CANTILEVER BEAMS OF UNIFORM STRENGTH*

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1. The object of the paper, its methods and results. The problem of shaping a beam from a given amount of material in such a manner as to obtain maximum strength requires that the maximum stress of each cross section be constant. In the case of bending, the classical treatment of this problem[1,2,3,4] is based on the theory of beams of constant cross section, the influence of shearing stresses and of the weight of the beam being neglected. A collection of solutions of this elementary problem, for rectangular and circular cross sections, is given in the Hütte handbook for engineers.[5] If the strength of the material is relatively low, the weight W of the beam cannot be neglected. This occurs in certain concrete structures, such as reinforced concrete bridges, and was demonstrated by Gaede[6] in his treatment of a cantilever of rectangular cross section and constant width, the external load being a force F at the free end.

In the present paper, we shall consider cantilevers of more general cross section but with the same type of loading, except in §6 where more general loading will be considered. Let us denote by x the distance from the free end, by $A(x)$ the area of the cross section and by $S(x)$ the section modulus ($S=M/\sigma$, where M is the bending moment and σ is the maximum stress). The bending moment $M(x)=\sigma S(x)$ at the distance x from the free end is then given by

$$Fx + \gamma \int_0^x (x-\xi)A(\xi)d\xi = \sigma S(x) \tag{1.1}$$

where γ denotes the density of the beam material. The total weight of the beam equals

$$\gamma \int_0^L A(\xi)d\xi = W \tag{1.2}$$

where L is the length of the beam. Since σ is constant along the beam, differentiation of (1.1) with respect to x yields

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[1] F. Grashof, *Theorie der Elasticität und Festigkeit*, R. Gaertner, Berlin, 2nd Edition, 1878, pp. 113–121.

[2] C. Bach, *Elasticität und Festigkeit*, J. Springer, Berlin, 2nd Edition, 1894, pp. 85–88.

[3] S. Timoshenko, *Strength of Materials*, part I, 2nd Edition, D. Van Nostrand Company, New York, 1940, pp. 209–210.

[4] C. Guidi, *Teoria dell' elasticità e resistenza dei materiali*, 11th Edition, Torino, 1925, pp. 135–142.

[5] *Hütte—Des Ingenieurs Taschenbuch*, vol. I, 25th Edition, Wilhelm Ernst, Berlin, 1925, pp. 626–629.

[6] Gaede, *Balkenträger von gleichem Widerstande gegen Biegung*, Bautechnik, **15**, 120–122 (1937).

$$F + \gamma \int_0^x A(\xi)d\xi = \sigma S'(x), \qquad S(0) = 0, \tag{1.1'}$$

$$\gamma A(x) = \sigma S''(x), \qquad S(0) = 0, \qquad \sigma S'(0) = F, \tag{1.1''}$$

where the primes denote derivatives with respect to x. By use of (1.1′) we can write (1.2) in the form

$$\sigma S'(L) = F + W. \tag{1.2'}$$

We note that (1.1′) and (1.1″) are forms of the well-known equations of equilibrium of a beam, $Q = M'$, $q = M''$, where Q, q are respectively the shearing force and load per unit length.

If the section modulus is assigned, $A(x)$ is given by (1.1″) and the problem is solved. In general however there are no criteria for the choice of the function $S(x)$; instead, some geometric characteristics of the cross section are assigned. Problems of this type are treated in the present paper in a general manner. They involve an integral equation (cf. Blasius[7]). Its solution may involve almost any of the classical special functions. Some simple cases leading to hyperbolic, Bessel and elliptic functions are discussed. The possibility of using Legendre, hypergeometric, Lamé and some other functions is indicated.

