# STRESSES IN THE DIAPHRAGMS OF DIAPHRAGM-PUMPS*

BY

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**1. Introduction.** Complete prevention of leakage from a reciprocating pump is difficult to ensure over a long period of working. When the fluid to be pumped is of such a nature that no leakage whatever is permissible, some modification in the design of the pump is essential, and under these circumstances a diaphragm-pump may conveniently be used. This in its essentials consists of two chambers (Fig. 1) attached to a modified reciprocating pump. The chambers are of conical form rounded off at the apex and at the base, and between them a diaphragm is clamped at its edge. For high-pressure operation the diaphragm is a very thin steel disc. The fluid to be pumped passes through one chamber, connexion to inlet and exhaust valves being made by means of a number of small ports. The other chamber is connected similarly to a single-acting reciprocating pump, which is not fitted with valves. This chamber, the pump cylinder and their connecting ports are filled with a liquid (commonly oil), and thus motion of the piston of the reciprocating pump causes the diaphragm to be pressed alternately against both conical surfaces, thereby producing the desired pumping action. The inevitable leakage of oil past the piston is made good by means of an auxiliary pump.

An approximate method of calculating the stresses in the diaphragm is explained below, hence the size of the chambers may be so designed that the fatigue strength of the diaphragm is not exceeded. In section 2 the deflexion of the diaphragm is taken as sinusoidal, in section 3 as a cubic, and in section 4 as following a Bessel-function relation Attention is confined to the stresses which result from distortion into the same shape as the chamber, no regard being paid to the local stresses round the ports.

**2. Stresses when the transverse displacement is sinusoidal.** In general the displacement of the diaphragm from its unstrained position has not only a transverse but also a radial component; therefore it does not seem possible (except by relaxation methods) to calculate the stresses for a specified shape of chamber. It is necessary to assume a reasonable expression for the transverse displacement $w$, from which the corresponding radial displacement $u$ and the stresses will be obtained; and, when both $u$ and $w$ are known, the shape of the chamber is determined.

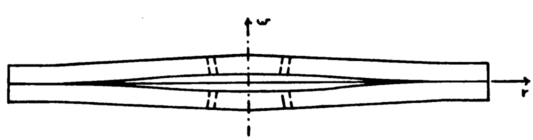

FIG. 1. Arrangement of diaphragm and chambers.

With the axes shown in Fig. 1 we shall in this section take $w$ as specified by

$$w = \frac{w_0}{2}\left(1 + \cos\frac{\pi r}{a}\right), \tag{1}$$

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where $a$ is the radius of the diaphragm and $w_0$ the maximum value of $w$. This expression satisfies the conditions that the slope must vanish at the centre and at the edge. For $u$ we assume as an approximation[1] that

$$u = r(a - r)(C_1 + C_2 r), \tag{2}$$

where $C_1$ and $C_2$ are constants which will be determined by the principle of minimum strain energy. The conditions that $u$ is zero at the centre and at the edge are automatically fulfilled. Now the transverse displacements are many times the thickness of the diaphragm, hence large-deflexion theory must be employed. In Timoshenko's notation (loc. cit.) the radial and tangential strains are thus

$$\left.\begin{aligned} e_r &= \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2, \\ e_t &= \frac{u}{r}. \end{aligned}\right\} \tag{3}$$

The diaphragm being very thin in comparison with its radius, the strain energy in it due to bending may be neglected in comparison with that due to the stretching of its middle plane. Hence the strain energy in the diaphragm is

$$V_1 = \frac{\pi E h}{1 - \nu^2}\int_0^a r(e_r^2 + e_t^2 + 2\nu e_r e_t)dr. \tag{4}$$

Here $E$ denotes Young's modulus, $\nu$ Poisson's ratio, and $h$ the uniform thickness of the diaphragm. On putting (1) and (2) into (3) and inserting the results in (4) we obtain

$$\begin{aligned} V_1 = \frac{\pi E h}{1 - \nu^2}\Bigg[&\frac{1}{4}C_1^2 a^4 + \frac{3}{10}C_1 C_2 a^5 + \frac{7}{60}C_2^2 a^6 + \frac{3\pi^4}{1024}w_0^2 \\ &+ \frac{w_0^2 a}{8}\left\{C_1\left(1 - \frac{\pi^2}{6} + \nu\left(\frac{\pi^2}{6} + \frac{1}{2}\right)\right) + C_2 a\left(\frac{5}{4} - \frac{\pi^2}{12} + \nu\left(\frac{\pi^2}{12} + \frac{1}{4}\right)\right)\right\}\Bigg]. \end{aligned} \tag{5}$$

