THE INTRINSIC THEORY OF THIN SHELLS AND PLATES PART II.—APPLICATION TO THIN PLATES*

BY

WEI-ZANG CHIEN

Department of Applied Mathematics, University of Toronto

7. The general equations for a thin plate. We shall now investigate the equations of equilibrium and compatibility for a thin plate, not necessarily of constant thickness. First, we shall introduce the condition that the system is a plate, i.e., its middle surface in the unstrained state is plane. We have therefore

$$b_{\alpha\beta} = 0, \qquad R_{\alpha\beta\gamma\rho} = 0. \tag{7.1}$$

Furthermore, in order to simplify the problem, we assume in the following sections that the body force forms a parallel vector field, and therefore (3.38), (3.39) are satisfied; this is true for most practical problems.

Substituting (7.1) and the conditions on body force into (6.34) and (6.35), we have three equations of equilibrium for a thin plate

$$\begin{aligned} -2A_{(1)}^{\rho\gamma\pi\lambda}q_{\rho\gamma}p_{\pi\lambda}h + \tfrac{2}{3}A_{(1)}^{\rho\gamma\pi\lambda}(q_{\pi\lambda}h^3)_{\underset{a}{|}\rho\gamma} + A_{(3)}^{\rho\gamma\pi\omega\lambda\delta}q_{\pi\omega}q_{\lambda\delta}q_{\rho\gamma}h^3 + P^0 \\ + 2X_{[0]}^0h + (Q^\pi h)_{\underset{a}{|}\pi} + \frac{1-2\sigma}{1-\sigma}a^{\pi\lambda}q_{\pi\lambda}Q^0h = O_{(42)}^0, \end{aligned} \tag{7.2a}$$

$$\begin{aligned} 2A_{(1)}^{\rho\alpha\pi\lambda}(p_{\pi\lambda}h)_{\underset{a}{|}\rho} - A_{(3)}^{\rho\alpha\pi\omega\delta\lambda}(q_{\pi\omega}q_{\lambda\delta}h^3)_{\underset{a}{|}\rho} + \tfrac{2}{3}a^{\alpha\pi}q_{\pi\gamma}A_{(1)}^{\gamma\rho\lambda\delta}(q_{\lambda\delta}h^3)_{\underset{a}{|}\rho} + P^\alpha \\ + 2X_{[0]}^\alpha h + \frac{\sigma}{1-\sigma}a^{\alpha\rho}(Q^0h)_{\underset{a}{|}\rho} + (a^{\pi\delta}a^{\alpha\gamma} + 2a^{\alpha\delta}a^{\pi\gamma})Q_\gamma q_{\pi\delta}h = O_{(43)}^\alpha, \end{aligned} \tag{7.2b}$$

where the O-symbols have the following magnitudes,

$$O_{(42)}^0;\ X^0ph, X^0\hat{Q}h, p^2qh, \hat{Q}^2qh, \hat{Q}qph, \hat{P}qh^2, X^0h^2, \hat{Q}h^3, q\hat{P}h^3, qph^3, qXh^3, qh^5. \tag{7.3a}$$

$$O_{(43)}^\alpha;\ p^2h, \hat{Q}ph, \hat{X}ph, \hat{X}\hat{Q}h, \hat{Q}^2h, Pqh^2, \hat{Q}h^3, \hat{X}h^3, ph^3, qh^5. \tag{7.3b}$$

We recall that

$$A_{(1)}^{\alpha\beta\pi\lambda} = \frac{1}{1-\sigma^2}\left\{\sigma a^{\alpha\beta}a^{\pi\lambda} + (1-\sigma)a^{\alpha\pi}a^{\beta\lambda}\right\}, \tag{7.3c}$$

$$A_{(3)}^{\alpha\beta\pi\gamma\lambda\delta} = \frac{2(2\sigma-1)}{3(1-\sigma)}a^{\delta\lambda}A_{(1)}^{\alpha\beta\pi\gamma} + \tfrac{5}{3}a^{\delta\pi}A_{(1)}^{\alpha\beta\lambda\gamma}. \tag{7.3d}$$

Similarly, substituting (7.1) into (6.43) and (6.44), we have three equations of compatibility for a thin plate

$$\begin{aligned} 2\mathfrak{n}_{[0]}^{\beta\gamma}q_{\alpha\beta\underset{a}{|}\gamma}(1 + 2\mathfrak{n}_{[0]}^{\pi\lambda}\mathfrak{n}_{[0]}^{\rho\delta}p_{\delta\lambda}p_{\pi\rho} + 2a^{\pi\lambda}p_{\pi\lambda}) \\ - 2\mathfrak{n}_{[0]}^{\beta\lambda}q_{\beta\pi}(a^{\pi\lambda} + 2\mathfrak{n}_{[0]}^{\pi\rho}\mathfrak{n}_{[0]}^{\lambda\delta}p_{\rho\delta})(p_{\alpha\lambda\underset{a}{|}\gamma} + p_{\gamma\lambda\underset{a}{|}\alpha} - p_{\alpha\gamma\underset{a}{|}\lambda}) = 0, \end{aligned} \tag{7.4a}$$

---

* Received June 12, 1943. Part I of this paper appeared in this Quarterly, 1, 297–327 (1944).

$$(2\mathbf{n}_{[0]}^{\rho\alpha}\mathbf{n}_{[0]}^{\beta\gamma}p_{\rho\gamma\underset{a}{|}\alpha\beta} + \mathbf{n}_{[0]}^{\rho\alpha}\mathbf{n}_{[0]}^{\beta\gamma}q_{\rho\gamma}q_{\alpha\beta})(1 + 2\mathbf{n}_{[0]}^{\pi\lambda}\mathbf{n}_{[0]}^{\omega\delta}p_{\lambda\delta}p_{\pi\omega} + 2a^{\pi\delta}p_{\pi\delta})$$
$$+ \mathbf{n}_{[0]}^{\rho\alpha}\mathbf{n}_{[0]}^{\beta\gamma}(2\mathbf{n}_{[0]}^{\pi\lambda}\mathbf{n}_{[0]}^{\omega\delta}p_{\lambda\delta} + a^{\pi\omega})(p_{\pi\rho\underset{a}{|}\gamma} + p_{\gamma\pi\underset{a}{|}\rho} - p_{\gamma\rho\underset{a}{|}\pi})(p_{\alpha\omega\underset{a}{|}\beta} + p_{\beta\omega\underset{a}{|}\alpha} - p_{\alpha\beta\underset{a}{|}\omega}) = 0. \quad (7.4b)$$

The macroscopic tensors in (6.29), (6.30), (6.31) can be written as

$$T^{\alpha\beta} = 2A_{(1)}^{\alpha\beta\pi\lambda}p_{\pi\lambda}h - A_{(3)}^{\alpha\beta\pi\rho\lambda\delta}q_{\pi\rho}q_{\lambda\delta}h^3 + \frac{\sigma}{1-\sigma}a^{\alpha\beta}Q^0h + O_{(44)}^{\alpha\beta}, \quad (7.5a)$$

$$L^{\alpha\beta} = \tfrac{2}{3}\mathbf{n}_{[0]}^{\rho\beta}a_{\pi\rho}A_{(1)}^{\alpha\pi\lambda\delta}q_{\lambda\delta}h^3 + O_{(45)}^{\alpha\beta}, \quad (7.5b)$$

$$T^{\alpha 0} = \tfrac{2}{3}A_{(1)}^{\pi\alpha\lambda\delta}(q_{\lambda\delta}h^3)_{\underset{a}{|}\pi} + Q^\alpha h + (a^{\pi\lambda}P^\alpha + a^{\alpha\pi}P^\lambda)q_{\pi\lambda}h^2 + O_{(46)}^{\alpha 0}, \quad (7.5c)$$

where

$$O_{(44)}^{\alpha\beta} = O_{(44)}^{\alpha\beta}(p^2h,\ \hat{Q}^2h,\ \hat{Q}ph,\ ph^3,\ \hat{Q}h^3,\ \hat{X}h^3,\ q^2h^5), \quad (7.6a)$$

$$O_{(45)}^{\alpha\beta} = O_{(45)}^{\alpha\beta}(qph^3,\ X^0h^3,\ \hat{Q}h^3,\ q^2h^5,\ Xqh^5), \quad (7.6b)$$

$$O_{(46)}^{\alpha 0} = O_{(46)}^{\alpha 0}(\hat{Q}ph,\ \hat{Q}^2h,\ \hat{Q}\hat{X}h^2,\ \hat{Q}qh^2,\ \hat{P}ph^2,\ \hat{P}\hat{Q}h^2,\ qph^3,\ X^0h^3,\ \hat{Q}h^3,\ Xqh^3,\ q^2h^5). \quad (7.6c)$$

Equations (7.2a, b) and (7.4a, b) are the six differential equations of a thin plate in the six unknowns $p_{\alpha\beta}$ and $q_{\alpha\beta}$. The next step is to introduce certain systematic approximations based upon the thinness of the plate, so as to obtain a set of differential equations in simpler form.

