SOME PRESENT NONLINEAR PROBLEMS OF THE ELECTRICAL AND AERONAUTICAL INDUSTRIES[1]

BY

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1. Introduction. The accelerated growth of research in the field of nonlinearity is due to different causes. The general advancement of science requires increasingly more precise expressions for the laws of science. Accurate nonlinear equations frequently depart from the linearized or postulated linear equations which have been previously used for approximate results. The quest for perfection and generalization and the love of difficult investigations by professional mathematicians play a large part in this growth. Another incentive is the increasingly exacting requirements of modern manufacturing. These requirements are born of the competitive necessity of producing ever improved machines and equipment in the most economical manner. The greatest incentive is necessity. Manufactured equipment and devices must be designed to work.

A nonlinear problem[2] has been defined as "one which, when formulated mathematically, reduces to (one or) a system of differential, integral, or integro-differential equations such that at least one of the three quantities, a derivative, an integral, or a dependent variable, is involved transcendentally or algebraically to a power other than the first in at least one equation of the system or in at least one boundary condition of the system." Of course, in dealing with applied problems, a physical definition independent of all mathematical concepts is preferable, but such is difficult to formulate.

Nonlinear problems resolve themselves into two general types, continuous and discrete. The first type deals with the behavior of quantities in a field or in at least one continuous region of space and, more often than not, reduces mathematically to systems of nonlinear partial differential equations. Problems relating primarily to this field have been treated by Dr. Theodore von Kármán in his Josiah Willard Gibbs Memorial lecture.[3] This paper is both a milestone and a beacon of progress in that it is an admirable exposition and inventory of the nonlinear problems of continuous fields and at the same time an inspiration and invitation to both the engineer and mathematician for further advancement in this difficult field. Among other subjects, the von Kármán lecture treated relaxation oscillations, subharmonic resonance, nonlinear problems in the theory of elasticity in which the hypotheses of (a) small deflections are abandoned, (b) Hooke's law no longer holds, plasticity, hydrodynam-

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[1] A Symposium Address before the American Mathematical Society at Stanford University, April 24, 1943. Manuscript received Aug. 16, 1943.

[2] E. G. Keller, *Analytical methods of solving discrete nonlinear problems in electrical engineering*, Transactions of the American Institute of Electrical Engineers, **60**, 1194 (1941).

[3] Theodore von Kármán, *The engineer grapples with nonlinear problems*, Bull. Am. Math. Soc., **46**, 615 (1940).

ics; and aerodynamics of (a) ideal fluids, (b) viscous fluids, and (c) compressible fluids. The bibliography of the paper contains 178 entries.

The second type of nonlinear problem is called discrete. Discrete nonlinear problems are characterized by the fact that they possess only a finite number of degrees of freedom. They are frequently reducible mathematically to systems of nonlinear ordinary (total) differential equations or to systems of nonlinear integral equations.

2. **Nature of industrial discrete nonlinear problems.** Solutions of nonlinear problems in industry are usually in the "small"; i.e., a solution of a system is not required for every magnitude whatsoever of the parameters involved. In the solution of such problems the greatest single body of theory contributing to nonlinear analysis of discrete systems is that which grew out of the attempts of the great French and German mathematicians of the last century to solve the problem of three bodies. While their objective, in its complete generality, was never realized, the pure mathematics developed (theory of differential equations, convergence, dominant functions, singularities, removable singularities, etc.) is today directly applicable in the study of nonlinear equations of electrical circuits, rotating electrical machines, and various nonlinear dynamical and aerodynamical devices. The two second largest bodies of theory are those of nonlinear integral equations as developed by T. Lalesco[4] and others[5] and the methods of Galerkin[6] and Ritz along with the modifications of these techniques.[7]

There are at least three salient characteristics of nonlinear engineering problems which distinguish them from purely theoretical problems. First, oscillograms, differential analyzer solutions,[8] or other records frequently indicate the nature of the solution of the mathematical systems in question. Such mechanical or electrical solutions for the same system often differ so much among themselves that there is the risk of concluding erroneously that the solution is not unique. (For example, the differential equations which yield the two solutions represented in Figs. 3 and 4 also possess sinusoidal solutions. Yet the solutions are unique; i.e., the solution in (4) is identical with the sinusoidal solution.) Of course, there are systems which do not possess a unique solution. In general, even when a solution is unique it may have so many manifestations that it is often necessary to integrate the system to determine the effect of the various parameters involved. A second characteristic of industrial nonlinear problems is that frequently the methods of mathematics are not powerful enough to yield a complete solution of the problem in sufficiently simple form to be usable. Tricks and devices, born of physical concepts, must *guide* the mathematics if a usable solution is to be attained. The mathematics is surely necessary and it is just as surely not sufficient. The solution is mathematics plus. A third distinction of industrial nonlinear problems is the fact that the derivation of the equations of performance requires, in addition to a knowledge of mathematics, mathematical physics,

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[4] V. Volterra, *Leçons sur les équations intégrales*, Gauthier-Villars, Paris, 1913, p. 90.

[5] H. Galajikian, Bull. Amer. Math. Soc., **19**, 342 (1913); also Ann. of Math., **16**, 172 (1915); E. Schmidt, Math. Ann., **65**, 370 (1908).

[6] A. N. Dinnik, *Galerkin's method for determining the critical strengths and frequencies of vibrations*, Aeronautical Engineering, U.S.S.R., **9**, No. 5, 99 (1935). Also W. J. Duncan, *Galerkin's method in mechanics and differential equations*, R&M 1798 (1938).

[7] For additional bibliographies see references 2 and 3 above, also the book, E. G. Keller, *Mathematics of modern engineering*, vol. II, Wiley and Sons, New York, 1942, pp. 303–304. These list a total of 302 papers; and these papers in turn possess bibliographies.

[8] V. Bush and H. L. Hazen, *Integraph of differential equations*, J. Franklin Inst., **204**, 575 (1927).

and engineering, inventive ability in thought. A system of equations may be an invention of the highest order. It is not always necessary to integrate a system of nonlinear equations. Often it is necessary only to determine under what conditions the physical system is stable. Of course, no single stability criterion exists for nonlinear systems such as exists for linear systems. When a solution of a nonlinear system cannot be obtained with sufficient rapidity or when it can be obtained *but is worthless because the time consumed in applying it is too great,* it may be possible to obtain the information desired by integrating a dominant and a "subordinate" system such that the solution of the actual problem is bounded or limited by the solutions of the dominant and subordinate system. The use of dominant and "subordinate" systems will be clear in the following problems.

