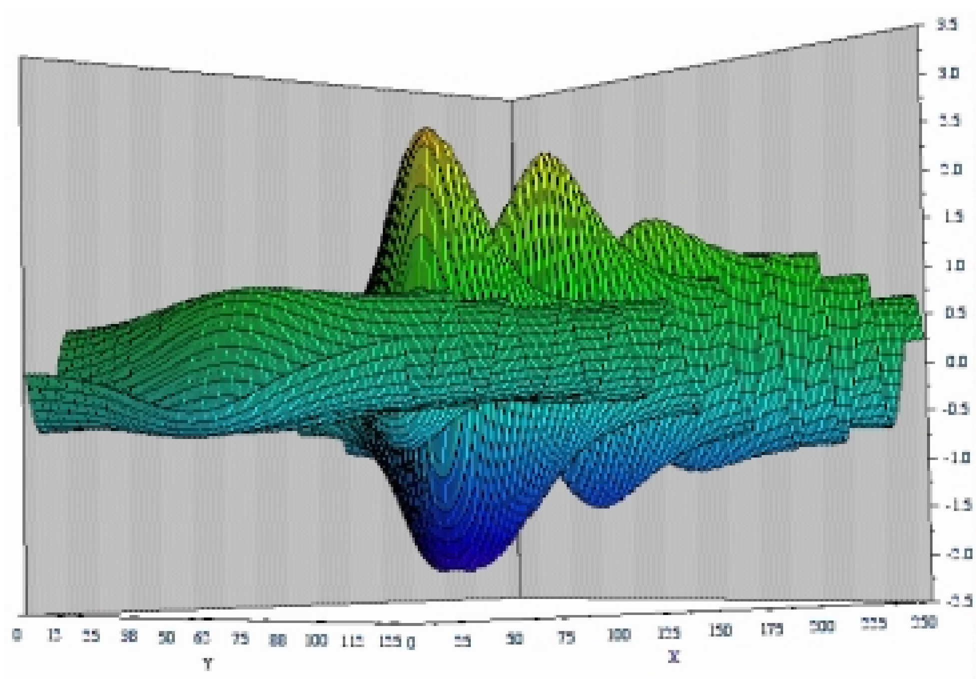

Figure 2 Extremely fast numerical simulation: Evolution of rogue wave, t = 240.

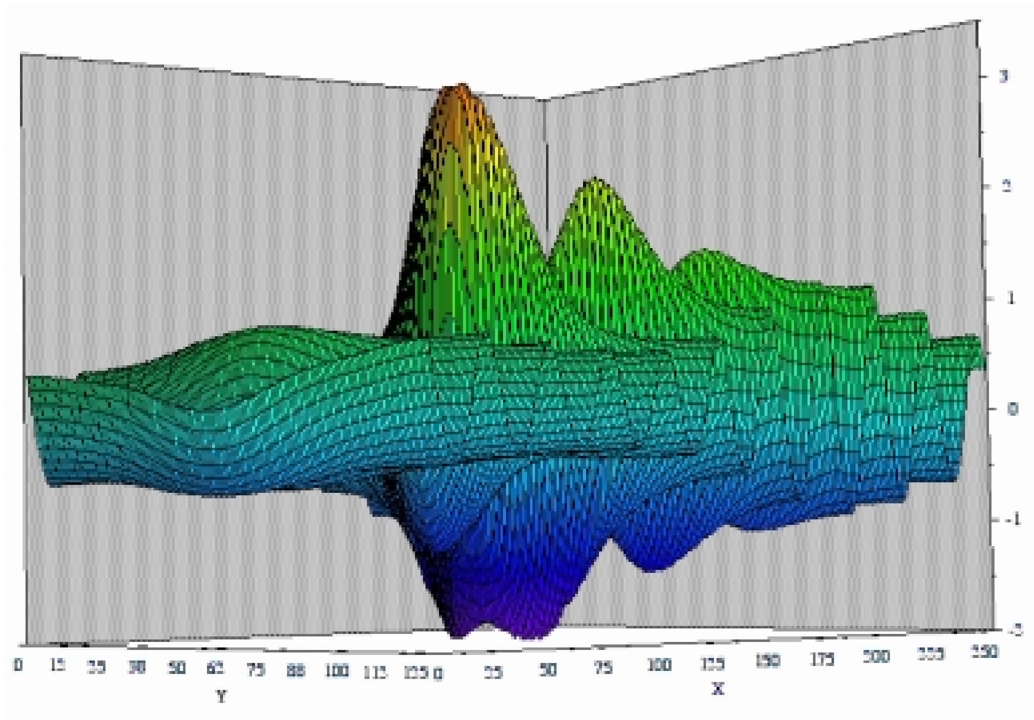

Figure 3 Extremely fast numerical simulation: Maximum rogue wave amplitude, t = 240.

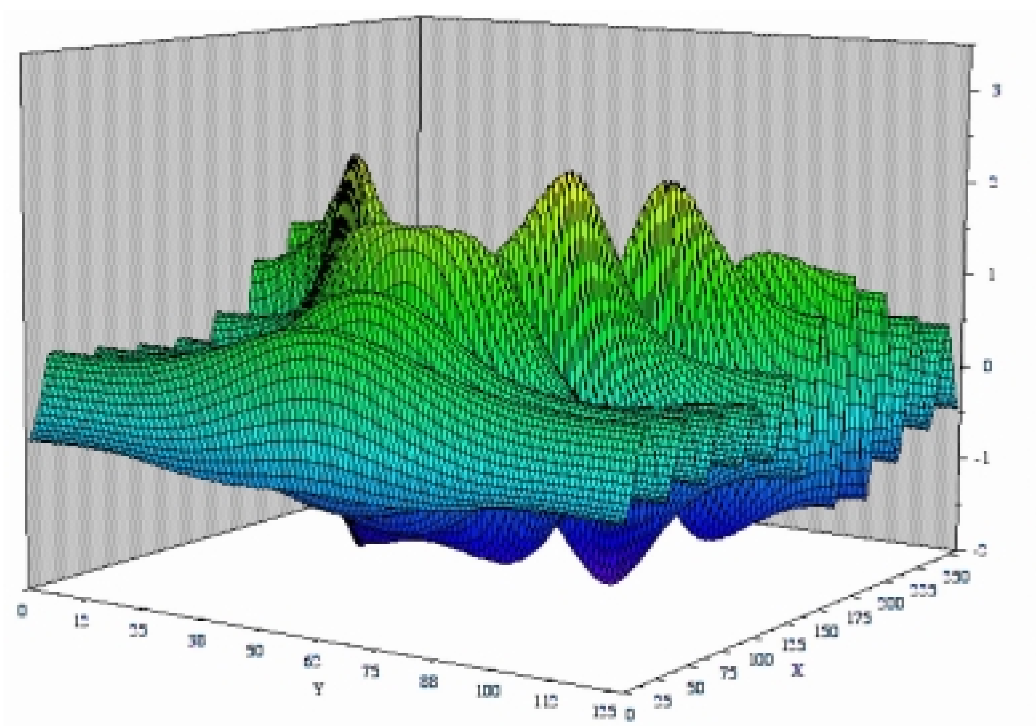

Figure 4 Extremely fast numerical simulation: Somewhat after the rogue wave maximum amplitude, t = 330.