8. Classification of all thin plate problems. We consider a family of ∞^1 thin plates of the same material, having an identical middle surface S_0' in the unstrained state, but different thicknesses; each is subject to the action of (i) external force systems applied at the edges, (ii) surface loadings on its two boundary surfaces, and (iii) uniform body force throughout the plate. (This includes gravity, but excludes a centrifugal field.) We attach to the middle surface of each plate the same system of coordinates x^α, so that the fundamental tensor $a_{\alpha\beta}$ is the same for all plates in this family. We assign to each plate a value of a parameter ϵ, so that the thickness of all the plates can be represented by

$$2h = 2\epsilon\bar{h}(x^1, x^2), \quad (8.1)$$

where $0<\epsilon<\epsilon_1$ and the function $\bar{h}$ is the same for all the plates; for thin plates, ϵ_1 is supposed to be small, but the basic idea of the method is that we seek solutions valid for all ϵ in the range $0<\epsilon<\epsilon_1$.

Equation (8.1) implies that the derivatives of the thickness at any point are of the same order of magnitude as the thickness itself. We shall call these plates "regular plates." On the other hand, if the thickness and its derivatives are of different orders of magnitude, we have an "irregular plate." The following theory is limited to regular plates only.

We may suppose ϵ chosen equal to the ratio of the average thickness to a selected lateral dimension (usually the smallest lateral dimension) of the plate. For a circular plate, ϵ is the ratio of the average thickness to the diameter of the plate. For a rectangular plate, it may be chosen equal to the ratio of the average thickness to the length of the shorter side.

It is important to observe that ϵ is the only parameter involved. Except the fundamental tensor $a_{\alpha\beta}$ and Poisson's ratio σ, all the other quantities occurring are functions of ϵ, and no quantity is "small" unless it tends to zero with ϵ. (Young's modulus does not appear, on account of the use of reduced stresses and body forces.) Thus for any "small quantity" Ψ, we must have

$$\lim_{\epsilon \to 0} \Psi = 0. \tag{8.2}$$

In order that a problem may belong to the theory of small strain, e_{ij} must be a small quantity, and therefore

$$\lim_{\epsilon \to 0} e_{ij} = 0. \tag{8.3}$$

It follows that $p_{\alpha\beta}$ must also be a "small quantity," depending on ϵ like Ψ in (8.2). But this is not necessarily true for $q_{\alpha\beta}$.

It is understood that all conditions (such as reduced edge forces, reduced surface loadings, and reduced body forces) depend on ϵ in such a way that (8.3) holds. We shall assume that Q^i, P^i, $X^i_{[0]}$ vanish at least as fast as ϵ, and are in fact power series in ϵ. This assumption implies that the derivatives of any of these quantities with respect to x^α are of the same order of magnitude as (or higher order of magnitude than) the quantity itself. Hence we write

$$Q^0 = \sum_{s=k_0}^{\infty} Q^0_{(s)}\epsilon^s, \qquad Q^\alpha = \sum_{s=k}^{\infty} Q_{(s)}\epsilon^s, \tag{8.4a}$$

$$P^0 = \sum_{s=n_0}^{\infty} P^0_{(s)}\epsilon^s, \qquad P^\alpha = \sum_{s=n}^{\infty} P^\alpha_{(s)}\epsilon^s, \tag{8.4b}$$

$$X^0_{[0]} = \sum_{s=j_0}^{\infty} X^0_{(s)[0]}\epsilon^s, \qquad X^\alpha_{[0]} = \sum_{s=j}^{\infty} X^\alpha_{(s)[0]}\epsilon^s. \tag{8.4c}$$

where k, k_0, n, n_0, j, j_0 are integers greater than zero, and $P^i_{(s)}$, $Q^i_{(s)}$, $X^i_{(s)[0]}$ are functions of x^α, independent of ϵ.

Similarly we assume that the traction, shearing force and bending moment applied on the edge curve can be represented by

$$\tilde{T}^{\alpha\beta} = \sum_{s=t}^{\infty} \tilde{T}^{\alpha\beta}_{(s)}\epsilon^s, \quad (8.5a) \qquad \tilde{L}^{\alpha\beta} = \sum_{s=u}^{\infty} \tilde{L}^{\alpha\beta}_{(s)}\epsilon^s, \quad (8.5b) \qquad \tilde{T}^{\alpha 0} = \sum_{s=l}^{\infty} \tilde{T}^{\alpha 0}_{(s)}\epsilon^s, \quad (8.5c)$$

where t, u, l are positive integers, and $\tilde{T}^{\alpha\beta}_{(s)}$, $\tilde{L}^{\alpha\beta}_{(s)}$, $\tilde{T}^{\alpha 0}_{(s)}$ are functions of position on the edge curve, independent of ϵ.

Now the problem is to find the behaviour of the family of ∞^1 thin plates under the action of a given family of external force systems (8.4), (8.5). Given an external force system defined by (8.4), (8.5), we seek solutions of the equations of equilibrium (7.2) and the equations of compatibility (7.4) of the form

$$p_{\alpha\beta} = \sum_{s=p}^{\infty} p_{(s)\alpha\beta}\epsilon^s, \quad (8.6a) \qquad q_{\alpha\beta} = \sum_{s=q}^{\infty} q_{(s)\alpha\beta}\epsilon^s, \quad (8.6b)$$

where p and q are zero or positive integers, and $p_{(s)\alpha\beta}$ and $q_{(s)\alpha\beta}$ are functions of x^α, independent of ϵ. Only those problems admitting solutions with $p>0$ belong to the theory of small strain. On the other hand, q may be zero; then we are dealing with a finite deflection problem.

The usual discussion of plate theory is based on the deflection, i.e. the normal displacement of a particle on the middle surface. The present method is intrinsic, and the general equations contain no explicit reference to the displacement. However, since $q_{\alpha\beta}$ corresponds to change of curvature (i.e. curvature of the middle surface after strain), it is clear that finite values of $q_{\alpha\beta}$ correspond to finite deflection and small values of $q_{\alpha\beta}$ to small deflection. Similar remarks apply in the case of shells. Hence, in classification, we may use the familiar word "deflection" when referring to the order of magnitude of $q_{\alpha\beta}$.

The assumed forms (8.6a, b) imply that the derivatives of $p_{\alpha\beta}$, $q_{\alpha\beta}$ with respect to x^α are of the same order of magnitude as the quantities themselves, or of higher order. In fact, $p_{\alpha\beta}$ and $q_{\alpha\beta}$ expressed by (8.6a, b) represent the behaviour of the family of ∞^1 thin plates under the action of the given family of P^i, Q^i, $X^i_{[0]}$, $\tilde{T}^{\alpha\beta}$, $\tilde{T}^{\alpha 0}$, $\tilde{L}^{\alpha\beta}$ defined by the equations (8.4), (8.5). It is understood that if P^i, Q^i, $X^i_{[0]}$, $\tilde{T}^{\alpha\beta}$, $\tilde{T}^{\alpha 0}$, $\tilde{L}^{\alpha\beta}$ are identically equal to zero (i.e., k, k_0, n, n_0, j, j_0, t, u, $l=\infty$), then $p_{\alpha\beta}$ and $q_{\alpha\beta}$ vanish (i.e., p, $q=\infty$) everywhere; this corresponds to the unstrained state of the plate. This means that self-strained plates are not discussed.

In a thin plate problem, we are to regard the numbers k, k_0, n, n_0, j, j_0, t, u, l as given; the initial step towards solution would appear to be the determination of p and q, for then we could simplify the equations of equilibrium and compatibility in the first approximation by picking out the principal terms in ϵ from equations (7.2a, b), (7.4a, b). But owing to the partial indeterminacy of p and q, this method is not successful.

It is much simpler to solve the problem in the reverse order. First we assign integral values to p and q. The values of k, k_0, n, n_0, j, j_0 are fixed by the conditions that $X^0_{(j_0)[0]}$, $X^\alpha_{(j)[0]}$, $P^0_{(n_0)}$, $P^\alpha_{(n)}$, $Q^0_{(k_0)}$, $Q^\alpha_{(k)}$ should contribute to the principal parts of (7.2a, b), without dominating these equations to the exclusion of $p_{\alpha\beta}$ and $q_{\alpha\beta}$. The equations of equilibrium and compatibility in the first approximation are then obtained by picking out the principal terms in ϵ from equations (7.2a, b), (7.4a, b). Then the values of t, u, l are automatically fixed through the expressions (7.5).

We shall now discuss the classification of thin plate problems based on assigned values of p and q, so that the principal parts of (7.2a, b), (7.4a, b) in the first approximation are different for different "Types." The classification is shown graphically in Fig. 3, where permissible pairs of (p, q)-values are represented by circles. As indicated in (8.6a, b), we consider only non-negative integral values of p and q. Since, however, $p=0$ corresponds to finite extension of the middle surface, we must omit the (p, q)-points on the q-axis.

It is found that the points in the (p, q)-plane break up into twelve groups depending on their positions relative to the *division lines* AD, AB, OC and the p-axis. For any point (except $q=0$) on the line AD, it is easily seen from inspection of (7.2a) that the first and second terms are of the same order of magnitude and prevail over all the other terms, with possible exception of those involving $X^i_{[0]}$, P^i, Q^i. For any (p, q)-point (except $q=0$) above AD, the second term in (7.2a) dominates, and for any (p, q)-point below AD, the first term dominates. For the point A, the first three terms in (7.2a) are of the same order of magnitude and prevail over the right hand side. For any point on the p-axis above A, the second and third terms in (7.2a) are of the same order of magnitude and prevail over the other terms. Thus the principal part of (7.2a) takes five different forms depending on the position of the (p, q)-point relative to the line AD and the p-axis.