Now

$$\frac{\partial V_1}{\partial C_1} = \frac{\partial V_1}{\partial C_2} = 0; \tag{6}$$

hence

$$\left.\begin{aligned} C_1 &= \frac{25 w_0^2}{128 a^3}\left\{\frac{\pi^2}{3} + \frac{17}{5} - \nu\left(\frac{\pi^2}{3} + 1\right)\right\}, \\ C_2 &= -\frac{15 w_0^2}{128 a^4}\left\{\frac{\pi^2}{3} + 13 - \nu\left(\frac{\pi^2}{3} + 1\right)\right\}. \end{aligned}\right\} \tag{7}$$

In the remaining calculations we will consider the case $\nu = 0.3$ when (7) reduces to

$$C_1 = 1.06\frac{w_0^2}{a^3}, \qquad C_2 = -1.76\frac{w_0^2}{a^4}. \tag{8}$$

[1] Cf. S. Timoshenko, *Theory of Plates and Shells*, McGraw-Hill, New York and London, 1940, chap. IX.

The radial and tangential tensile stresses are

$$\sigma_r = \frac{E}{1-\nu^2}(e_r + \nu e_t), \qquad \sigma_t = \frac{E}{1-\nu^2}(e_t + \nu e_r). \tag{9}$$

Then, with the aid of (3) and (8), (9) becomes

$$\left.\begin{aligned} \sigma_r &= \frac{Ew_0^2}{a^2}\left\{1.51 - 7.11\frac{r}{a} + 6.38\frac{r^2}{a^2} + 1.36\sin^2\frac{\pi r}{a}\right\}, \\ \sigma_t &= \frac{Ew_0^2}{a^2}\left\{1.51 - 4.95\frac{r}{a} + 3.67\frac{r^2}{a^2} + 0.41\sin^2\frac{\pi r}{a}\right\}. \end{aligned}\right\} \tag{10}$$

These expressions are plotted in Fig. 2, from which it will be seen that the maximum stress occurs at the centre and is given by 1.51 $Ew_0^2/a^2$. If $w_0/a = 1/35$ and $E = 13000$ tons/sq. in., the maximum stress is 17 tons/sq. in., which for a good quality steel is a reasonable working stress. Finally we will examine the shape of the chamber corresponding to (10). It will be noticed from (2) that $u$ is zero not only at the centre and at the edge but also at $r/a = -C_1/(C_2 a) = 0.60$. On differentiating (2) it appears that $u$