3. **Some representative discrete nonlinear problems of industry.** In this paper a number of representative nonlinear systems are treated which illustrate the principles enumerated in the last section. These systems are either original, appearing here for the first time, or else of very recent date. Some of them pertain to electrical manufacturing, others to aircraft development. Although, as stated above, the derivation of the equations of a system is often more important than the solution, none of the equations considered are derived here. Some systems are derived in the literature and to these references are given. The derivations of the remaining ones can not be given for military reasons. These are viewed here merely as hypothetical nonlinear systems.

1. *Nonlinear control circuits.* As is well known, the volt-ampere characteristic of a nonlinear series circuit (Fig. 1) is represented by the curve in Fig. 2. Such circuits have numerous industrial applications due to their rugged mechanical simplicity and at the same time their electrical sensitivity. The characteristic in Fig. 2 displays the fact that there exists a so-called critical or resonant voltage E_0 at which the R.M.S. value of the current suddenly increases many fold. For a value of $E<E_0$ (see $E \sin wt$ in Fig. 1) the current is sinusoidal. For $E>E_0$ the current has the wave form displayed in Fig. 3.

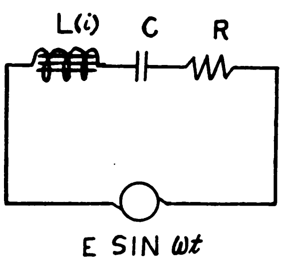

FIG. 1.

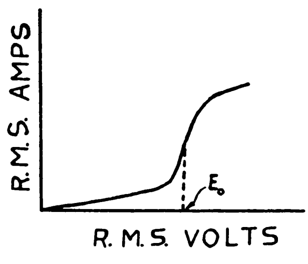

FIG. 2.

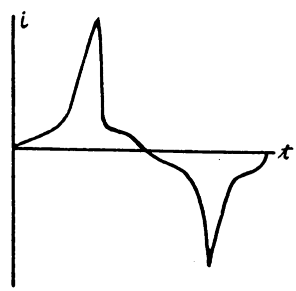

FIG. 3.

In industrial applications E_0 is prescribed. It is required to design a circuit which will be sensitive for this prescribed value of E_0. A simple slide rule formula is desired which will express E_0 as a function of the circuit parameters and of the nonlinear reactor employed. The equation of performance for the circuit in Fig. 1 is

$$L(i)\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = -E\cos\omega(t - t_0). \tag{1}$$

For the range of interest the current i is such that the saturation curve of the reactor is single-valued and represented by the equation

$$H = ki = x - a_3x^3 + a_5x^5. \tag{2}$$

With $\theta = \omega t$, Eqs. (1) and (2) yield

$$M\frac{dx}{d\theta} + R(x - a_3x^3 + a_5x^5) + x_c\int(x - a_3x^3 + a_5x^5)d\theta = -Ek\cos(\theta - \theta_0), \tag{3}$$

where, for a given ω, M and x_c are constants. The integration of the nonlinear Eq. (3) and the development of E_0 as a function of the parameters of the physical problem are carried out elsewhere and need not be repeated here.[9]

2. *Nonlinear transmission line phenomena.* If a series capacitor is employed in the primary side of a transmission line to improve the power factor, curious wave forms of voltage and current ensue. The system becomes unstable as far as possessing a periodic solution is concerned. This is to be expected since the maximum flux density attains a value close to that of the knee of the saturation curve if the transformers

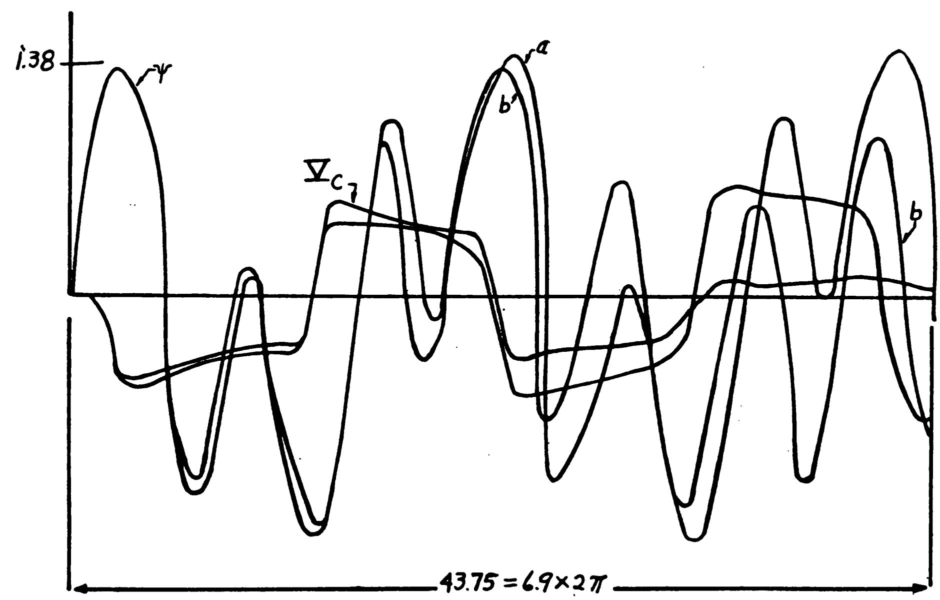

FIG. 4.

are operating efficiently. The addition of series capacitance is thus likely to create an unstable system. In this unstable system, the current and voltage taken on an indefinitely large number of wave forms such as shown in Fig. 4. Synchronous motors which require sinusoidal applied voltages cannot operate on currents and voltages of the type shown in Fig. 4.

If the capacitor of the system is shunted by a resistor as indicated in Fig. 5, then the equations of performance are

[9] E. G. Keller, *Resonance theory of non-linear control circuits*, J. Franklin Inst., 225, 561 (1938)

$$\frac{d\psi}{d\theta} + R_1 i_1 + L\frac{di_1}{dt} + \frac{1}{C}\int (i_1 - i_2)dt = E\sin(\theta + \theta_0),$$

$$i_2 R_2 = \bar{x}_c \int (i_1 - i_2)d\theta, \quad (\bar{x}_c = 1/C), \tag{4}$$

$$\alpha i_1 = x - a_3x^3 + a_5x^5;\ H = \alpha i;\ x = B/B_0;\ B_0 = dB/dH \text{ at } H = 0,$$

or

$$\alpha B_0 \frac{d^2x}{d\theta^2} + \left[\frac{\bar{x}_c B_0 \alpha}{R_2} + R_1(1 - 3a_3x^2 + 5a_5x^4)\right]\frac{dx}{d\theta} + \frac{\bar{x}_c(R_1 + R_2)}{R_2}(x - a_3x^3 + a_5x^5)$$
$$= \frac{\alpha}{R_2}[R_2^2 + \bar{x}_c^2]^{1/2}E\cos\left(\theta + \theta_0 - \tan^{-1}\frac{\bar{x}_c}{R_2}\right).$$

Now R_2 must have the smallest possible value consistent with stability, since it represents a perpetual loss of power. There are ten parameters and two variables.