Similarly, the form of the dominant part of (7.2b) depends on the position of the (p, q)-point relative to the line AB and the p-axis. Finally, the form of the dominant part of (7.4b) depends on the position of the (p, q)-point relative to the line OC and the p-axis. The equation (7.4a) has no division line, since the term $n_{[0]}^{\beta\gamma} q_{\alpha\beta|\gamma}$ dominates for any position of the (p, q)-point.

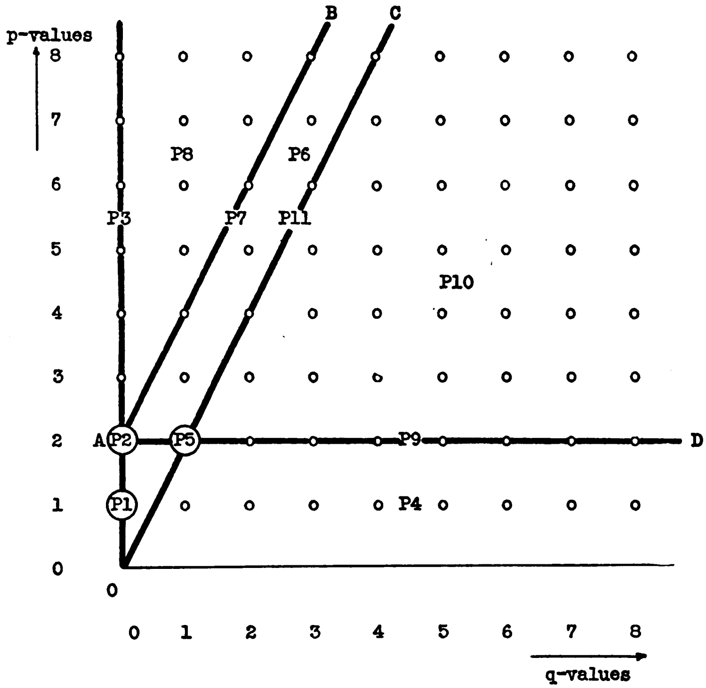

FIG. 3. Classification of thin plate problems.
p = order of extension of middle surface, q = order of change of curvature of middle surface.
(Type $P12$ is not indicated in the diagram, since for these problems, $q=\infty$, and consequently the corresponding points lie at infinity to the right hand side.)

It follows that the (p, q)-plane is divided into twelve regions, so far as permissible non-negative integral values of p and q are concerned, and so the complete classification of all thin plate problems involves consideration of twelve types (Types $P1$–$P12$). Type $P12$ is not indicated in the diagram, since for these problems, $q=\infty$, and consequently the corresponding points lie at infinity to the right hand side.

Although the classification gives twelve types, four of these (Types $P3$, $P6$, $P7$, $P8$) are less important than the others. They represent *overdetermined* problems, in which the number of equations exceed the number of unknowns. Such cases can occur only when very special relations connect the body forces and surface forces.

These twelve types may be described as follows:

(1) Problems of finite deflection $(q=0)$, Types $P1$–$P3$.

(2) Problems of small deflection ($q \geqq 1$, $p=1$; $q=1$, $p=2$; $q \geqq 1$, $p>2q$), Types $P4$–$P8$.
(3) Problems of very small deflection ($q \geqq 2$, $2q \geqq p \geqq 2$), Types $P9$–$P11$.
(4) Problems of zero deflection ($q=\infty$), Type $P12$.

In order to save space, we shall not discuss all the twelve types in detail. The discussion of Types $P1$, $P2$, $P3$ will serve as an example. The results for all types are summarized in the tables in the Appendices at the end of this paper. The principal parts of the equations of equilibrium and compatibility are shown in Table I, and the orders of magnitude of the external forces and the principal parts of the macroscopic tensors in Table II.

It should be noted that the theory of generalized plane stress [1, 2], the Lagrange-Kirchhoff theory of small deflection [3, 4, 5], and the von Kármán theory of "large" deflection [6] can be derived respectively from the Types $P12$, $P11$, $P5$.

We shall devote the next section to discussing the problems of finite deflection ($P1$–$P3$). All results for these types are new, and may prove particularly interesting.

9. Problems of finite deflection ($q=0$), types $P1-P3$.

(a) TYPE $P1$: $q=0$, $p=1$. FINITE DEFLECTION WITH DOMINANT EXTENSION IN THE MIDDLE SURFACE

General equations. By the condition that, in the first approximation, (7.2a, b) receive significant contributions from $P^0_{(n_0)}$, $P^\alpha_{(n)}$, $X^0_{(j_0)[0]}$, $X^\alpha_{(j)[0]}$, $Q^0_{(k_0)}$, $Q^\alpha_{(k)}$, we must have

$$n_0 = n = 2, \qquad i_0 = j = 1, \qquad k_0 = k = 1. \tag{9.1}$$

Therefore, we obtain from (6.23)

$$h_{(+)} = \bar{h}\epsilon + 0(\epsilon^2), \qquad h_{(-)} = \bar{h}\epsilon + 0(\epsilon^2); \tag{9.2}$$

consequently, the common assumption that the middle surface of the unstrained plate is deformed into the middle surface of the strained state is justified in the first approximation.

We now substitute (8.1), (8.4)–(8.6) into (7.2), (7.4). The lowest power of ϵ occurring is ϵ^2 in (7.2), and ϵ^0 in (7.4). The corresponding coefficients give rise to equations of equilibrium and compatibility in the first approximation as follows:

$$-2A^{\rho\gamma\pi\lambda}_{(1)} q_{(0)\rho\gamma} p_{(1)\pi\lambda}\bar{h} + P^0_{(2)} + 2X^0_{(1)[0]}\bar{h} + (Q^\pi_{(1)}\bar{h})_{\underset{a}{|}\pi} + \frac{1-2\sigma}{1-\sigma} a^{\pi\lambda} q_{(0)\pi\lambda} Q^0_{(1)}\bar{h} = 0, \tag{9.3a}$$

$$2A^{\rho\alpha\pi\lambda}_{(1)}(p_{(1)\pi\lambda}\bar{h})_{\underset{a}{|}\rho} + P^\alpha_{(2)} + 2X^\alpha_{(1)[0]}\bar{h} + \frac{\sigma}{1-\sigma} a^{\alpha\rho}(Q^0_{(1)}\bar{h})_{\underset{a}{|}\rho}$$
$$+ (a^{\pi\lambda}a^{\alpha\gamma} + 2a^{\alpha\lambda}a^{\pi\gamma}) q_{(0)\pi\lambda} Q_{(1)\gamma}\bar{h} = 0, \tag{9.3b}$$

$$n^{\gamma\beta}_{[0]} q_{(0)\alpha\gamma\underset{a}{|}\beta} = 0, \tag{9.3c}$$

$$n^{\alpha\beta}_{[0]} n^{\rho\gamma}_{[0]} q_{(0)\rho\beta} q_{(0)\alpha\gamma} = 0. \tag{9.3d}$$

We may remark that all quantities in the above equations are finite, i.e. independent of ϵ. The macroscopic tensors in (7.5) can be written as

$$T^{\alpha\beta} = (2A_{(1)}^{\alpha\beta\pi\lambda}p_{(1)\pi\lambda}\bar{h} + \frac{\sigma}{1-\sigma}a^{\alpha\beta}Q_{(1)}^{0}\bar{h})\epsilon^2 + O(\epsilon^3), \tag{9.4a}$$

$$L^{\alpha\beta} = \tfrac{2}{3}n_{[0]}^{\omega\beta}a_{\pi\omega}A_{(1)}^{\alpha\pi\lambda\delta}q_{(0)\lambda\delta}\bar{h}^3\epsilon^3 + O(\epsilon^4), \tag{9.4b}$$

$$T^{\alpha 0} = Q_{(1)}^{\alpha}\bar{h}\epsilon^2 + O(\epsilon^3) \quad \text{if} \quad Q_{(1)}^{\alpha} \neq 0,$$

$$T^{\alpha 0} = \left\{Q_{(2)}^{\alpha}\bar{h} + \tfrac{2}{3}A_{(1)}^{\pi\alpha\lambda\delta}(q_{(0)\lambda\delta}\bar{h}^3)_{\underset{a}{|}\pi}\right\}\epsilon^3 + O(\epsilon^4) \quad \text{if} \quad Q_{(1)}^{\alpha} = 0. \tag{9.4c}$$

Equations (9.3a, b, c, d) form a set of six equations for the six unknowns $q_{(0)\alpha\beta}$ and $p_{(1)\alpha\beta}$. From (9.3d), we see that the total curvature of the middle surface does not change for the first approximation. Consequently the strained middle surface in this type of problem is to be regarded as a developable surface.

