THE THERMAL-STRESS AND BODY-FORCE PROBLEMS OF THE INFINITE ORTHOTROPIC SOLID*

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1. Introduction. Elastic problems dealing with orthotropic materials have had considerable investigation in recent years,[1] but up to the present time, such investigation has been largely limited to a consideration of the problems involving thin plates of this material.

In the present paper, two problems dealing with the stresses and displacements in an infinite elastic orthotropic solid are solved, and in each case the results are obtained in terms of three independent displacement potentials. The two solutions are: 1) the displacement potentials arising from an arbitrary distribution of temperature within a finite region of the solid (the temperature being measured from an arbitrary datum) and 2) the potentials arising from an arbitrary distribution of body force within a finite region. Each of these problems reduces to the solution of three simultaneous partial differential equations, which are transformed, through the use of Fourier integrals, into individual solutions for each potential. The expressions for these potentials are reduced to the form of Newtonian potential integrals for those cases where sufficient symmetry of the material properties exists to allow such a reduction. In the more complicated cases, the results are still expressed in closed form in terms of definite integrals.

2. The thermo-elastic problem. The conditions under which the thermo-elastic problem will be formulated and solved are the following. The material is to be homogeneous, orthotropic, and elastic, throughout the infinite region, and is to be within that class of orthotropic materials which has three coefficients of temperature expansion, α_j, associated with the three principal directions of the material. The body forces will be taken as vanishing, since any problem involving both thermal and body force effects has a solution which is merely the superposition of the two individual solutions. The temperature distribution is to be an arbitrary function of position with the restrictions that this function must vanish everywhere outside some finite region, be continuous everywhere and be differentiable everywhere except on a finite number of surfaces.

The fundamental relations needed to formulate the problem mathematically are: the equations of equilibrium of an element of the material; the thermo-elastic equations, that is, the relations between strains, stresses and temperature; and the relations between strains and displacements.

The equations of equilibrium are found by a consideration of the equilibrium of a rectangular parallelepiped of the material under general loading. Since these equations are independent of the type of material under consideration, they are given, as

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[1] See, for example, A. E. Green, and G. I. Taylor, *Stress distributions in aeolotropic plates*, Proc. Roy. Soc. A **173**, 163 (1939).

in the isotropic case for zero body force,[2] by three equations of the type,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \tag{1}$$

where the notation is the conventional one.

The orthotropic material has been defined as one whose Hooke's law has the form indicated by equations (2), when T is identically zero. The effect of temperatures, different from datum, is to produce normal strains in the three principal directions of the material, as specified under the conditions of the problem. Hence, when the coordinate axes are taken parallel to the principal directions, the general formulas for the strains have the form,

$$e_x = a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z + \alpha_1 T, \cdots ; \qquad \gamma_{yz} = a_{44}\tau_{yz}, \cdots . \tag{2}$$

If we now define three displacement potentials, ϕ_j, such that

$$u = \frac{\partial \phi_1}{\partial x}, \qquad v = \frac{\partial \phi_2}{\partial y}, \qquad w = \frac{\partial \phi_3}{\partial z},$$

and such that ϕ_j and its derivatives vanish at infinity, the conventional definitions of the strains become,

$$e_x = \frac{\partial u}{\partial x} = \frac{\partial^2 \phi_1}{\partial x^2}, \cdots ; \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{\partial^2}{\partial z \partial y}(\phi_2 + \phi_3), \cdots \tag{3}$$

Combining now, equations (1), (2), and (3), we obtain three equations of which the following is the first:

$$\frac{\partial}{\partial x}\left[\left(b_{11}\frac{\partial^2}{\partial x^2} + b_{66}\frac{\partial^2}{\partial y^2} + b_{55}\frac{\partial^2}{\partial z^2}\right)\phi_1 + c_{12}\frac{\partial^2}{\partial y^2}\phi_2 + c_{13}\frac{\partial^2}{\partial z^2}\phi_3\right] = -\beta_1\frac{\partial T}{\partial x}\cdot \tag{4}$$

Each of these may be integrated once to give,[3]

$$\left(b_{11}\frac{\partial^2}{\partial x^2} + b_{66}\frac{\partial^2}{\partial y^2} + b_{55}\frac{\partial^2}{\partial z^2}\right)\phi_1 + c_{12}\frac{\partial^2}{\partial y^2}\phi_2 + c_{13}\frac{\partial^2}{\partial z^2}\phi_3 = -\beta_1 T, \cdots \tag{4a}$$

The arbitrary functions which appear in each of the foregoing integrations must each vanish, since, for example, in the first equation, all terms vanish when x is infinite and y, z are finite, implying that all functions independent of x must vanish identically.