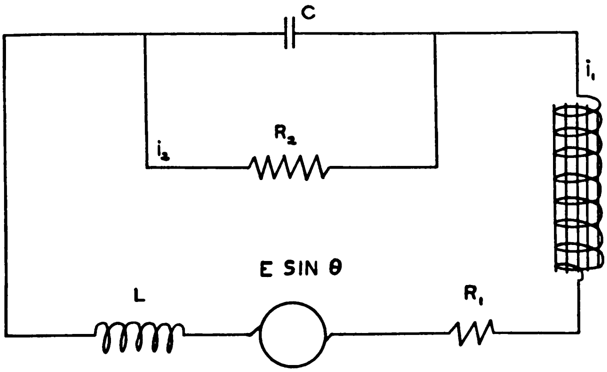

FIG. 5.

There are infinitely many values of the parameters for which the system is unstable and equally as many for which it is stable, i.e., for which the solutions are sinusoidal. A convenient slide rule formula is desired giving the above smallest value of R_2 as functions of the other nine parameters of the system. The equations of the system are derived and solved elsewhere.[10]

3. *Nonlinear springs.* It is sufficient to say that, in general, the differential equations involving nonlinear springs are integrable by hyperelliptic functions.[11] If damping is large a combination of variation of parameters and hyperelliptic functions will usually afford sufficient accuracy.

4. *Electric locomotive oscillations.* Experience classifies the five oscillatory motions of an electric locomotive as pitch, roll, plunge, nose, and rear-end lash. The last two are especially important because their pronounced existence in a locomotive produces a tendency to derail. Considered superficially, characteristic oscillations of an electric locomotive would seem to be similar to those of an ordinary vehicle such as an automobile, but experimental data and observation indicate the existence of dangerous

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[10] E. G. Keller, *Beat theory of non-linear circuits*, J. Franklin Inst., **228**, No. 3, 319–337 (1939).

[11] A theory of hyperelliptic functions in usable form is given by F. R. Moulton, *On certain expansions of elliptic, hyperelliptic and related functions*, Am. J. Math., **34**, 177–202 (1912).

nose and rear-end lash which are not common to an automobile. If the tendency to nose exists in an electric locomotive and if the locomotive noses for a speed V_0, then it will nose for all speeds greater than V_0. Consequently, nosing is not a resonance phenomenon and cannot be avoided by running at a slightly different speed. It might be supposed that nosing is due to the coning of the wheels or to the staggering of the rails or to a combination of these two possible causes. Such causes, however, would produce resonance frequencies for definite discrete values of V instead of instability for all values of V exceeding V_0. Rails on European railroads are not staggered and yet electric locomotive nosing still persists. The tendency to nose and the violence of the oscillation increase with the *weight* and *power* of the locomotive.

In seeking the source of the phenomenon, consider first an elementary experiment. Let a miniature set of driving wheels and axle be constructed from two rubber paste bottle stoppers and a lead pencil. If the miniature drivers are forced down against two rulers as rails, if a torque is applied tending to rotate the wheels, and if further in the forward motion slight lateral motion is permissible then an oscillating torque will be experienced tending to rotate the axle about a line through the center of axle and perpendicular to the plane of the track. The creepage forces between the rubber wheels and the rails produce an oscillatory torque.

The weight of an electric locomotive is so great that it effectively rolls on elastic wheels on elastic rails. Making use of this fact and whatever additional facts are necessary the equations of motion[12] can be shown to be

$$
\begin{aligned}
&M\ddot{x}_0 = 0,\\
&M\ddot{y}_0 = -F_2 - f_2 - 2f\left(\frac{\dot{y}_2}{V} - \zeta\right) - 2f\left(\frac{\dot{\bar{y}}}{V} - \zeta\right) - F_1 - f_1 - 2f\left(\frac{\dot{y}_1}{V} - \zeta\right),\\
&M\ddot{z}_0 + \lambda_1(z_0 - b_1\eta) + \lambda_2(z_0 - c\xi + b_2\eta) + \lambda_2(z_0 - c\xi + b_2\eta) + k_1^2\dot{z}_0 = 0,\\
&A\ddot{\xi} + \lambda_2 c(z_0 + c\xi + b_2\eta) - \lambda_2 c(z_0 - c\xi + b_3\eta) + k_2^2\dot{\xi}\\
&\qquad = -b_5(F_1 + F_2 + f_1 + f_2) - 2b_5\frac{f}{V}(\dot{y}_1 + \dot{\bar{y}} + \dot{y}_2) + 6b_5 f\zeta,\\
&B\ddot{\eta} - \lambda_1 b_1(z_0 - b_1\eta) + \lambda_2 b_2(z_0 + c\xi + b_2\eta) + \lambda_2 b_2(z_0 - c\xi + b_2\eta) + k_3^2\dot{\eta} = 0,\\
&C\ddot{\zeta} = -d_3(F_1 - F_2) - d_4(f_1 - f_2) - \frac{2fd_3}{V}(\dot{y}_1 - \dot{y}_2) - \frac{6fb^2}{V}\dot{\zeta}\\
&\qquad\qquad = \frac{2f\lambda b}{r}(\bar{y} + y_1 + y_2) + F_1(\bar{y}_1).
\end{aligned}
\tag{5}
$$

To the accuracy required, the flange forces are given by

$$
\begin{aligned}
F_1 &= H\left(\frac{y_1}{\delta_1}\right)^3 + I\left(\frac{y_1}{\delta_1}\right)^5 + J\left(\frac{y_1}{\delta_1}\right)^7 + \cdots,\\
F_2 &= H\left(\frac{y_2}{\delta_1}\right)^3 + I\left(\frac{y_2}{\delta_1}\right)^5 + J\left(\frac{y_2}{\delta_1}\right)^7 + \cdots,
\end{aligned}
\tag{6'}
$$

[12] E. G. Keller, *Mathematics of modern engineering*, vol. II, p. 72.

$$
\begin{aligned}
f_1 &= h\left(\frac{y_3}{\delta_2}\right)^3 + i\left(\frac{y_3}{\delta_2}\right)^5 + j\left(\frac{y_3}{\delta_2}\right)^7 + \cdots, \\
f_2 &= h\left(\frac{y_4}{\delta_2}\right)^3 + i\left(\frac{y_4}{\delta_2}\right)^5 + j\left(\frac{y_4}{\delta_2}\right)^7 + \cdots,
\end{aligned}
\tag{6''}
$$

where the constants H, I, J, j, i, h are determined from force curves, and δ_1 and δ_2 are lengths shown in Fig. 6.

