where

$$\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\beta^2} d\beta.$$

It should be noted that in (18′) the function $v(x, t)$ is expressed in terms of tabulated functions.

The final solution of our problem is given by (4) in conjunction with (11) and (18) or (11′) and (18′).

THE SPHERICAL GYROCOMPASS*

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In the existing literature on gyroscopes[1] the theory of the gyrocompass is developed for the case of a rotor whose ellipsoid of inertia is an ellipsoid of revolution. The mathematics of this treatment is somewhat involved and, in deducing the differential equations of motion, approximations based on the smallness of the earth's angular velocity are made. In the present communication we shall treat a gyrocompass the rotor of which has a spherical ellipsoid of inertia. The motion of such a gyrocompass is, of course, covered by the more general theory usually given, but owing to the symmetry of the sphere this case allows a considerably simpler, separate treatment in which, moreover, no approximations are necessary. At the same time the essential features of gryoscopic motion are preserved.

The following system will serve as a simple model of a spherical gyrocompass. The rotor is a rigid homogeneous sphere rotating freely about a light axle which passes through its centre. The ends of this axle can slide in a smooth horizontal ring which is concentric with the rotor and rigidly attached to the earth. When the rotor is set in rapid revolution about its axle the latter executes oscillations about the meridian which will now be examined.

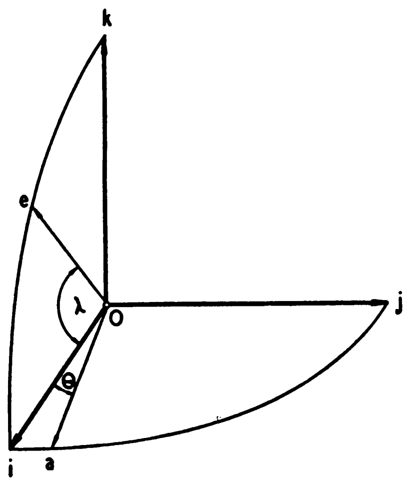

FIG. 1.

In the figure the right-handed unit triad, **i**, **j**, **k**, which is fixed relative to the earth is defined as follows: O is the center of the rotor; **k** lies in the direction of the upward vertical; **i** lies along the meridian and points north; **j**, pointing west, completes the triad. The unit vector, **a**, lies along the axle of the gyrocompass and the unit vector, **e** (in the **i**, **k** plane), is parallel to the earths' axis; thus the angle λ, between **i** and **e**, is the latitude of the observer.

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[1] Cf. T. Levi Civita and U. Amaldi, *Lezioni di meccanica razionale*, vol. 2, Zanichelli, Bologna, 1927, pp. 191–195; or J. L. Synge and B. A. Griffith, *Principles of mechanics*, McGraw-Hill Book Co., New York, 1942, pp. 430–433.

It is clear that the couple exerted by the ring on the rotor must be of the form

$$\mathbf{G} = G(\mathbf{a} \times \mathbf{k}).$$

Further, if A is the moment of inertia of the rotor, $\boldsymbol{\omega}$ its angular velocity and $\mathbf{h}$ its angular momentum, we have the relation

$$\mathbf{G} = \dot{\mathbf{h}} = A\dot{\boldsymbol{\omega}}.$$

Consequently

$$\dot{\boldsymbol{\omega}}\cdot\mathbf{a} = 0, \qquad (1); \qquad\qquad \dot{\boldsymbol{\omega}}\cdot\mathbf{k} = 0. \qquad (2)$$

Since the gyrocompass has only two degrees of freedom, equations (1) and (2), together with initial conditions, completely determine its motion.

We observe that $\boldsymbol{\omega}$ is made up of three parts: the spin of the sphere about its axle; the rotation of the axle relative to the frame $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$; and finally the absolute rotation of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ or of the earth to which it is attached. Therefore we may write: $\boldsymbol{\omega} = s\mathbf{a} + \dot{\theta}\mathbf{k} + \Omega\mathbf{e}$, where s is the spin of the rotor, θ the angle between the meridian $\mathbf{i}$ and the axle $\mathbf{a}$, and Ω the angular velocity of the earth. Differentiation of this relation gives $\dot{\boldsymbol{\omega}} = \dot{s}\mathbf{a} + s\dot{\mathbf{a}} + \ddot{\theta}\mathbf{k} + \dot{\theta}\dot{\mathbf{k}}$, and since the angular velocity of $\mathbf{a}$ is $\dot{\theta}\mathbf{k} + \Omega\mathbf{e}$ and that of $\mathbf{k}$ is $\Omega\mathbf{e}$, this equation becomes

$$\dot{\boldsymbol{\omega}} = \dot{s}\mathbf{a} + s(\dot{\theta}\mathbf{k} + \Omega\mathbf{e}) \times \mathbf{a} + \ddot{\theta}\mathbf{k} + \dot{\theta}\Omega(\mathbf{e} \times \mathbf{k}). \qquad (3)$$

Substituting (3) into (1) and (2), we immediately arrive at

$$\dot{s} - \dot{\theta}\Omega\cos\lambda\sin\theta = 0, \qquad (4); \qquad\qquad s\Omega\cos\lambda\sin\theta + \ddot{\theta} = 0 \qquad (5)$$

as the required equations of motion.

The solution of these equations may be obtained in the usual way. From (4) it follows that $s = s_0 + \Omega\cos\lambda(\cos\theta_0 - \cos\theta)$, where s_0 and θ_0 are the initial values of s and θ. Inserting this value of s into (5), we obtain a differential equation for θ alone, which is of the classical type $\ddot{\theta} = f(\theta)$; this can be solved in terms of hyperelliptic functions. If the initial spin s_0 is great we may replace s in (5) by s_0 to obtain the well known result: the motion of the axle $\mathbf{a}$ is identical with the motion of a simple pendulum, the position of equilibrium being in the direction of the meridian.

An interesting property of the spherical gyrocompass can be deduced directly from Eqs. (4) and (5) *which are exact*. For, if we multiply by s and $\dot{\theta}$ respectively and add, we obtain $s\dot{s} = \dot{\theta}\ddot{\theta} = 0$, which on integration becomes $s^2 + \dot{\theta}^2 = \text{constant}$.[2] This shows that a spherical gyrocompass has an angular velocity of strictly constant magnitude relative to the earth.

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[2] Levi Civita and Amaldi, l.c., derive a corresponding result, namely

$$\tfrac{1}{2}A\dot{\theta}^2 + \tfrac{1}{2}Cs^2 = \text{const.}$$

for the general gyrocompass. (A and C are the transverse and axial moments of inertia respectively.) They then explain it by energy considerations. In fact, however, this equation and therefore also their reasoning is not strictly accurate. The exact form is

$$\tfrac{1}{2}A\dot{\theta}^2 + \tfrac{1}{2}Cs^2 + \tfrac{1}{2}(C - A)\Omega^2\cos^2\lambda\sin^2\theta = \text{const.}$$