Due to the convenient form of the boundary conditions, these equations are easily integrated by the following procedure. Multiply each equation through by $e^{-i(x\xi+y\eta+z\zeta)}$ and integrate over the whole region, integrating by parts those terms containing derivatives of ϕ_j. This operation produces the following three equations, using the abbreviated forms defined below in equations (6).

[2] A. E. H. Love, *A treatise on the mathematical theory of elasticity*, Cambridge, 1934, p. 125.

[3] The b_{ij}, c_{ij} and β_j are combinations of elastic and thermal constants arising from the above operation. The manner in which these constants appear in the second and third of these is easily deduced from equations (5).

$$
\begin{aligned}
(b_{11}\xi^2 + b_{66}\eta^2 + b_{55}\zeta^2)E_1 + c_{12}\eta^2 E_2 + c_{13}\zeta^2 E_3 &= \beta_1 S,\\
c_{12}\xi^2 E_1 + (b_{66}\xi^2 + b_{22}\eta^2 + b_{44}\zeta^2)E_2 + c_{23}E_3 &= \beta_2 S,\\
c_{13}\xi^2 E_1 + c_{23}\eta^2 E_2 + (b_{55}\xi^2 + b_{44}\eta^2 + b_{33}\zeta^2)E_3 &= \beta_3 S,
\end{aligned}
\tag{5}
$$

where,

$$
\begin{aligned}
E_j &= \iiint_{-\infty}^{\infty} \phi_j e^{-i(x\xi+y\eta+z\zeta)} dx\, dy\, dz,\\
S &= \iiint T e^{-i(x\xi+y\eta+z\zeta)} dx\, dy\, dz.
\end{aligned}
\tag{6}
$$

Equations (5) are easily solved for the E_j, and yield the expressions,

$$
E_j = F_j(\xi, \eta, \zeta) S, \tag{7}
$$

where the F_j become ratios of homogeneous polynomials in ξ^2, η^2 and ζ^2.

Noting now, that by their definitions, the E_j are the Fourier transforms (in three dimensions) of the ϕ_j, we may write

$$
\begin{aligned}
\phi_j(x, y, z) &= \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} F_j(\xi, \eta, \zeta) e^{i(x\xi+y\eta+z\zeta)} S(\xi, \eta, \zeta) d\xi\, d\eta\, d\zeta\\
&= \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} F_j(\xi, \eta, \zeta) e^{i(x\xi+y\eta+z\zeta)} d\xi\, d\eta\, d\zeta \cdot \iiint_{-\infty}^{\infty} T(r, s, t) e^{-i(r\xi+s\eta+t\zeta)} dr\, ds\, dt,
\end{aligned}
\tag{8}
$$

and the order of the indicated integrations may be changed to give,

$$
\phi_j = \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} T(r, s, t) dr\, ds\, dt \iiint_{-\infty}^{\infty} F_j(\xi, \eta, \zeta) e^{i[(x-r)\xi+(y-s)\eta+(z-t)\zeta]} d\xi\, d\eta\, d\zeta. \tag{9}
$$

Since each F_j (as defined by equation 7) is a ratio of second order polynomial in ξ^2, η^2, and ζ^2, to one of third order, we may write,

$$
F_j = B \frac{R_4^2 R_5^2}{R_1^2 R_2^2 R_3^2},
$$

where $R_k^2 = \lambda_k^2 \xi^2 + \mu_k^2 \eta^2 + \zeta^2$, and where the λ_k, μ_k, and B, are constants depending on the values of the constants appearing in the determinants defining F_j, and hence, may be considered as known. Note that the λ_k^2, μ_k^2, for $k = 1, 2, 3$, must be non-negative, since no singularity may exist except at the origin.

In many cases, the expressions for the F_j may be reduced to the form,

$$
F_j = \sum_{k=1,2,3} \frac{A_{jk}}{R_k^2}. \tag{10}
$$

This will always be true when the problem involves a material which is isotropic in a certain plane (for example, a laminated plastic) unless identical values of R^2 recur in the denominator. This may be seen by noting that since the denominator of F_j must be invariant under a rotation about the z axis due to this isotropy (the plane of isotropy is here taken as the x, y plane), ξ^2 and η^2 must occur in the combination $\xi^2 + \eta^2$, and hence, $\lambda_k = \mu_k$, and the R_k^2 become essentially binomials. The reduction of F_j to the form of equation (10) is, in this case, merely a matter of evaluating A_{jk}.