The variables y_1, y_2, y_3, y_4, and $\bar{y}$ are eliminated from (5) by means of the relations

$$
\begin{aligned}
y_1 &= y_0 + b_5\xi + d_3\zeta, \qquad & y_2 &= y_0 + b_5\xi - d_3\zeta, \\
y_3 &= y_0 + h_1\xi + d_4\zeta, \qquad & y_4 &= y_0 + h_1\xi - d_4\zeta,
\end{aligned}
$$
$$
\bar{y} = y_0 + b_5\xi,
$$

where b_5, d_3, d_4, h_1 and h_2 are lengths defined in reference 12.

If in (5) $F_1 = F_2 = f_1 = f_2 = 0$, then the equations are linear; and the solution can be written down at once. This solution is either stable or unstable as indicated by the roots of the characteristic equation. The nature of the roots are, of course, a function of V, the operating speed of the locomotive. Even if the locomotive is unstable with vanishing flange forces, it is stable with non-vanishing flange forces. In this case the locomotive is operating roughly and damaging the track needlessly.

In practical applications, then, it is not necessary to integrate the nonlinear system (5). As a check on the validity of the theory, however, it is necessary to integrate the nonlinear system and compare the predicted motion with actual motion as determined by runs on a test track. Evidently the solution of (5) for $F_1 = F_2 = f_1 = f_2 = 0$ cannot be used as a generating solution for the case of the non-vanishing of the flange forces because the stability or unstability of this generating solution is carried over into the complete solution.

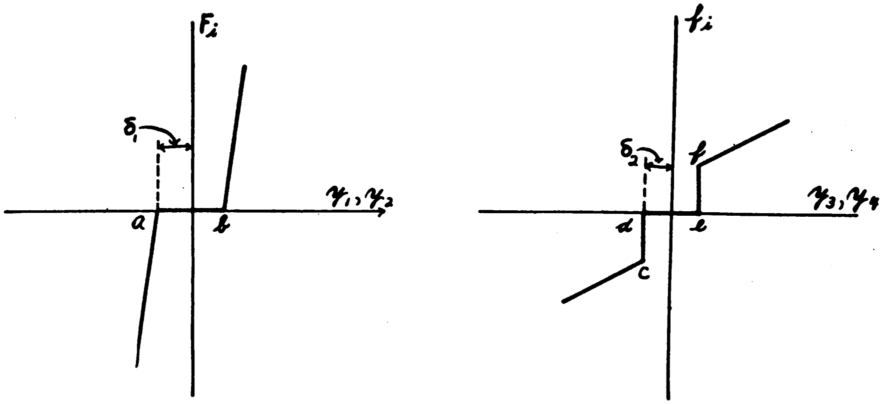

FIG. 6.

Since nosing and rear end lash are the two motions of most importance, it is sufficient to resort to elementary means. Consequently, F_i and f_i [Eqs. (6)] are replaced by segments of straight lines as shown in Fig. 6, and the second and sixth equations of (5) are solved for y_0 and ζ by operational methods (general operational methods where both initial charges and initial currents exist must be used.) Since the flange forces are taken as functions with discontinuous slopes, the system of differential equations and its characteristic equation change as the flange forces, as functions of y_1, y_2, y_3, y_4, change at points a, b, c, d, e, f shown in Fig. 6. The first set of initial conditions are chosen by trial and error in such a way that the resulting motion is periodic in y_0 and ζ. The solution for a complete cycle is sufficient. The check of the theory is the approximate agreement of computed and test periods.

5. *Nonlinear differential equations of dynamic braking of a synchronous machine.* The equations of dynamic braking are

$$\begin{aligned}\frac{dI}{dt} &= \frac{(E - IR)}{L}\frac{[(rs_0/s)^2 + x_dx_q]}{[(rs_0/s)^2 + x_{d'}x_q]} - \frac{\mu_0}{s^4}\frac{I^3[x_q^2 + (rs_0/s)^2]}{[(rs_0/s)^2 + x_{d'}x_q][(rs_0/s)^2 + x_dx_q]^3},\\ \frac{ds}{dt} &= -\frac{\mu_1 I^2}{s}\frac{x_q^2 + (rs_0/s)^2}{[x_dx_q + (rs_0/s)^2]^2},\end{aligned} \tag{7}$$

where

$$\mu_0 = \frac{2KPr^3x_q(x_d - x_{d'})s_0^2}{JI_0^2}, \quad \mu_1 = \frac{KPr}{JI_0^2},$$

I being the field current, s the rotor speed, t the time in seconds, all other symbols being constant parameters. It is desired to obtain an expression for the time of stopping of the rotor as a function of the parameters of the machine.

The last term in the right member of the first of Eqs. (7) has in all cases a magnitude of approximately ten per cent of its predecessor. Thus a solution as a power series in a parameter which vanishes with μ_0 is to be expected. Neglecting the term containing μ_0 in (7) and dividing the first equation by the second, we obtain a solution of the resulting equation immediately. However, this solution is an implicit function of I and s and such that it is solvable explicitly for either I or s only as a slowly convergent series. An attempted solution by the method of variation of parameters is equally cumbersome.

It is known from oscillograms, however, that both I and s are monotone decreasing functions of the time for the interval within which (7) is valid. Change of dependent variables by

$$I = \frac{E}{R} + I_1e^{-y} \quad \text{and} \quad s = s_0e^{-z}$$

yields

$$\begin{aligned}\frac{dy}{dt} &= \frac{R}{L} + \frac{R(A^2 - A_0^2)}{L[A_0^2 + (re^z)^2]} + \mu_0\frac{[x_q^2 + (re^z)^2][E/R + I_1e^{-y}]^3}{[A_0^2 + (re^z)^2\ [A^2 + (re^z)^2]^3}\frac{e^{y+4z}}{I_1s_0^4},\\ \frac{dz}{dt} &= \mu_1\frac{[(E/R) + I_1e^{-y}]^2[x_q^2 + (re^z)^2]}{[A^2 + (re^z)^2]^2s_0^2}e^{2z},\end{aligned} \tag{8}$$

where

$$A_0^2 = x_{d'}x_q, \qquad A^2 = x_dx_q, \qquad A > A_0.$$

The number of revolutions before the rotor of the machine comes to rest is

$$N = \frac{1}{2\pi}\int_0^\infty s\,dt = \frac{1}{2\pi}\int_0^\infty s_0e^{-z}dt. \tag{9}$$