Equations (9.3a, b, c, d) can be further simplified. Since for a plate, $R_{\rho\alpha\beta\gamma}=0$, the order of the operations of covariant differentiation is immaterial; consequently, from (9.3c), we have

$$q_{(0)\alpha\beta} = w_{(0)\underset{a}{|}\alpha\beta}. \tag{9.5}$$

Here $w_{(0)}$ is an unknown function of x^α, which satisfies, in consequence of (9.3d),

$$n_{[0]}^{\alpha\beta}n_{[0]}^{\rho\gamma}w_{(0)\underset{a}{|}\rho\beta}w_{(0)\underset{a}{|}\alpha\gamma} = 0. \tag{9.6}$$

The existence of $w_{(0)}$, satisfying (9.5), is easily proved by temporary use of special coordinates (rectangular Cartesians). The last equation is, in fact, the famous differential equation [7] of a developable surface in the curvilinear coordinate system, and $w_{(0)}$ may be called the *deflection function*. If (9.6) is satisfied, $q_{(0)\pi\gamma}$ is given by (9.5). There still remain the three equations (9.3a, b) for the three unknowns $p_{(1)\pi\gamma}$. We can handle the problem indirectly by means of $T_{(2)}^{\alpha\beta}$. This is the coefficient of the lowest power, ϵ^2, in the series for $T^{\alpha\beta}$, and by (9.4a)

$$T_{(2)}^{\alpha\beta} = 2A_{(1)}^{\alpha\beta\pi\lambda}p_{(1)\pi\lambda}\bar{h} + \frac{\sigma}{1-\sigma}a^{\alpha\beta}Q_{(1)}^{0}\bar{h}. \tag{9.7}$$

We note that this is a symmetrical tensor, so that it has only three independent components. Substituting (9.5), (9.7) into (9.3a, b), we have

$$-T_{(2)}^{\pi\lambda}w_{(0)|\pi\lambda} + P_{(2)}^{0} + 2X_{(1)[0]}^{0}\bar{h} + (Q_{(1)}^{\pi}\bar{h})_{\underset{a}{|}\pi} + a^{\pi\lambda}w_{(0)\underset{a}{|}\pi\lambda}Q_{(1)}^{0}\bar{h} = 0, \tag{9.8a}$$

$$T_{(2)\underset{a}{|}\pi}^{\pi\alpha} + P_{(2)}^{\alpha} + 2X_{(1)[0]}^{\alpha}\bar{h} + (a^{\pi\lambda}a^{\alpha\gamma} + 2a^{\alpha\lambda}a^{\pi\gamma})w_{(0)\underset{a}{|}\pi\lambda}Q_{(1)\gamma}\bar{h} = 0. \tag{9.8b}$$

To sum up, for problems of type $P1$, we have a set of four equations (9.6), (9.8a, b) in the four unknowns, $w_{(0)}$ and $T_{(2)}^{\alpha\beta}$.

Special case. The following special case is interesting. If

$$P_{(2)}^{\alpha} = X_{(1)[0]}^{\alpha} = Q_{(1)}^{\alpha} = 0, \tag{9.9}$$

then by (9.8b) there exists an Airy function $\chi_{(2)}$, so that

$$T_{(2)}^{\alpha\beta} = n_{[0]}^{\alpha\pi}n_{[0]}^{\beta\lambda}\chi_{(2)\underset{a}{|}\pi\lambda}. \tag{9.10}$$

This is easily proved by temporary use of special coordinates (rectangular Cartesians [2]). Consequently, (9.8a) can be reduced to the form

$$-\, \mathbf{n}_{[0]}^{\rho\pi}\mathbf{n}_{[0]}^{\lambda\delta}\chi_{(2)\underset{a}{|}\pi\delta}w_{(0)\underset{a}{|}\rho\lambda} + P_{(2)}^{0} + 2X_{(1)[0]}^{0}\bar{h} + a^{\pi\lambda}w_{(0)\underset{a}{|}\pi\lambda}Q_{(1)}^{0}\bar{h} = 0. \tag{9.11}$$

The problem is now to find $\chi_{(2)}$ and $w_{(0)}$ as functions of x^α satisfying the two non-linear partial differential equations (9.6) and (9.11). In rectangular Cartesian coordinates, the equations (9.6) and (9.11) may be written as

$$w_{(0),12}w_{(0),12} - w_{(0),11}w_{(0),22} = 0, \tag{9.12a}$$

$$2\chi_{(2),12}w_{(0),12} - \chi_{(2),11}w_{(0),22} - \chi_{(2),22}w_{(0),11} + Q_{(1)}^{0}\bar{h}\Delta w_{(0)} + P_{(2)}^{0} + 2X_{(1)[0]}^{0}\bar{h} = 0, \tag{9.12b}$$

where the comma indicates partial differentiation with respect to x^α, and Δ is the two-dimensional Laplace operator. The macroscopic tensors are given by

$$\left.\begin{aligned} T^{11} &= \chi_{(2),22}\epsilon^2 + O(\epsilon^3), \qquad T^{22} = \chi_{(2),11}\epsilon^2 + O(\epsilon^3),\\ T^{12} &= T^{21} = -\,\chi_{(2),12}\epsilon^2 + O(\epsilon^3), \end{aligned}\right\} \tag{9.13a}$$

$$\left.\begin{aligned} L^{11} &= -\,L^{22} = -\,D(1-\sigma)w_{(0),12}\epsilon^3 + O(\epsilon^4),\\ L^{12} &= D(w_{(0),11} + \sigma w_{(0),22})\epsilon^3 + O(\epsilon^4),\\ L^{21} &= -\,D(w_{(0),22} + \sigma w_{(0),11})\epsilon^3 + O(\epsilon^4), \end{aligned}\right\} \tag{9.13b}$$

$$\left.\begin{aligned} T^{10} &= \left\{(1-\sigma)(Dw_{(0),12})_{,2} + (Dw_{(0),11} + \sigma Dw_{(0),22})_{,1}\right\}\epsilon^3 + O(\epsilon^4),\\ T^{20} &= \left\{(1-\sigma)(Dw_{(0),12})_{,1} + (Dw_{(0),22} + \sigma Dw_{(0),11})_{,2}\right\}\epsilon^3 + O(\epsilon^4). \end{aligned}\right\} \tag{9.13c}$$

Here the symbol D is defined by

$$D = \frac{2\bar{h}^3}{3(1-\sigma^2)}\cdot \tag{9.14}$$

This is a finite quantity; the ordinary flexual rigidity is $D\epsilon^3E$ (where E is Young's modulus). An example of this type of problem is given below.

Example. A long rectangular plate is subjected to a uniform tension $T_{(2)}\epsilon^2$ on the two long edges, and a normal load $P_{(2)}^{0}\epsilon^2$ on one face; this normal load does not vary along the length of the plate. Find the form of the plate in the strained state.

In this example, we can neglect the edge effect near the end of the plate by considering the plate infinitely long. We assume that the middle surface in the strained state is cylindrical, with the generators of the cylinder parallel to the length of the plate, that is

$$q_{(0)11} = w_{(0),11} = \Omega(x^1), \qquad q_{(0)22} = q_{(0)12} = 0. \tag{9.15}$$

Here x^α are rectangular Cartesian coordinates, such that the x^2-axis is parallel to the long edges, and the x^1-axis perpendicular to them; Ω is an unknown function. Furthermore, in this example,

$$P_{(2)}^{\alpha} = X_{(1)[0]}^{\alpha} = Q_{(1)}^{0} = Q_{(1)}^{\alpha} = X_{(1)[0]}^{0} = 0. \tag{9.16}$$

Then from (9.8b) and the condition that $T_{(2)}^{\alpha\beta}$ are functions of x^1 only, we have, in consequence of the boundary conditions on the two long edges,

$$T_{(2)}^{11} = T_{(2)}, \quad (9.17\text{a}) \qquad T_{(2)}^{12} = T_{(2)}^{21} = 0, \quad (9.17\text{b}) \qquad T_{(2)}^{22} = 0. \quad (9.17\text{c})$$

Substituting (9.15)–(9.17) into (9.8a), we obtain

$$\Omega(\mathrm{x}^1) = \frac{P^0_{(2)}}{T_{(2)}}. \tag{9.18}$$

Therefore the curvature at any point of the cylindrical surface is proportional to the normal pressure at the point. For uniformly distributed pressure, the strained middle surface is circular cylindrical. It should be noted that the above conclusion holds in general for plates of non-uniform thickness, with the limitation that $\bar{h}$ is independent of $\mathbf{x}^2$.