When equation (10) does hold, the integration proceeds as follows: Using the conventional vector notation and the new coordinates with the subscript k (where $\xi_k=\lambda_k\xi$, $x_k=x\lambda_k^{-1}$, $r_k=r\lambda_k^{-1}$, etc. and where $\bar{m}_k=\bar{\imath}(x_k-r_k)+\bar{\jmath}(y_k-s_k)+\bar{k}(z_k-t_k)$), the integral over ξ, η, and ζ, of equation (9) defining Green's function G, may be written,

$$G_j(x, y, z, r, s, t) = \iiint_{-\infty}^{\infty} \sum_{k=1,2,3} \frac{A_{jk}}{R_k^2} e^{i\bar{m}_k\cdot\bar{R}_k} \frac{d\xi_k}{\lambda_k} \frac{d\eta_k}{\mu_k} d\zeta_k. \tag{11}$$

If we now change to a spherical coordinate system in which γ is the angle between $\bar{m}_k$ and $\bar{R}_k$ and δ is the polar angle about $\bar{m}_k$, this integral becomes,

$$G_j = \iiint \sum \frac{A_{jk}}{\lambda_k\mu_k} e^{im_kR_k\cos\gamma} \sin\gamma\, d\gamma\, d\delta\, dR_k,$$

where the integration now takes place over, $0\leqq\gamma\leqq\pi$, $0\leqq\delta\leqq 2\pi$, $0\leqq R_k<\infty$. The elementary integrations over γ and δ produce

$$G_j = \sum_{k=1,2,3} \frac{4\pi A_{jk}}{\lambda_k\mu_k m_k} \int_0^\infty \frac{\sin m_kR_k}{R_k} dR_k,$$

which is known to have the value,

$$G_j = 2\pi^2 \sum_{k=1,2,3} \frac{A_{jk}}{\lambda_k\mu_k} \frac{1}{m_k}.$$

Now transforming the remaining terms of equation (9) to the coordinates with the subscript k, and substituting the above value for Green's function, we obtain,

$$\phi_j = \frac{1}{4\pi}\iiint \sum_{k=1,2,3} A_{jk} \frac{T(\lambda_k r_k, \mu_k s_k, t_k)}{\sqrt{(x_k-r_k)^2+(y_k-s_k)^2+(z_k-t_k)^2}} dr_k\, ds_k\, dt_k. \tag{12}$$

Hence, the problem, wherein $T(x, y, z)$ represents the temperature distribution, becomes the problem of evaluating the Newtonian potential function corresponding to a mass distribution of,

$$\rho = \frac{A_{ik}}{4\pi} T(\lambda_k x_k, \mu_k y_k, z_k).$$

For an isotropic material, the ϕ_k become alike, and are given by,[4]

$$\phi_i = \frac{\alpha}{4\pi}\frac{1+v}{1-v}\iiint \frac{T(r, s, t)}{\sqrt{(x-r)^2+(y-s)^2+(z-t)^2}} dr\, ds\, dt.$$

In the evaluation of Green's function for those cases where the denominator of F_j has a multiple root, it is convenient to introduce the notation

$$G_j = \sum_n G_{jn}, \qquad \Delta_k = \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} + \frac{\partial^2}{\partial z^2}.$$

In this case, integrals of the form,

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[4] J. N. Goodier, *On the integration of the thermo-elastic equations*, Phil. Mag. (7), **23**, 1017 (1937).

$$G_{jn} = \int\int\int_{-\infty}^{\infty} \frac{A_{jn}R_1^2}{R_2^4} e^{i[(x-r)\xi+(y-s)\eta+(z-t)\zeta]}d\xi\, d\eta\, d\zeta \tag{13}$$

must be evaluated, provided $R_1^2 \neq R_2^2$. Using the above notation, the equivalence of the following to equation (13) may be verified by substitution:

$$\Delta_2 G_{jn} = \Delta_1 \int\int\int \frac{A_{jn}}{R_2^2} e^{i[(x-r)\xi+(y-s)\eta+(z+t)\zeta]}d\xi\, d\eta\, d\zeta. \tag{13a}$$

The integral involved in this equation is, however, the same as that appearing in equation (11), so (13a) becomes,

$$\Delta_2 G_{jn} = \Delta_1 \frac{2C_{jn}}{\sqrt{(x_2 - r_2)^2 + (y_2 - s_2)^2 + (z_2 - t_2)^2}},$$

where C_{jn} is an easily evaluated constant. Substitution will again show that,

$$G_{jn} = C_{jn}\Delta_1\sqrt{(x_2 - r_2)^2 + (y_2 - s_2)^2 + (z_2 - t_2)^2},$$

is equivalent to the above equation, and hence the ϕ_j are given by,

$$\phi_j = \frac{1}{8\pi^3}\sum_n \int\int\int T(r, s, t)G_{jn}(r, s, t, x, y, z)dr\, ds\, dt. \tag{14}$$

In those cases where F_j cannot be reduced to one of the foregoing convenient forms, G_j is more difficult to evaluate. Since no explicit form has been found for this function, other than complicated definite integrals, it is believed best to leave it in the form defined by equation (9).