Now it is sufficient for practical purposes to set an upper limit to N as given by (9) provided the upper limit is sufficiently small and provided the results display the effect of each parameter of the system. To accomplish this (8) may be replaced by a simpler system of equations. Evidently,

$$[A_0^2 + (re^z)^2] \leqq (A_0^2 + r^2)e^{2z}, \qquad [A^2 + (re^z)^2] \leqq (A^2 + r^2)e^{2z} \tag{10}$$

for $z \geqq 0$. Employing (10) in (7) and integrating, we have

$$\begin{aligned} y &= \frac{R}{L}t + \frac{R(A^2 - A_0^2)}{L(A_0^2 + r^2)}\int_0^t e^{-2z}dt \\ &\quad + \frac{1}{s_0^4 I_1}\frac{(r^2 + x_q^2)\mu_0}{(A_0^2 + r^2)(A^2 + r^2)^3}\int_0^t \left[\frac{E}{R} + I_1 e^{-y}\right]^3 e^{y-4z}dt, \\ z &= \frac{\mu_1(r^2 + x_q^2)}{s_0^2(A^2 + r^2)^2}\int_0^t \left[\frac{E}{R} + I_1 e^{-y}\right]^2 e^{-2z}dt, \end{aligned} \tag{11}$$

where the instantaneous values of y and z as given by (11) are always less than those given by the solution of (7) for $0 < t < \infty$.

The system (11) is of the form

$$u_k(t) = \phi_k(t) + \int_0^t K_k[t, \xi, u_1(\xi), \cdots, u_n(\xi)]d\xi \quad (k = 1, 2, \cdots, n)$$

which is Lalesco's system of nonlinear integral equations. The solution of this is the limit of the sequences

$$\begin{gathered} u_k^{(0)} = \phi_k(t), \\ u_k^{(1)} = \phi_k(t) + \int_0^t K_k[t, \xi, \phi_1(\xi), \cdots, \phi_n(\xi)]d\xi \quad (k = 1, 2, \cdots, n), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{gathered}$$

In the present application $\phi_1(t) = Rt/L$ and $\phi_2(t) = 0$. For small synchronous machines the second approximations $u_1^{(1)}$ and $u_2^{(2)}$ give values of y and z such that N in (9) is in error by five per cent. The integration in (9) is carried out numerically. Because of bearing friction and other decelerating factors not included in (7) the upper limit in (9) is finite.

6. *A double-valued nonlinear problem.* Consider the integration of the equation

$$I\ddot{\theta} + \beta\dot{\theta} + \mu[k_1\theta + k_2 \tan^{-1} k_3(\theta \pm a)] = 0. \tag{12}$$

This equation was derived ingeniously by W. W. Beman to express an important phenomenon in aerodynamics. The quantities I, β, μ, k_1, k_2, k_3, and a are all positive numbers and the plus or minus sign in $(\theta \pm a)$ is used according as $\dot{\theta} < 0$ or $\dot{\theta} > 0$.

Evidently, for a particular amplitude of θ, Eq. (12) possesses a periodic solution. The period and amplitude of this solution are desired. Eq. (12) in the normal form is

$$\dot{\theta} = \theta_1, \qquad \dot{\theta}_1 = -(\mu/I)[k_1\theta + k_2 \tan^{-1} k_3(\theta \pm a)] - \beta\theta_1/I. \tag{13}$$

An integral of (13) for $\beta = 0$ is

$$\theta_1^2 = c - \frac{2\mu}{I}\left\{\frac{k_1\theta^2}{2} + k_2\left[(\theta \pm a)\tan^{-1} k_3(\theta \pm a) - \frac{1}{2k_3}\log\left(1 + k_3^2(\theta_1 \pm a)^2\right)\right]\right\}$$

or

$$\theta_1 \equiv \pm \sqrt{c - f(\theta)}, \tag{14}$$

where $c = f(\theta_0)$ and where θ_0 is the maximum positive displacement for $t = t_0$. If (14) is used as an equation of change of variable, the method of variation of parameters yields

$$\frac{\partial \theta_1}{\partial c}\frac{\partial c}{\partial t} + \frac{\partial \theta_1}{\partial t} = -\frac{\mu}{I}\left[k_1\theta + k_2 \tan^{-1} k_3(\theta \pm a)\right] - \frac{\beta}{I}\theta_1;$$

whence

$$\dot{c} = -2\beta\theta_1^2/I, \tag{15}$$

or

$$c = -\frac{2\beta}{I}\int_{t_0}^{t} \theta_1^2 dt + d = -\frac{2\beta}{I}\int_{\theta_0}^{\theta} \dot{\theta}\, d\theta + d,$$

where d is an arbitrary constant. From the last equation

$$\frac{dc}{d\theta} = -\frac{2\beta}{I}\theta_1 = -\frac{2\beta}{I}(\pm\sqrt{c - f(\theta)}). \tag{16}$$

To determine the signs in (16) it is evident from (15) that c is a decreasing function of the time. Consequently, for $\dot{\theta} < 0$

$$\frac{dc}{d\theta} = \frac{dc}{dt}\frac{dt}{d\theta} = \frac{dc}{dt}(-\sqrt{c - f(\theta)}).$$

Thus Eq. (16) is

$$\frac{dc}{d\theta} = \pm\frac{2\beta}{I}\sqrt{c - f(\theta)} \tag{17}$$

according as $\dot{\theta}_1 < 0$ or $\dot{\theta}_1 > 0$.

For the integration of (17) it is sufficient to replace $\sqrt{c - f(\theta)}$ by $k[c - f(\theta)]$ where k is determined graphically by

$$\int \sqrt{c - f(\theta)}\, d\theta = k\int [c - f(\theta)]d\theta,$$

$c = f(\theta_0)$ or $c = f(\theta_0')$ according as $\theta_0 \leqq \theta \leqq \theta_0'$ or $\theta_0' \leqq \theta \leqq \theta_0''$, and θ_0, θ_0', and θ_0'' are shown in Fig. 7. The curve in Fig. 7 is the solution (14). With this replacement and simple integration

$$\begin{aligned} c &= e^{2\beta k(\theta - \theta_0)/I}\left[c_0 - \frac{2\beta k}{I}\int_{\theta_0}^{\theta} e^{-2\beta k\theta/I} f(\theta) d\theta\right] \qquad (\theta_0 \leqq \theta \leqq \theta_0'), \\ c &= e^{-2\beta k(\theta - \theta_0')/I}\left[c_0' - \frac{2\beta k}{I}\int_{\theta_0'}^{\theta} e^{2\beta k\theta/I} f(\theta) d\theta\right] \qquad (\theta_0' \leqq \theta \leqq \theta_0''), \end{aligned} \tag{18}$$

where $c_0 = f(\theta_0)$ and $c_0' = f(\theta_0')$. The above values of c are substituted in Eq. (14). The solution is periodic when θ_0 is chosen such that θ_0'' turns out to be equal to θ_0. The period of the motion is then given by

$$P = 2\int_{\theta_0}^{\theta_0'} \frac{d\theta}{\theta_1}. \tag{19}$$

The numerical integration of (19) presents no difficulty at the limits θ_0 and θ_0' since θ_1 in the vicinity of θ_0 and θ_0' can be replaced by an integrable function f_0 such that the limit of $(f_0/\theta_1)=1$ at $\theta_1=\theta_0$ and $\theta_1=\theta_0'$.