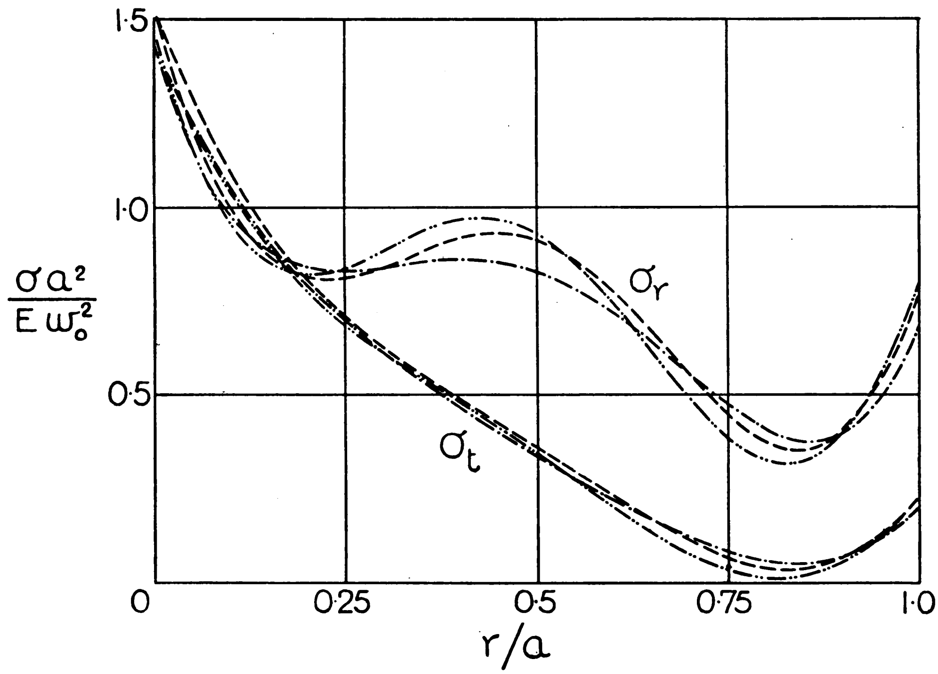


FIG. 2. Radial and tangential tensile stresses in the diaphragm.
— — — — sinusoidal displacement.
—·—·— cubic displacement.
—··—··— Bessel-function displacement.

has a maximum value $0.12w_0^2/a$ at $r/a = 0.24$ and a minimum value $-0.06w_0^2/a$ at $r/a = 0.82$. Hence the greatest radial difference between the shape of the chamber and $w$ as given by (1) is $0.12\ w_0^2/a$, which for $w_0/a=1/35$ is only $0.98\times10^{-4}a$. For a chamber of normal diameter this difference is too small to be appreciable in manufacture.

It is of interest to determine the strain undergone by a radius of the diaphragm, and for this purpose an accurate method of rectification is available. For a chamber of sinusoidal shape a radius is extended to a length $s$ given by

$$s = 2\int_0^{a/2}\left(1+\frac{\pi^2w_0^2}{4a^2}\cos^2\frac{\pi r}{a}\right)^{1/2}dr$$
$$= \frac{2a}{\pi}(1-p^2)^{-1/2}\int_0^{\pi/2}\left(1-p^2\sin^2\frac{\pi r}{a}\right)^{1/2}d\left(\frac{\pi r}{a}\right), \tag{11}$$

where $$p^2 = \frac{\pi^2w_0^2}{4a^2}\Big/\left(1+\frac{\pi^2w_0^2}{4a^2}\right).$$

Since $p$ is small, the first bracket in (11) may be expanded by the binomial theorem and the complete elliptic integral replaced by

$$E = \frac{\pi}{2}\left(1-\frac{p^2}{4}-\frac{3p^4}{64}-\cdots\right).$$

The strain of the radius then reduces to

$$\frac{s-a}{a} = \frac{p^2}{4}+\frac{13p^4}{64}+\cdots. \tag{12}$$

For $w_0/a=1/35$ this strain amounts to 0.05%, hence in a steel wire distorted into this sinusoidal form the tensile stress would be only $0.0005\times13000=6.5$ tons/sq.in.

**3. Stresses when the transverse displacement follows a cubic relation.** To estimate how far the stresses depend on the expression assumed for $w$, we will in this section replace (1) by the cubic

$$w = w_0\left(1-\frac{3r^2}{a^2}+\frac{2r^3}{a^3}\right). \tag{13}$$

This equation satisfies the same four boundary conditions as (1), and the greatest difference between the two is approximately $0.010w_0$ at $r/a=0.28$ and 0.72. After employing (2), (3), (4) and (6) we find that

$$V_1 = \frac{\pi Eh}{1-\nu^2}\left[\frac{1}{4}C_1^2a^4+\frac{3}{10}C_1C_2a^5+\frac{7}{60}C_2^2a^6+\frac{9}{35}\frac{w_0^4}{a^2}\right.$$
$$\left.+\frac{3}{70}w_0^2a\left\{2C_1(3\nu-1)+C_2a(3\nu+1)\right\}\right], \tag{14}$$

$$C_1 = \frac{3w_0^2}{56a^3}(23-15\nu),\qquad C_2 = -\frac{9w_0^2}{56a^4}(11-3\nu). \tag{15}$$

For $\nu=0.3$, (15) reduces to

$$C_1 = 0.99\frac{w_0^2}{a^3}, \qquad C_2 = -1.62\frac{w_0^2}{a^4}, \tag{16}$$

and the stresses obtained from (3) and (9) are

$$\left.\begin{aligned}\sigma_r &= \frac{Ew_0^2}{a^2}\left\{1.42 - 6.61\frac{r}{a} + 25.67\frac{r^2}{a^2} - 39.56\frac{r^3}{a^3} + 19.78\frac{r^4}{a^4}\right\},\\ \sigma_t &= \frac{Ew_0^2}{a^2}\left\{1.42 - 4.60\frac{r}{a} + 9.32\frac{r^2}{a^2} - 11.87\frac{r^3}{a^3} + 5.93\frac{r^4}{a^4}\right\}.\end{aligned}\right\} \tag{17}$$

From Fig. 2, in which these expressions also are shown, it will be seen that the maximum stress is slightly smaller than that obtained in section 2.