3. The body-force problem. As in dealing with isotropic materials, the solution of the body force problem may be shown to reduce to a form analogous to that of the thermo-elastic problem. To show this, we shall consider only the problem where the body force is directed parallel to the x axis, noting that the general solution is obtained by the superposition of three such problems.

Equations (1) and (2) are modified to contain the body-force function, X, and to eliminate the temperature terms. Equations (4) are then obtained again, where now the right hand sides are replaced respectively by, X, 0, and 0.

The ϕ_j will not, in general, vanish at infinity in this problem, hence the procedure needs a slight modification. The second and third of these equations are integrated with respect to y and z respectively and then differentiated with respect to x. This yields equations (4a), where again, X, 0, 0, appear on the right and where the ϕ_j are replaced by $\partial\phi_j/\partial X$. The procedure is now identical with that of the thermal problem, and the ϕ_j are found by the expressions analogous to equation (14).

4. The two-dimensional problem. If we carry through in two dimensions the procedure used in the previous sections of this paper, we arrive at an equation which is identical to equation (8) except that z, t, and ζ, no longer appear. The expressions for F_j are now simpler in form, being given by,

$$F_j = \frac{\lambda_1^2\xi^2 + \mu_1^2\eta^2}{(\lambda_2^2\xi^2 + \mu_2^2\eta^2)(\lambda_3^2\xi^2 + \mu_3^2\eta^2)},$$

which may always be reduced to the form

$$F_j = \sum_{k=1,2} \frac{A_{jk}}{R^2}$$

unless $R_2 = R_3$. ($R_k^2 = \lambda_k^2\xi^2 + \mu_k^2\eta^2$).

Before changing the order of integration, we differentiate equation (8) with respect to y. The integral form of Green's function becomes then

$$\frac{\partial G_{jk}}{\partial y} = \iint_{-\infty}^{\infty} A_{jk} \frac{i\eta e^{i(x\xi+y\eta)}}{\lambda_k^2\xi^2 + \mu_2^2\eta^2} d\xi\, d\eta, \tag{15a}$$

unless $R_2 = R_3$, in which case,

$$\Delta_2 \frac{\partial G_j}{\partial y} = \Delta_1 \iint A_j \frac{i\eta e^{i(x\xi+y\eta)}}{\lambda_2\xi^2 + \mu_2\eta^2} d\xi\, d\eta. \tag{15b}$$

This latter expression is, of course, derived by the same reasoning used in the three dimensional problem.

Equation (15a), after the introduction of the coordinates with the subscript k, can be written in the iterated integral form,

$$\frac{\partial G_{jk}}{\partial y} = \int_0^\infty \frac{4A_{jk}}{\lambda_k} \sin \eta_k\, y_k\, d\eta_k \int_0^\infty \frac{\cos \eta_k x_k \dfrac{\xi_k}{\eta_k}}{1 + \left(\dfrac{\xi_k}{\eta_k}\right)^2} d\left(\frac{\xi_k}{\eta_k}\right)$$

which is known to be equivalent to,

$$\frac{\partial G_{jk}}{\partial y} = \int_0^\infty \frac{4A_{jk}}{\lambda_k} \sin y_k\eta_k \cdot \frac{\pi}{2} e^{-|x_k\eta_k|} d\eta_k$$

and this integral yields,

$$\frac{\partial G_{jk}}{\partial y} = \frac{2\pi A_{jk}}{\lambda_k} \frac{y_k}{x_k^2 + y_k^2}$$

or

$$G_{jk} = \frac{\pi A_{jk}}{\lambda_k\mu_k} \ln (x_k^2 + y_k^2), \tag{16}$$

and we obtain the familiar two-dimensional logarithmic potential.

Equation (15b), then becomes, in an analogous manner,

$$\Delta_2 G_j = \frac{\pi A_j}{\lambda_k\mu_k} \Delta_1 \ln (x_2^2 + y_2^2)$$

or,

$$G_j = \frac{\pi A_j}{2\lambda_k\mu_k} \Delta_2\left[(x_2^2 + y_2^2) \ln (x_2^2 + y_2^2)\right]. \tag{17}$$

Hence, Green's functions are determined for each two-dimensional problem involving thermal stress or body forces in the infinite plate. The usual methods of superimposing plane stress (or strain) solutions may be utilized, of course, to solve the corresponding problems for the finite body.