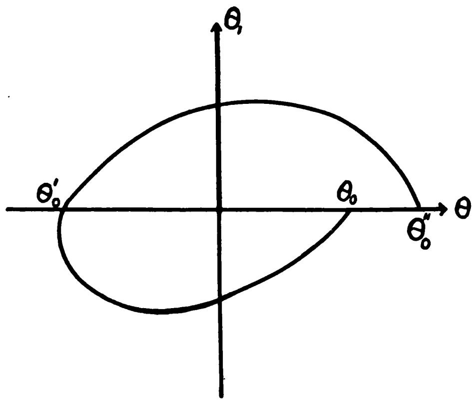

FIG. 7.

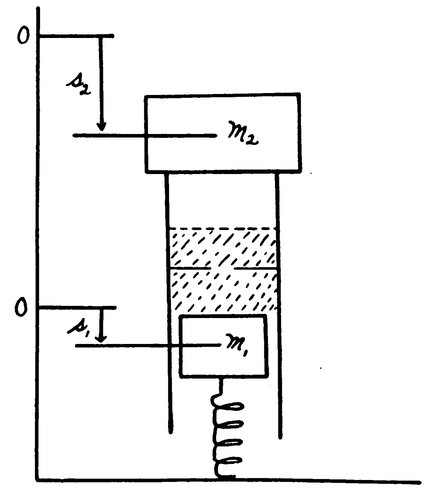

FIG. 8.

7. *A nonlinear problem of two oleo-pneumatically coupled masses one of which is subject to impact.* It can be shown without difficulty that the nonlinear differential equations of motion of m_1 and m_2 shown in Fig. 8 are

$$\begin{aligned} &m_1\ddot{s}_1 - w_1 - \frac{p_0S}{\left[1-\dfrac{(s_2-s_1)}{D}\right]^{1.2}} - \frac{\rho(S-s_m)^3(\dot{s}_2-\dot{s}_1)^2}{2gc^2[A(r)]^2} + k_2\dot{s}_1 + f(s_1) = 0,\\ &m_2\ddot{s}_2 - nw_2 + \frac{p_0S}{\left[1-\dfrac{(s_2-s_1)}{D}\right]^{1.2}} + \frac{\rho(S-s_m)^3(\dot{s}_2-\dot{s}_1)^2}{2gc^2[A(r)]^2} = 0, \end{aligned} \tag{20}$$

where

$$[A(r)]^2 = \pi^2\{R^2 - [r_i + b_i(s_2-s_1)]^2\}, \qquad f(s_1) = (K_0 + k_0s_1), \tag{21}$$

and $n=(1-m)$, $0\leqq m\leqq 1$. In Eqs. (20) and (21), s_1, s_2, and t are the dependent and independent variables, the remaining symbols being constants.

A solution of (20) is desired for the initial conditions $\dot{s}_1(0)=\dot{s}_2(0)=v_0$. The time t is counted from the instant when the lower end of the spring is in contact with a fixed horizontal surface. For suitable values of R, r_i, and b_i a graph of $[A(r)]^2$ is either Fig. 9a or 9b.

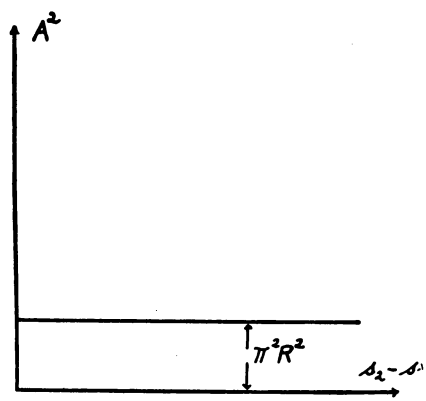

FIG. 9a.

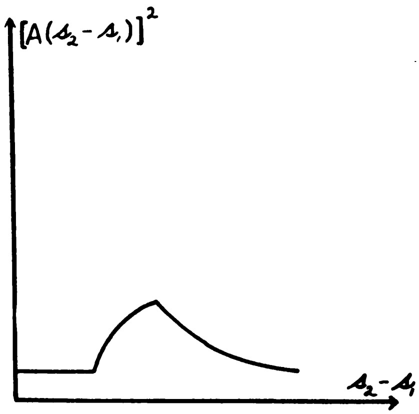

FIG. 9b.

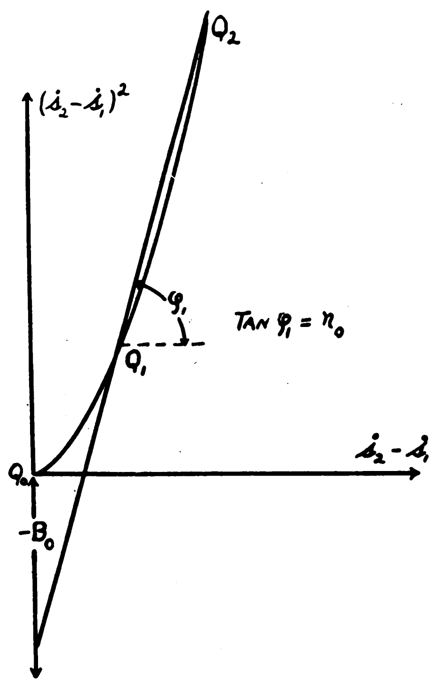

FIG. 10.

Even in the case where $A(r) = \pi R$ the following methods, (a) power series solutions in the time, (b) numerical integration, (c) successive integrations, (d) Galerkin's, (e) curvature, and (f) expansions in power series in parameters, become so laborious that they fail for practical purposes. Lord Rayleigh[13] has given methods of handling differential equations linear in all terms except containing a damping coefficient which depends on the square of the velocity. The velocity term is supposed small. In Eq. (20) the velocity term is small or large dependent upon the stage of the motion.

Consider the curves shown in Figs. 10, 11, and 12. It is evident that the forces relevant to these curves can be approximated by the arc Q_0Q_1 and the small number of secant lines Q_1Q_2, R_0R_1, R_1R_2, and R_2R_3. The location of Q_1, R_0, R_1, R_2, and R_3 will become evident from physical conditions presently discussed.