(b) TYPE $P2$: $q=0$, $p=2$. FINITE DEFLECTION WITH SMALL EXTENSION IN THE MIDDLE SURFACE

General equations. As in Type $P1$, we have

$$n_0 = n = 3, \qquad j_0 = j = 2, \qquad k_0 = k = 2. \tag{9.19}$$

By substituting the ϵ series from (8.1), (8.4)–(8.6) into (7.2a, b), (7.4a, b), it is found that the lowest power of ϵ occurring in (7.2a, b) is ϵ^3, and in (7.4a, b) is ϵ^0. The corresponding coefficients give rise to the equations of equilibrium and compatibility in the first approximation as follows:

$$\begin{aligned} -\,2A_{(1)}^{\rho\gamma\pi\lambda}q_{(0)\rho\gamma}p_{(2)\pi\lambda}\bar{h} + \tfrac{2}{3}A_{(1)}^{\rho\gamma\pi\lambda}(q_{(0)\pi\lambda}\bar{h}^3)_{\underset{a}{|}\rho\gamma} + P^0_{(3)} + 2X^0_{(2)[0]}\bar{h} \\ + A_{(3)}^{\rho\gamma\pi\omega\lambda\delta}q_{(0)\pi\omega}q_{(0)\lambda\delta}q_{(0)\rho\gamma}\bar{h}^3 + (Q^\pi_{(2)}\bar{h})_{\underset{a}{|}\pi} + \frac{1-2\sigma}{1-\sigma}a^{\pi\lambda}q_{(0)\pi\lambda}Q^0_{(2)}\bar{h} = 0, \end{aligned} \tag{9.20a}$$

$$\begin{aligned} 2A_{(1)}^{\rho\alpha\pi\lambda}(p_{(2)\pi\lambda}\bar{h})_{\underset{a}{|}\rho} + A_{(3)}^{\rho\alpha\pi\omega\lambda\delta}(q_{(0)\pi\omega}q_{(0)\lambda\delta}\bar{h}^3)_{\underset{a}{|}\rho} + \tfrac{2}{3}a^{\alpha\tau}q_{(0)\tau\gamma}A_{(1)}^{\gamma\beta\lambda\delta}(q_{(0)\lambda\delta}\bar{h}^3)_{\underset{a}{|}\beta} + P^\alpha_{(3)} \\ + 2X^\alpha_{(2)[0]}\bar{h} + (a^{\pi\lambda}a^{\alpha\gamma} + 2a^{\alpha\lambda}a^{\pi\gamma})q_{(0)\pi\lambda}Q_{(2)\gamma}\bar{h} + \frac{\sigma}{1-\sigma}a^{\alpha\beta}(Q^0_{(2)}\bar{h})_{\underset{a}{|}\beta} = 0, \end{aligned} \tag{9.20b}$$

$$\mathfrak{n}^{\gamma\beta}_{[0]}q_{(0)\alpha\gamma\underset{a}{|}\beta} = 0, \quad (9.20\text{c}); \qquad \mathfrak{n}^{\alpha\beta}_{[0]}\mathfrak{n}^{\rho\gamma}_{[0]}q_{(0)\rho\beta}q_{(0)\alpha\gamma} = 0. \tag{9.20d}$$

The macroscopic tensors in (7.5a, b, c) can be written as

$$T^{\alpha\beta} = \left\{2A_{(1)}^{\alpha\beta\pi\lambda}p_{(2)\pi\lambda}\bar{h} + \frac{\sigma}{1-\sigma}a^{\alpha\beta}Q^0_{(2)}\bar{h} - A_{(3)}^{\alpha\beta\pi\omega\lambda\delta}q_{(0)\pi\omega}q_{(0)\lambda\delta}\bar{h}^3\right\}\epsilon^3 + O(\epsilon^4), \tag{9.21a}$$

$$L^{\alpha\beta} = \tfrac{2}{3}\mathfrak{n}^{\omega\beta}_{[0]}a_{\pi\omega}A_{(1)}^{\alpha\pi\lambda\delta}q_{(0)\lambda\delta}\bar{h}^3\epsilon^3 + O(\epsilon^4), \tag{9.21b}$$

$$T^{\alpha 0} = \left\{Q^\alpha_{(2)}\bar{h} + \tfrac{2}{3}A_{(1)}^{\pi\alpha\lambda\delta}(q_{(0)\lambda\delta}h^3)_{\underset{a}{|}\pi}\right\}\epsilon^3 + O(\epsilon^4). \tag{9.21c}$$

Here $A_{(1)}^{\alpha\beta\pi\lambda}$, $A_{(3)}^{\alpha\beta\pi\omega\lambda\delta}$ are given as in (6.33a, b). Equations (9.20a, b, c, d) form a set of six fundamental equations for the six unknowns $p_{(2)\alpha\beta}$ and $q_{(0)\alpha\beta}$. We see that from (9.20d) that the middle surface in the strained state is a developable surface.

As in Type $P1$, the problem can be further simplified by introducing $w_{(0)}$, such that

$$q_{(0)\alpha\beta} = w_{(0)\underset{a}{|}\alpha\beta}. \tag{9.22}$$

We have also

$$T^{\alpha\beta}_{(3)} = 2A_{(1)}^{\alpha\beta\pi\lambda}p_{(2)\pi\lambda}\bar{h} - A_{(3)}^{\alpha\beta\pi\omega\lambda\delta}q_{(0)\pi\omega}q_{(0)\lambda\delta}\bar{h}^3 + \frac{\sigma}{1-\sigma}a^{\alpha\beta}Q^0_{(2)}\bar{h}. \tag{9.23}$$

We note that $T^{\alpha\beta}_{(3)}$ is the coefficient of the lowest power of ϵ in $T^{\alpha\beta}$. It is a symmetric tensor, and consequently it has only three independent components. Substituting (9.22) and (9.23) into (9.20a, b, c, d), we find that (9.20c) is identically satisfied, while the other three equations become

$$- T^{\rho\gamma}_{(3)} w_{(0)\underset{a}{|}\rho\gamma} + \tfrac{2}{3} A^{\rho\gamma\pi\lambda}_{(1)} (w_{(0)\underset{a}{|}\pi\lambda}\bar{h}^3)_{\underset{a}{|}\rho\gamma} + P^0_{(3)} + 2X^0_{(2)[0]}\bar{h} + (Q^\pi_{(2)}\bar{h})_{\underset{a}{|}\pi} + a^{\pi\lambda} w_{(0)\underset{a}{|}\pi\lambda} Q^0_{(2)}\bar{h} = 0, \tag{9.24a}$$

$$T^{\rho\alpha}_{(3)\underset{a}{|}\rho} + \tfrac{2}{3} a^{\alpha\pi} w_{(0)\underset{a}{|}\pi\omega} A^{\omega\rho\lambda\delta}_{(1)} (w_{(0)\underset{a}{|}\lambda\delta}\bar{h}^3)_{\underset{a}{|}\rho} + P^\alpha_{(3)} + 2X^\alpha_{(2)[0]}\bar{h} + (a^{\pi\lambda}a^{\alpha\gamma} + 2a^{\alpha\lambda}a^{\pi\gamma}) w_{(0)\underset{a}{|}\pi\lambda} Q_{(2)\gamma}\bar{h} = 0, \tag{9.24b}$$

$$\mathbf{n}^{\alpha\beta}_{[0]} \mathbf{n}^{\rho\gamma}_{[0]} w_{(0)\underset{a}{|}\rho\beta} w_{(0)\underset{a}{|}\alpha\gamma} = 0. \tag{9.24c}$$

To sum up, we have for problems of Type $P2$ a set of four equations (9.24a, b, c), in the four unknowns, $w_{(0)}$ and $T^{\alpha\beta}_{(3)}$.

The special case of uniform thickness will now be treated. Since $\bar{h}$ is constant, (9.24a, b) may be written in the form

$$- T^{\rho\gamma}_{(3)} w_{(0)\underset{a}{|}\rho\gamma} + D\Delta\Delta w_{(0)} + P^0_{(3)} + 2X^0_{(2)[0]}\bar{h} + Q^\pi_{(2)\underset{a}{|}\pi} h + Q^0_{(2)}\bar{h}\Delta w_{(0)} = 0, \tag{9.25a}$$

$$\left\{ T^{\beta\alpha}_{(3)} + D a^{\alpha\pi} a^{\beta\gamma} w_{(0)\underset{a}{|}\pi\gamma}\Delta w_{(0)} - \tfrac{1}{2} D a^{\alpha\beta}(\Delta w_{(0)})^2 \right\}_{\underset{a}{|}\beta} + P^\alpha_{(3)} + 2X^\alpha_{(2)[0]}\bar{h} + (a^{\alpha\beta}\Delta w_{(0)} + 2a^{\alpha\lambda}a^{\beta\pi} w_{(0)\underset{a}{|}\pi\lambda}) Q_{(2)\beta}\bar{h} = 0, \tag{9.25b}$$

and (9.24c) remains unchanged. Here Δ is the two-dimensional Laplace operator, and D is the reduced flexual rigidity as in (9.14).