**4. Stresses when the transverse displacement follows a Bessel-function relation.** Lastly we will take $w$ as given by

$$w = W_0\{J_0(kr) - m\}, \tag{18}$$

where $k=\alpha/a$, $\alpha=3.83\cdots$ being the first positive root of $J_1(x)=0$,

$$m = J_0(\alpha) = -0.402\cdots,$$

and

$$W_0 = w_0/\{J_0(0) - J_0(\alpha)\} = w_0/1.402\cdots.$$

This equation satisfies the four boundary conditions, and it gives a displacement which, unlike those previously considered, is unsymmetrical about the line $w=w_0/2$. Except at $r=0$ and $r=a$ the displacement is everywhere less than that specified by (1), the greatest difference between the two being approximately $0.019\,w_0$ at $r/a=0.53$. The same procedure as before leads to

$$\begin{aligned}V_1 = \frac{\pi Eh}{1-\nu^2}\bigg[&\frac{1}{4}C_1^2a^4 + \frac{3}{10}C_1C_2a^5 + \frac{7}{60}C_2^2a^6 + \frac{W_0^4k^4}{4}\int_0^a rJ_1^4(kr)dr\\ &+ W_0^2\bigg\{\frac{k^2\nu a^3J_0^2(ka)}{12}(C_2a + 3C_1)\\ &+ \frac{3}{8}(2+\nu)(C_2a - C_1)\left(\int_0^a J_0^2(kr)dr - aJ_0^2(ka)\right)\bigg\}\bigg],\end{aligned} \tag{19}$$

$$\left.\begin{aligned}C_1 &= \frac{5W_0^2}{4a^4}\left[6(2+\nu)\left\{\int_0^a J_0^2(kr)dr - aJ_0^2(ka)\right\} - k^2\nu a^3J_0^2(ka)\right],\\ C_2 &= -\frac{5W_0^2}{4a^5}\left[9(2+\nu)\left\{\int_0^a J_0^2(kr)dr - aJ_0^2(ka)\right\} - k^2\nu a^3J_0^2(ka)\right].\end{aligned}\right\} \tag{20}$$

If we take $\nu=0.3$ and $\int_0^\alpha J_0^2(x)dx=1.2599$,[3] (20) reduces to

[3] After some fruitless attempts to evaluate this integral, I asked Professor G. N. Watson whether it was expressible in any simple form; his reply was that he thought not, and he computed its value to 15 places of decimals, his result being 1.25990 97359 05768. The value 1.2599 is sufficiently accurate for our present purpose.

$$C_1 = 1.01\frac{w_0^2}{a^3}, \quad C_2 = -1.74\frac{w_0^2}{a^4} \tag{21}$$

and the stresses are

$$\left.\begin{aligned} \sigma_r &= \frac{Ew_0^2}{a^2}\left\{1.44 - 6.95\frac{r}{a} + 6.31\frac{r^2}{a^2} + 4.10J_1^2(kr)\right\}, \\ \sigma_t &= \frac{Ew_0^2}{a^2}\left\{1.44 - 4.84\frac{r}{a} + 3.63\frac{r^2}{a^2} + 1.23J_1^2(kr)\right\}. \end{aligned}\right\} \tag{22}$$

These stress distributions, which are plotted in Fig. 2, are in close accord with the results obtained in sections 2 and 3.

**5. Conclusions.** The following conclusions emerge from the above calculations:—

(*i*) For the three kinds of displacement considered, the maximum stress in the diaphragm is at the centre and is about 1.5 $Ew_0^2/a^2$.

(*ii*) The stress distributions due to the three kinds of displacement do not differ widely. Hence, if it is decided to use one kind, and small errors are made in the difficult process of machining the chambers, no great alteration in the stresses will result.