For the first interval of motion replace Eq. (20) by the equations

$$\begin{aligned} \ddot{s}_1 + K_0 + k_0 s_1 - \alpha[A_0 + m_0(s_2 - s_1)] - g &= \beta(\dot{s}_2 - \dot{s}_1)^2, \\ \ddot{s}_2 + \gamma[A_0 + m_0(s_2 - s_1)] - ng &= -\delta(\dot{s}_2 - \dot{s}_1)^2, \end{aligned} \tag{22}$$

where

$$\alpha = \frac{p_0 S}{m_1}, \qquad \beta = \frac{\rho(S - s_m)^3}{2w_1c^2\pi^2R^2}, \qquad \gamma = \frac{p_0 S}{m_2}, \qquad \delta = \frac{\rho(S - s_m)^2}{2w_2c^2\pi^2R^2},$$

[13] Lord Rayleigh, *Theory of sound*, (2nd ed.) vol. I, Macmillan, London, 1894, p. 81.

and A_0, m_0, K_0, and k_0 are shown in the figures. The ordinate of R_3 represents the value of $(1-\xi)^{-1.20}$ when the system is at rest under the force of gravity. The points R_2 and R_4 are located such that no ordinate on the secant lines exceeds the corresponding ordinate on the arc by more than ten per cent.

The solution of Eq. (20) is now broken up into two time intervals. We reduce Eq. (21) to the normal form by the substitutions

$$s_1 = \xi_1 + a_1, \qquad s_2 = \xi_2 + a_2, \qquad \dot{\xi}_1 = \xi_3, \qquad \dot{\xi}_2 = \xi_4,$$

where a_1 and a_2 are constants such that no constant term remains in the resulting differential equations. Then the equations are

$$\begin{aligned} \dot{\xi}_1 &= \xi_3, \\ \dot{\xi}_2 &= \xi_4, \\ \dot{\xi}_3 &= -(\alpha m_0 + k_0)\xi_1 + \alpha m_0 \xi_2 + \beta(\xi_4 - \xi_3)^2, \\ \dot{\xi}_4 &= \gamma m_0 \xi_1 - \gamma m_0 \xi_2 - \delta(\xi_4 - \xi_3)^2, \end{aligned} \tag{23}$$

and

$$\begin{aligned} a_1 &= \gamma[g + (\alpha A_0 - K_0)] - \alpha[\gamma A_0 - ng]/\gamma k_0, \\ a_2 &= \{m_0[\gamma(g + \alpha A_0 - K_0) - \alpha(\gamma A_0 - ng)] - k_0(\gamma A_0 - ng)\}/m_0\gamma k_0, \end{aligned}$$

and $\alpha A_0 = K_0$ in order that $\dot{s}_1$ may not be positive in its initial motion. That is, the origin of time is taken to be the instant at which the upward force of the spring S is equal to the downward force due to gas pressure on m_1.

The general solution of (23) (with squared terms suppressed) is

$$\begin{aligned} \xi_1 &= A_1 \sin \omega_1 t + A_2 \cos \omega_1 t + A_3 \sin \omega_2 t + A_4 \cos \omega_2 t, \\ \xi_2 &= b_1 A_1 \sin \omega_1 t + b_1 A_2 \cos \omega_1 t + b_2 A_3 \sin \omega_2 t + b_2 A_4 \cos \omega_2 t, \\ \xi_3 &= \omega_1 A_1 \cos \omega_1 t - \omega_1 A_2 \sin \omega_1 t + \omega_2 A_3 \cos \omega_2 t - \omega_2 A_4 \sin \omega_2 t, \\ \xi_4 &= b_1\omega_1 A_1 \cos \omega_1 t - b_1\omega_1 A_2 \sin \omega_1 t + b_2\omega_2 A_3 \cos \omega_2 t - b_2\omega_2 A_4 \sin \omega_2 t, \end{aligned} \tag{24}$$

where ω_1 and ω_2 are the roots of the characteristic equation and

$$b_1 = (\alpha m_0 + k_0 - \omega_1^2)/\alpha m_0, \qquad b_2 = (\alpha m_0 + k_0 - \omega_2^2)/\alpha m_0.$$

The nonlinear terms in (22) are taken into account by the method of variation of parameters. Employing (24) as equations of change of variables and remembering that (24) satisfies

$$\frac{\partial \xi_1}{\partial t} = \xi_3, \qquad \frac{\partial \xi_2}{\partial t} = \xi_4,$$

$$\frac{\partial \xi_3}{\partial t} = -(\alpha m_0 + k_0)\xi_1 + \alpha m_0 \xi_2, \qquad \frac{\partial \xi_4}{\partial t} = \gamma m_0 \xi_1 - \gamma m_0 \xi_2$$

we have

$$\begin{aligned} \frac{\partial \xi_1}{\partial A_1}\dot{A}_1 + \frac{\partial \xi_1}{\partial A_2}\dot{A}_2 + \frac{\partial \xi_1}{\partial A_3}\dot{A}_3 + \frac{\partial \xi_1}{\partial A_4}\dot{A}_4 &= 0, \\ \frac{\partial \xi_2}{\partial A_1}\dot{A}_1 + \frac{\partial \xi_2}{\partial A_2}\dot{A}_2 + \frac{\partial \xi_2}{\partial A_3}\dot{A}_3 + \frac{\partial \xi_2}{\partial A_4}\dot{A}_4 &= 0, \end{aligned} \tag{25'}$$

$$\frac{\partial \xi_3}{\partial A_1}\dot{A}_1 + \frac{\partial \xi_3}{\partial A_2}\dot{A}_2 + \frac{\partial \xi_3}{\partial A_3}\dot{A}_3 + \frac{\partial \xi_3}{\partial A_4}\dot{A}_4 = \beta(\xi_4 - \xi_3)^2, \qquad (25'')$$
$$\frac{\partial \xi_4}{\partial A_1}\dot{A}_1 + \frac{\partial \xi_4}{\partial A_2}\dot{A}_2 + \frac{\partial \xi_4}{\partial A_3}\dot{A}_3 + \frac{\partial \xi_4}{\partial A_4}\dot{A}_4 = -\delta(\xi_4 - \xi_3)^2.$$

The solutions of (25), after some rather lengthy trigonometric manipulations, are

$$\dot{A}_1 = \frac{\delta + b_2\beta}{\omega_1(b_2 - b_1)}(\xi_4 - \xi_3)^2 \cos \omega_1 t, \qquad \dot{A}_2 = -\frac{\delta + b_2\beta}{\omega_1(b_2 - b_1)}(\xi_4 - \xi_3)^2 \sin \omega_1 t,$$
$$\dot{A}_3 = -\frac{\delta + b_1\beta}{\omega_2(b_2 - b_1)}(\xi_4 - \xi_3)^2 \cos \omega_2 t, \quad \dot{A}_4 = \frac{\delta + b_1\beta}{\omega_2(b_2 - b_1)}(\xi_4 - \xi_3)^2 \sin \omega_2 t. \qquad (26)$$