Furthermore, when

$$P^\alpha_{(3)} = X^\alpha_{(2)[0]} = Q^\alpha_{(2)} = 0, \tag{9.26}$$

the equation (9.25b) will be satisfied by putting ($\phi_{(3)}$ is an arbitrary function of x^α)

$$T^{\alpha\beta}_{(3)} = \mathbf{n}^{\alpha\pi}_{[0]} \mathbf{n}^{\beta\lambda}_{[0]} \phi_{(3)\underset{a}{|}\pi\lambda} - D a^{\alpha\pi} a^{\beta\lambda} w_{(0)\underset{a}{|}\pi\lambda}\Delta w_{(0)} + \tfrac{1}{2} D a^{\alpha\beta}(\Delta w_{(0)})^2. \tag{9.27}$$

And consequently, (9.25a) can be reduced to the form

$$- \mathbf{n}^{\rho\pi}_{[0]} \mathbf{n}^{\gamma\delta}_{[0]} \phi_{(3)\underset{a}{|}\pi\delta} w_{(0)\underset{a}{|}\rho\gamma} + D\Delta\Delta w_{(0)} + \tfrac{1}{2} D(\Delta w_{(0)})^3 + P^0_{(3)} + 2X^0_{(2)[0]}\bar{h} + Q^0_{(2)}\bar{h}\Delta w_{(0)} = 0. \tag{9.28}$$

Therefore for a plate of uniform thickness under the condition (9.26), we have in this type of problem a set of two equations, (9.24c) and (9.28), with two unknowns $w_{(0)}$ and $\phi_{(3)}$. In rectangular Cartesians, these two equations may be written as

$$w_{(0),12} w_{(0),12} - w_{(0),11} w_{(0),22} = 0, \tag{9.29a}$$

$$2w_{(0),12}\phi_{(3),12} - w_{(0),11}\phi_{(3),22} - w_{(0),22}\phi_{(3),11} + D\Delta\Delta w_{(0)} + \tfrac{1}{2} D(\Delta w_{(0)})^3 + P^0_{(3)} + 2X^0_{(2)[0]}\bar{h} + Q^0_{(2)}\bar{h}\Delta w_{(0)} = 0. \tag{9.29b}$$

The macroscopic tensors are given by

$$\left.\begin{aligned} T^{11} &= \left\{\phi_{(3),22} + \tfrac{1}{2}D(w_{(0),22} - w_{(0),11})\Delta w_{(0)}\right\}\epsilon^3 + O(\epsilon^4), \\ T^{12} &= T^{21} = -\left\{\phi_{(3),21} + Dw_{(0),12}\Delta w_{(0)}\right\}\epsilon^3 + O(\epsilon^4), \\ T^{22} &= \left\{\phi_{(3),11} + \tfrac{1}{2}D(w_{(0),11} - w_{(0),22})\Delta w_{(0)}\right\}\epsilon^3 + O(\epsilon^4), \end{aligned}\right\} \tag{9.30a}$$

$$\left.\begin{aligned} L^{11} &= -L^{22} = -D(1-\sigma)w_{(0),12}\epsilon^3 + O(\epsilon^4), \\ L^{12} &= D(w_{(0),11} + \sigma w_{(0),22})\epsilon^3 + O(\epsilon^4), \\ L^{21} &= -D(w_{(0),22} + \sigma w_{(0),11})\epsilon^3 + O(\epsilon^4), \end{aligned}\right\} \tag{9.30b}$$

$$T^{10} = D(\Delta w_{(0)})_{,1}\epsilon^3 + O(\epsilon^4), \qquad T^{20} = D(\Delta w_{(0)})_{,2}\epsilon^3 + O(\epsilon^4). \tag{9.30c}$$

An interesting example of this type of problem is given below.

Example: A long rectangular plate of uniform thickness is deformed under the actions of (a) uniform tensions $T_{(3)A}\epsilon^3$, $T_{(3)B}\epsilon^3$ and uniform bending moments $L_{(3)A}\epsilon^3$, $L_{(3)B}\epsilon^3$ on the two long edges, (b) a normal load $P^0_{(3)}\epsilon^3$ on one face (this load does not vary along the length of the plate). Assuming that $p=2$, $q=0$, find the form of the middle surface in the strained state.

In this example, we can neglect the edge effect at the two ends by considering the plate infinitely long. Since the given external force system does not vary along the length of the plate, we shall assume that strain and stress are constant along this direction. Hence in the first place, the deformed surface is cylindrical, with the generators of the cylinder parallel to the length of the plate:

$$q_{(0)11} = w_{(0),11} = \Omega(x^1), \quad q_{(0)12} = w_{(0),12} = q_{(0)22} = w_{(0),22} = 0. \tag{9.31}$$

Here x^α are rectangular Cartesians, so that x^2-axis is parallel to the long sides and x^1-axis is perpendicular to them. Ω is a function of x^1, to be determined.

In the second place, $T^{\alpha\beta}$ is a function of x^1 only. Since the ends of the plate are free from tractions, it follows that T^{12} and T^{22} vanish everywhere to the third order:

$$T^{22}_{(3)} = T^{12}_{(3)} = 0. \tag{9.32}$$

The component T^{11} can be written as

$$T^{11} = T^{11}_{(3)}\epsilon^3 + O(\epsilon^4), \tag{9.33}$$

where $T^{11}_{(3)}$ is a function of x^1, to be determined.

The problem is to determine two unknowns Ω and $T^{11}_{(3)}$ as functions of x^1 through Eqs. (9.25a, b) under the conditions

$$P^\alpha_{(3)} = X^0_{(2)[0]} = X^\alpha_{(2)[0]} = Q^0_{(2)} = Q^\alpha_{(2)} = 0. \tag{9.34}$$

Substituting (9.31)–(9.34) into (9.25a, b), we have

$$-\Omega T^{11}_{(3)} + D\Omega_{,11} + P^0_{(3)} = 0, \tag{9.35}$$

$$(T^{11}_{(3)} + \tfrac{1}{2}D\Omega^2)_{,1} = 0. \tag{9.36}$$

Integration of (9.36) gives

$$T^{11}_{(3)} + \tfrac{1}{2}D\Omega^2 = C. \tag{9.37}$$

Here C is a constant to be determined by the conditions on the long edges. Substituting $w_{(0),\alpha\beta}$ from (9.31) into (9.30b, c), we get

$$L^{11} = -L^{22} = O(\epsilon^4), \quad L^{12} = D\Omega\epsilon^3 + O(\epsilon^4), \quad L^{21} = -\sigma D\Omega\epsilon^3 + O(\epsilon^4), \tag{9.38}$$

$$T^{10} = D\Omega_{,1}\epsilon^3 + O(\epsilon^4), \qquad T^{20} = O(\epsilon^4). \tag{9.39}$$

Then (9.37) becomes

$$T^{11}_{(3)} = C - \frac{(L^{12}_{(3)})^2}{2D}, \tag{9.40}$$

where, by definition, $L^{12}_{(3)} = D\Omega$. This equation is satisfied everywhere throughout the plate. Therefore it is also satisfied at the two long edges, and consequently $T_{(3)A}$, $T_{(3)B}$, $L_{(3)A}$, $L_{(3)B}$ must satisfy the following relation:

$$T_{(3)A} + \frac{(L_{(3)A})^2}{2D} = T_{(3)B} + \frac{(L_{(3)B})^2}{2D} = C. \tag{9.41}$$

Therefore we conclude that among $T_{(3)A}$, $T_{(3)B}$, $L_{(3)A}$, $L_{(3)B}$ only three quantities are independent; when any three are given, the fourth can be calculated through (9.41).

Substituting $T^{11}_{(3)}$ from (9.37) into (9.35), we obtain

$$\tfrac{1}{2}D\Omega^3 - C\Omega + D\Omega_{,11} = -P^0_{(3)}. \tag{9.42}$$

This is a non-linear differential equation of the second order and third degree in Ω. When the boundary values of Ω are given (or $L_{(3)A}$, $L_{(3)B}$ are given), the solution is uniquely determined.

If $P^0_{(3)} = 0$, the problem is identical with the problem of the elastica [8]. For then, if we introduce the new variable θ, so that

$$\Omega = \theta_{,1}, \tag{9.43}$$

equation (9.42) can be written as

$$\tfrac{1}{2}D(\theta_{,1})^3 - C\theta_{,1} + D\theta_{,111} = 0. \tag{9.44}$$

The second integral of this equation is

$$\tfrac{1}{2}D(\theta_{,1})^2 - C = F\cos\theta. \tag{9.45}$$

Equation (9.45) is in the same form as the well known equation for the elastica. The constant F can be determined by the boundary conditions on the long edges; θ is a physical quantity which denotes the direction of the tangent to the middle surface in the strained state.

The bending of a rectangular sheet of paper into a cylindrical surface by forces and couples applied to two opposite edges may be considered as a problem of the above type. There is, however, an edge effect in the neighborhood of the free edges.

(c) Type $P3$: $q=0$, $p>2$. Finite deflection with negligible extension in the middle surface

General equations. As in type $P1$, $P2$, we have

$$n_0 = n = 3, \qquad j_0 = j = 2, \qquad k_0 = k = 2. \tag{9.46}$$

By substituting the ϵ series from (8.1), (8.4)–(8.6) into (7.2a, b), (7.4a, b), we find that the lowest power in (7.2a, b) is ϵ^3, and in (7.4a, b) is ϵ^0. The corresponding coefficients give rise to the equations of equilibrium and compatibility in the first approximation as follows:

$$\tfrac{2}{3}A_{(1)}^{\rho\gamma\pi\lambda}(q_{(0)\pi\lambda}\bar{h}^3)_{\underset{a}{|}\rho\gamma} + A_{(3)}^{\rho\gamma\pi\omega\lambda\delta}q_{(0)\lambda\delta}q_{(0)\rho\gamma}q_{(0)\pi\omega}\bar{h}^3 + P_{(0)}^0$$
$$+ (Q_{(2)}^{\pi}\bar{h})_{\underset{a}{|}\pi} + 2X_{(2)[0]}^0\bar{h} + \frac{1-2\sigma}{1-\sigma}a^{\pi\lambda}q_{(0)\pi\lambda}Q_{(2)}^0\bar{h} = 0, \tag{9.47a}$$

$$- A_{(3)}^{\rho\alpha\pi\omega\delta\lambda}(q_{(0)\pi\omega}q_{(0)\delta\lambda}\bar{h}^3)_{\underset{a}{|}\rho} + \tfrac{2}{3}a^{\alpha\pi}q_{(0)\pi\gamma}A_{(1)}^{\gamma\rho\lambda\delta}(q_{(0)\lambda\delta}\bar{h}^3)_{\underset{a}{|}\rho} + P_{(3)}^{\alpha} + 2X_{(2)[0]}^{\alpha}\bar{h}$$
$$+ \frac{\sigma}{1-\sigma}a^{\alpha\rho}(Q_{(2)}^0\bar{h})_{\underset{a}{|}\rho} + (a^{\pi\lambda}a^{\alpha\gamma} + 2a^{\alpha\lambda}a^{\pi\gamma})q_{(0)\pi\lambda}Q_{(2)\gamma}\bar{h} = 0, \tag{9.47b}$$

$$\mathfrak{n}_{[0]}^{\gamma\beta}q_{(0)\alpha\gamma\underset{a}{|}\beta} = 0, \quad (9.47\text{c}) \qquad \mathfrak{n}_{[0]}^{\alpha\beta}\mathfrak{n}_{[0]}^{\rho\gamma}q_{(0)\rho\beta}q_{(0)\alpha\gamma} = 0. \tag{9.47d}$$

The macroscopic tensors in (7.5a, b, c) can be written as

$$T^{\alpha\beta} = \left\{\frac{\sigma}{1-\sigma}a^{\alpha\beta}Q_{(2)}^0\bar{h} - A_{(3)}^{\alpha\beta\pi\gamma\lambda\delta}q_{(0)\pi\gamma}q_{(0)\lambda\delta}\bar{h}^3\right\}\epsilon^3 + O(\epsilon^4), \tag{9.48a}$$

$$L^{\alpha\beta} = \tfrac{2}{3}\mathfrak{n}_{[0]}^{\omega\beta}a_{\pi\omega}A_{(1)}^{\alpha\pi\lambda\delta}q_{(0)\lambda\delta}\bar{h}^3\epsilon^3 + O(\epsilon^4), \tag{9.48b}$$

$$T^{\alpha 0} = \left\{\tfrac{2}{3}A_{(1)}^{\pi\alpha\lambda\delta}(q_{(0)\lambda\delta}\bar{h}^3)_{\underset{a}{|}\pi} + Q_{(2)}^{\alpha}\bar{h}\right\}\epsilon^3 + O(\epsilon^4). \tag{9.48c}$$

Equations (9.46a, b, c, d) form a set of six equations involving only three unknowns $q_{(0)\alpha\beta}$, so the problem is overdetermined. Let us suppose that $q_{(0)\alpha\beta}$ can be eliminated from these six equations; we get a set of three conditions, which may be written in the form

$$\Psi_{(j)} = \Psi_{(j)}(\bar{h}, Q_{(2)}^i, P_{(3)}^i, X_{(2)[0]}^i) = 0, \qquad (j = 1, 2, 3). \tag{9.49}$$

Equations (9.49) represent the three necessary conditions on the external force system in order that a plate may undergo *finite* deflection with *negligible* extension in the middle surface. A special example will be considered as follows.

Example. Under what circumstances can a portion of a plate of uniform thickness be bent by normal pressure into a cylindrical surface of finite curvature with negligible extension in the middle surface? The normal pressure is assumed to be constant along the generators of the cylinder.

In this case,

$$X_{(2)[0]}^i = Q_{(2)}^{\alpha} = P_{(3)}^{\alpha} = 0. \tag{9.50}$$

Let us choose the $\mathbf{x}^2$-axis in the direction of the generators of the assumed cylindrical surface, and the $\mathbf{x}^1$-axis in the perpendicular direction. Then we have as in the example of Types $P1$, $P2$

$$q_{(0)11} = \Omega(\mathbf{x}^1), \qquad q_{(0)12} = q_{(0)22} = 0, \tag{9.51}$$

and the equations (9.47a, b) become

$$D\Omega_{,11} + \frac{3-\sigma}{2(1-\sigma)}D\Omega^3 + P_{(3)}^0 + \frac{1-2\sigma}{1-\sigma}\Omega Q_{(2)}^0\bar{h} = 0, \tag{9.52a}$$

$$- D(\Omega^2)_{,1} + \sigma\bar{h}Q_{(2),1}^0 = 0. \tag{9.52b}$$

Integration of (9.52b) gives

$$\Omega^2 = \frac{\sigma\bar{h}}{D}(C_1 + Q_{(2)}), \tag{9.53}$$

where C_1 is an integration constant. Substituting Ω^2 from (9.53) into (9.52a), we get

$$(\sigma\bar{h}D)^{1/2}\{(C_1 + Q^0_{(2)})^{1/2}\}_{,11} + \frac{(3-\sigma)(\sigma\bar{h})^{3/2}}{2(1-\sigma)D^{1/2}}(C_1 + Q^0_{(2)})^{3/2}$$
$$+ P^0_{(3)} + \frac{1-2\sigma}{1-\sigma}\left(\frac{\sigma}{D}\right)^{1/2}\bar{h}^{3/2}Q^0_{(2)} = 0. \tag{9.54}$$

This is the required condition to be satisfied by $Q_{(2)}$, $P^0_{(3)}$.

Let us assume that Q^0 and P^0 are of the same order of magnitude; then, since $P_{(2)} = 0$, we have

$$Q^0_{(2)} = 0. \tag{9.55}$$

Then the condition (9.54) becomes

$$P^0_{(3)} = -\frac{(3-\sigma)(\sigma\bar{h})^{3/2}}{2(1-\sigma)D^{1/2}}C_1^{3/2} = \text{constant}. \tag{9.56}$$

Furthermore, since the right hand side of (9.53) is constant, the plate is bent into a circular cylindrical surface; its curvature is given by

$$\Omega = -\left\{\frac{2(1-\sigma)P_{(3)}}{(3-\sigma)D}\right\}^{1/3}. \tag{9.57}$$

When $P^0_{(3)} = 0$, we get from (9.57) $\Omega = 0$. Therefore we conclude that it is ***impossible*** to bend any portion of a plate of uniform thickness into a cylindrical surface of ***finite*** curvature with ***negligible*** extension in the middle surface, if on that portion of the plate the surface force is of the fourth order, and the body force of the third order, with respect to the thickness of the plate.

CONCLUSIONS

A systematic method of approximation based upon the thinness of the plate has been developed in this paper. It is found that thin plate problems may be classified into twelve types ($P1$–$P12$) according to the relative orders of magnitude of $\mathbf{p}_{\alpha\beta}$, $\mathbf{q}_{\alpha\beta}$ and $\mathbf{h}$. In each case, the problem reduces to the solution of a set of partial differential equations, different for different types. These differential equations are given in Table I. Furthermore, the principal parts of the macroscopic tensors and the orders of magnitude of the external forces for each case are given in Table II. Among these twelve types, $P1$–$P3$ represent the problems of finite deflection, $P4$–$P8$ the problems of small deflection, $P9$–$P11$ the problems of very small deflection and $P12$ the problems of zero deflection. The problems of finite deflection are discussed in section 9; these are new problems, and a simple example for each of these types is solved. The problems of small deflections, very small deflection, and zero deflection are familiar; the detailed discussion of these types is therefore not necessary. However, we may note that the theory of generalized plane stress, the Lagrange-Kirchhoff theory of "small" deflection, the von Kármán theory of "large" deflection and the membrane problem can be derived respectively from Types $P12$, $P11$, $P5$, $P4$.

APPENDICES

(i) TABLE I.—Table of the equations of equilibrium and compatibility of thin plate problems.

Types	q	p	(7.2a)					(7.2b)				(7.4b)		(7.4a)
			I_1^0	I_2^0	I_3^0	I_4^0	I_5^0	I_1^α	I_2^α	I_3^α	I_4^α	J_1^0	J_2^0	$J_{\alpha 1}$
$P1$	0	1	x			x	x	x		x	x		x	x
$P2$	0	2	x	x	x	x	x	x	x	x	x		x	x
$P3^*$	0	>2		x	x	x	x		x	x	x		x	x
$P4$	$\geqq 1$	1	x			x	x	x		x		x		x
$P5$	1	2	x	x		x	x	x		x		x	x	x
$P6^*$	$\geqq 1$	$2q+1$		x		x		x		x			x	x
$P7^*$	$\geqq 1$	$2q+2$		x		x		x	x	x	x		x	x
$P8^*$	$\geqq 1$	$>2q+2$		x		x			x	x	x		x	x
$P9$	2	>1	x	x		x	x	x		x		x		x
$P10$	$\geqq 2$	$2<p<2q$		x		x		x		x		x		x
$P11$	$\geqq 2$	$2q$		x		x		x		x		x	x	x
$P12$	∞	$\geqq 1$						x		x		x		

In this table, the following notation is used:
The terms occurring in the first equation of equilibrium (7.2a) are

$$I_1^0 = -2A_{(1)}^{\rho\gamma\pi\lambda} q_{\rho\gamma} p_{\pi\lambda} h, \qquad I_2^0 = \tfrac{2}{3} A_{(1)}^{\rho\gamma\pi\lambda} (q_{\pi\lambda} h^3)_{\underset{a}{|}\rho\gamma}, \qquad I_3^0 = A_{(3)}^{\rho\gamma\pi\omega\lambda\delta} q_{\rho\gamma} q_{\pi\omega} q_{\lambda\delta} h^3$$

$$I_4^0 = P^0 + 2X_{[0]}^0 h + (Q^\pi h)_{\underset{a}{|}\pi}, \qquad I_5^0 = \frac{1-2\sigma}{1-\sigma} a_{\pi\lambda} q^{\pi\lambda} Q^0 h.$$