The solution of (26) is obtained with sufficient approximation by using a device common in celestial mechanics; i.e., for small values of the time, the A_i entering (26) through ξ_3 and ξ_4 may be considered constants having the values obtained by the solution of (24) for $\xi_1 = -a_1$, $\xi_2 = -a_2$, $\xi_3 = \xi_4 = v_0$ at $t = 0$. Thus the solution of (26), to the accuracy required, is reduced to quadratures. Moreover, since the interval for which this solution is valid is small ($0 \leqq t \leqq 0.01$) the trigonometric functions involved may be expanded as power series in t before the quadratures are performed. The solution of (25) is

$$A_i = C_i + f_i(t) \qquad (i = 1, \cdots, 4), \qquad (27)$$

where $f_i(0) = 0$. The substitution of (27) in (24) gives the complete solution for $0 \leqq t \leqq t_1$, where $\omega_2 t_1 < \frac{1}{2}$ and $\omega_2 > \omega_1$. The values of $C_i = A_i$ as determined above.

The value of $[\xi_4(t_1) - \xi_3(t_1)]^2$ locates the point Q_1 in Fig. 10. The ordinate of Q_2 is v_0^2. The ordinate of R_1 is given by $s_2(t_1) - s_1(t_1)$. The ordinate of R_4 is the value of $(1-\xi)^{-1.2}$ when the air chamber is decreased to 0.7 of its initial value.

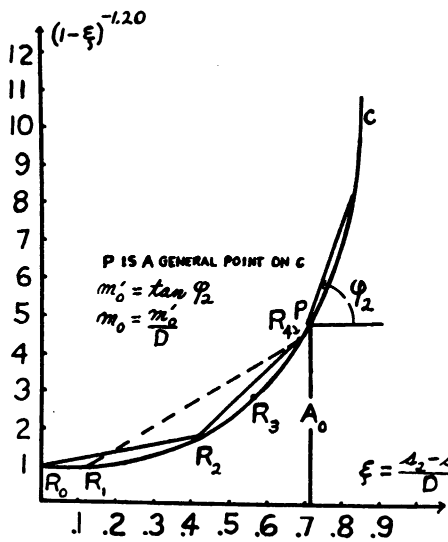

FIG. 11.

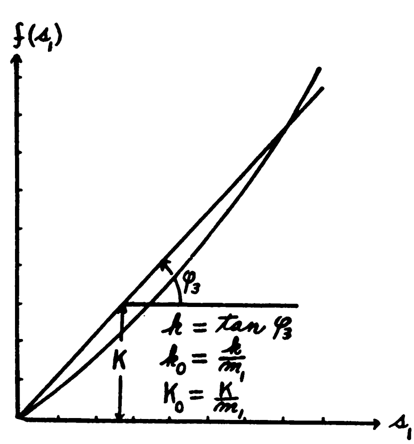

FIG. 12.*

* In Fig. 12 the origin should be marked S_0 and the second point of intersection of the straight line through the origin and the curve should be marked S_1.

In the solution for the second interval ($t_1 \leqq t < t_2$) of motion it is sufficient to replace the arcs $R_1R_2R_3R_4$ and Q_1Q_2 by the secant lines R_1R_4 and Q_1Q_2. The quantity $(\dot{s}_2 - \dot{s}_1)^2$ on the interval Q_1Q_2 may be written

$$(\dot{s}_2 - \dot{s}_1)^2 = -B_0 + n_0(\dot{s}_2 - \dot{s}_1), \tag{28}$$

and (22) becomes

$$\begin{aligned} (p^2 + \alpha m_0 + k_0)s_1 \qquad - \alpha m_0 s_2 &= g + \alpha A_0 - \beta B_0 + \beta n_0(\dot{s}_2 - \dot{s}_1) - K_0, \\ -\gamma m_0 s_1 + (p^2 + \gamma m_0)s_2 &= ng - \gamma A_0 + \delta B_0 - \delta n_0(\dot{s}_2 - \dot{s}_1), \end{aligned} \tag{29}$$

where any additional constants are shown in the figures. At the new origin of time for (29), $s_1(0) = s_2(0) = 0$ and $\dot{s}_1(0) = \dot{s}_1(t_1) = v_1$, $\dot{s}_2(0) = \dot{s}_2(t_1) = v_2$.

The proper determination of the constants b_1 and b_2 in the substitution $s_1 = \xi_1 + b_1$, $s_2 = \xi_2 + b_2$ in (29) yields

$$\begin{aligned} (p^2 + \beta n_0 p + \alpha m_0 + k_0)\xi_1 - (\beta n_0 p + \alpha m_0)\xi_2 &= 0, \\ -(\delta n_0 p + \gamma m_0)\xi_1 + (p^2 + \delta n_0 p + \gamma m_0)\xi_2 &= 0. \end{aligned} \tag{30}$$

While the characteristic equation of (30) is of the fourth degree, yet its roots are widely separated in practical cases and quickly found by Graeffe's method.

The values of $s_1 = \xi_1 + b_1$, $s_2 = \xi_2 + b_2$ as given by the solution of (30) do not yield the equilibrium positions of m_1 and m_2, because when $(\dot{s}_2 - \dot{s}_1)^2$ becomes small the relation (28) and Eqs. (29) are no longer valid. This is no defect of the solution because its purpose is the determination of the maximum accelerations acting on m_1 and m_2. These maxima occur in the interval $0 \leqq t \leqq t_2$. The equilibrium positions of m_1 and m_2 are determined from static considerations.

A point of special interest is the determination of the effects of the factor ng upon the solution. The above solution is constructed with this in mind.

The roots of the characteristic equation of (30) have special physical significance. In practical cases these are usually one or two pairs of complex roots. If there are four complex roots, one pair gives a high frequency oscillation of moderate magnitude for m_1. This is to be avoided.

If $[A(r)]^2$ is given by the graph shown in Fig. 9b the above method is still applicable. The solution is very sensitive with respect to $[A(r)]$.[2] Of course, the intervals of solution will exceed two in number, but in each interval the value of $(\dot{s}_2 - \dot{s}_1)^2$ will be given by the ordinates of the arc Q_0Q_1 or the secant Q_1Q_2.

The most complicated process involved in solving (20) is the solution of a quartic equation.

4. **Concluding remarks.** The seven problems presented above are representative of the nonlinear discrete problems of industry in so far as one nonlinear problem can represent a group the members of which differ greatly. No bibliography is given for the reason stated in footnote 7.

Methods of handling industrial nonlinear problems of continuous systems arising in industry are reserved for a subsequent paper.