The terms occurring in the second and third equations of equilibrium (7.2b) are

$$I_1^\alpha = 2A_{(1)}^{\rho\alpha\pi\lambda} (p_{\pi\lambda} h)_{\underset{a}{|}\rho},$$

$$I_2^\alpha = \tfrac{2}{3} a^{\alpha\pi} q_{\pi\gamma} A_{(1)}^{\gamma\rho\lambda\delta} (q_{\lambda\delta} h^3)_{\underset{a}{|}\rho} - A_{(3)}^{\rho\alpha\pi\omega\lambda\delta} (q_{\pi\omega} q_{\lambda\delta} h^3)_{\underset{a}{|}\rho},$$

$$I_3^\alpha = P^\alpha + 2X_{[0]}^\alpha h + \frac{\sigma}{1-\sigma} a^{\alpha\rho} (Q^0 h)_{\underset{a}{|}\rho}, \qquad I_4^\alpha = (a^{\pi\lambda} q_{\pi\lambda} a_\gamma^\alpha + 2a^{\alpha\pi} q_{\pi\gamma}) Q^\gamma h.$$

The terms occurring in the first equation of compatibility (7.4b) are

$$J_1^0 = 2\mathfrak{n}_{[0]}^{\rho\alpha} \mathfrak{n}_{[0]}^{\beta\gamma} p_{\rho\gamma \underset{a}{|} \alpha\beta}, \qquad J_2^0 = \mathfrak{n}_{[0]}^{\rho\alpha} \mathfrak{n}_{[0]}^{\beta\gamma} q_{\rho\gamma} q_{\alpha\beta}.$$

The term occurring in the second and third equations of compatibility (7.4a) is

$$J_{\alpha 1} = 2\mathfrak{n}_{[0]}^{\beta\gamma} q_{\alpha\beta \underset{a}{|} \gamma}.$$

On account of the conditions which hold in the various types of problem, some of these terms may be negligible in comparison with others. The table shows by the symbol 'x' those terms which are to be retained in the first approximation for the various types. (The overdetermined problems are denoted by '*'.) Thus for example, for problems of Type $P1$, we having the following equations of equilibrium and compatibility in the first approximation:

$$I_1^0 + I_4^0 + I_5^0 = 0, \qquad I_1^\alpha + I_3^\alpha + I_4^\alpha = 0, \qquad J_2^0 = 0, \qquad J_{\alpha 1} = 0.$$

These equations are written in terms of the small principal parts instead of in terms of the finite coefficients of the lowest power in ϵ (see (9.3a, b, c, d)).

(ii) TABLE II.—Table of the external force system and the macroscopic tensors for various types of thin plate problems.

Types	n_0	n	j_0	j	k_0	k	$T^{\alpha\beta}$: t	$T_1^{\alpha\beta}$	$T_2^{\alpha\beta}$	$T_3^{\alpha\beta}$	$L^{\alpha\beta}$: u	$L_1^{\alpha\beta}$	$T^{\alpha 0}$: l	T_1^α	T_2^α
P1	2	2	1	1	1	1	2	x		x	3	x	2	x	x
	2	2	1	1	1	2	2	x		x	3	x	3	x	x
P2	3	3	2	2	2	2	3	x	x	x	3	x	3	x	x
P3	3	3	2	2	2	2	3		x	x	3	x	3	x	x
P4	$q+2$	2	$q+1$	1	1	$q+1$	2	x		x	$q+3$	x	$q+2$	x	
	$q+2$	2	$q+1$	1	1	$q+2$	2	x		x	$q+3$	x	$q+3$	x	x
P5	4	3	3	2	2	3	3	x		x	4	x	4	x	x
P6	$q+3$	$2q+2$	$q+2$	$2q+1$	$2q+1$	$q+2$	$2q+2$	x		x	$q+3$	x	$q+3$	x	x
P7	$q+3$	$2q+3$	$2q+2$	$q+2$	$q+2$	$2q+2$	$2q+3$	x	x	x	$q+3$	x	$q+3$	x	x
P8	$q+3$	$2q+3$	$2q+2$	$q+2$	$q+2$	$2q+2$	$2q+3$		x	x	$q+3$	x	$q+3$	x	x
P9	$q+3$	3	$q+2$	2	2	$q+2$	3	x		x	$q+3$	x	$q+3$	x	x
P10	$q+3$	$p+1$	$q+2$	p	p	$q+2$	$p+1$	x		x	$q+3$	x	$q+3$	x	x
P11	$q+3$	$2q+1$	$q+2$	$2q$	$2q$	$q+2$	$2q+1$	x		x	$q+3$	x	$q+3$	x	x
P12	∞	$p+1$	∞	p	p	∞	$p+1$	x		x	∞		∞		

In this table, the following notation is used:

The terms occurring in the expression (7.5a) for the membrane stress tensor $T^{\alpha\beta}$ are denoted by

$$T_1^{\alpha\beta} = 2A_{(1)}^{\alpha\beta\pi\lambda} p_{\pi\lambda} h, \qquad T_2^{\alpha\beta} = -A_{(3)}^{\alpha\beta\pi\omega\lambda\delta} q_{\lambda\delta} q_{\pi\omega} h^3, \qquad T_3^{\alpha\beta} = \frac{\sigma}{1-\sigma} a^{\alpha\beta} Q^0 h.$$

The term occurring in the expression (7.5b) for the bending moment tensor $L^{\alpha\beta}$ is denoted by

$$L_1^{\alpha\beta} = \tfrac{2}{3} n_{[0]}^{\omega\beta} a_{\pi\omega} A_{(1)}^{\alpha\pi\lambda\delta} q_{\lambda\delta} h^3.$$

The terms occurring in the expression (7.5c) for the shearing stress tensor $T^{\alpha 0}$ are denoted by

$$T_1^\alpha = Q^\alpha h, \qquad T_2^\alpha = \tfrac{2}{3} A_{(1)}^{\pi\alpha\lambda\delta} (q_{\lambda\delta} h^3)_{|\pi}.$$

Furthermore,

n_0 = order of sum of the normal forces acting on the upper and lower boundary surfaces, or order of P^0,

n = order of sum of the tangential forces acting on the upper and lower boundary surfaces, or order of P^α,

j_0 = order of normal component of body force, or order of $X_{[0]}^0$,

j = order of tangential component of body force, or order of $X_{[0]}^\alpha$,

k_0 = order of difference of normal forces acting on the upper and lower surfaces, or order of Q^0,

k = order of difference of tangential components of forces acting on the upper and lower boundary surfaces, or order of Q^α,

t = order of membrane stress tensor $T^{\alpha\beta}$,

u = order of bending moment tensor $L^{\alpha\beta}$,

l = order of shearing stress tensor $T^{\alpha 0}$.

This table gives (a) the values of n_0, n, j_0, j, k_0, k, t, u, l, (b) the principal terms in the expressions for $T^{\alpha\beta}$, $L^{\alpha\beta}$, $T^{\alpha 0}$ (denoted by 'x'). The terms not marked with 'x' are negligible in comparison with those principal terms. It will be noted that there are two lines in the table for $P1$ and also for $P4$. This is because, in each case, k may have two values.

For example, in the case of Type $P1$, we have for $T^{\alpha\beta}$, $L^{\alpha\beta}$,

$$T^{\alpha\beta} = T_1^{\alpha\beta} + T_3^{\alpha\beta}, \qquad L^{\alpha\beta} = L_1^{\alpha\beta},$$

while for $T^{\alpha 0}$,

$$T^{\alpha 0} = T_1^\alpha \qquad (\text{if } k = 1),$$

$$T^{\alpha 0} = T_1^\alpha + T_2^\alpha \qquad (\text{if } k = 2).$$

(To be continued)

References

1. G. B. Airy, *On the strains in the interior of beams*, Phil. Trans. Roy. Soc. London. A **153**, 49–80 (1863).

2. E. G. Coker and L. M. G. Filon, *A treatise on photo-elasticity*, C mbridge, 1931, p. 126.

3. G. Kirchhoff, *Über das Gleichgewicht und die Bewegung einer elastischen Scheibe*, J. Reine und Angew. Math. (Crelle), **40**, 51–88 (1850); Comptes Rendus Ac. Sci. Paris, **84**, 1157 (1877).

4. M. J. Boussinesq, *Étude nouvelle sur l'équilibre et le mouvement des corps solides élastiques dont certaines dimensions sont très petites par rapport à d'autres*, J. de Mathématique (2), **16**, 125–247 (1871).

5. H. Marcus, *Die Theorie elastischer Gewebe und ihre Anwendung auf die Berechnung biegsamer Platten*, Springer, 1932.

6. Th. von Kármán, *Festigkeitsprobleme im Maschinenbau*, Enzyklopädie der mathematischen Wissenschaften, **IV**, 4 (1910), art. 27.

7. C. F. Gauss, *General investigations of curved surfaces*, Princeton, 1902.

8. A. E. H. Love, *A treatise on the mathematical theory of elasticity*, Cambridge, 1927, p. 401.