2. The type of beam. Throughout this paper we shall limit ourselves to cantilevers satisfying the following conditions: the line of centroids is a horizontal straight line (x-axis); each cross section has a vertical axis of symmetry (V-axis). In the plane of the cross section we choose a system of orthogonal Cartesian coordinates (U, V) with origin at the centroid C. In the vertical plane through the x-axis, we choose a system of Cartesian coordinates (x, y) with origin at the free end and y-axis directed downward. We assume that the curves bounding the cross sections are representable by the equations

$$U = u(x)u_1(t), \qquad V = v(x)v_1(t), \tag{2.1}$$

t being a parameter. The functions $u_1(t)$, $v_1(t)$ determine the shape of the cross section, whereas the functions $u(x)$ and $v(x)$ represent the change of the cross section along the axis of the beam. Any two cross sections are obtainable from each other by a transformation of dilatation[8] which depends on the position of the cross sections. We will choose $u_1(t)$ and $v_1(t)$ in such a manner that $u(x)$ and $v(x)$ be $\geqq 0$.

3. General equations. It is easily seen that $S = I/V_m$, where I is the moment of inertia of the cross section about the U-axis, and V_m is the maximum value of V. Thus, if α is the area enclosed by the curve $U = u_1(t)$, $V = v_1(t)$ and β the corresponding section modulus, we have

$$S = \beta u(x)[v(x)]^2, \qquad A = \alpha u(x)v(x). \tag{3.1}$$

If we set $a = \alpha\gamma/(\sigma\beta)$, the substitution of (3.1) into (1.1), (1.1″), (1.2), (1.2′) gives

$$Fx + \alpha\gamma \int_0^x (x - \xi)u(\xi)v(\xi)d\xi = \sigma\beta u(x)[v(x)]^2, \tag{3.2}$$

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[7] H. Blasius, *Träger kleinster Durchbiegung und Stäbe grösster Knickfestigkeit bei gegebenem Materialverbrauch*, Zeit. f. Math. u. Phys., **62**, 182–197 (1914).

[8] The writer is indebted to Prof. H. Busemann for this geometric terminology.

$$(uv^2)'' = auv, \qquad (uv^2)_{x=0} = 0, \qquad \sigma\beta(uv^2)'_{x=0} = F, \tag{3.2'}$$

$$\alpha \int_0^L u(x)v(x)dx = W/\gamma, \quad (3.3); \qquad \sigma\beta(uv^2)'_{x=L} = F + W. \tag{3.3'}$$

If $v(x)$ is known, (3.2) is a Volterra integral equation in $u(x)$ with the kernel $(x-\xi)/[v(x)]^2$. This kernel is a continuous function, within the interval of integration, if $v(0)\neq 0$, because we assume $v(x)$ continuous and by its physical meaning it must be $\neq 0$ for $x>0$. Therefore, according to the general theory of integral equations,[9] if $v(0)\neq 0$, Eq. (3.2) has one and only one solution $u(x)$ if $F\neq 0$ and only a meaningless solution $u=0$ if $F=0$. In other words, a cantilever of uniform strength under the action of its own weight alone must be such that $v(0)=0$.

4. Particular types of cantilevers of uniform strength. These are obtained by assuming particular forms for $u(x)$ or $v(x)$.

I). *v is constant.* The cross sections have constant height. If $F\neq 0$ the integral of (3.2′) is

$$u = Fr(\alpha\gamma v)^{-1} \sinh (rx) \tag{4.1}$$

where $r=(a/v)^{1/2}$. Substitution from this equation in (3.3′) gives the following condition for W:

$$\cosh (rL) = 1 + (W/F). \tag{4.2}$$

If $F=0$ no solution exists, which is in accordance with the general statement of §3, because here $v(0)\neq 0$.

II). *$v(x)$ is a linear function of x.* The cross sections have linearly varying height. We may restrict ourselves to the case

$$v(x) = c \pm x \tag{4.3}$$

since if x had a coefficient different from ± 1, the coefficient could be factored out and included in the function $v_1(t)$. Also, since we agreed to take $v(x)\geqq 0$ (cf. §2) and $x=0$ represents a point of the beam, c must be $\geqq 0$. Since $dv=\pm dx$, the solution of (3.2′), (3.3′) is[10]

$$u = v^{-3/2}Z_1(2ia^{1/2}v^{1/2}), \tag{4.4}$$

where Z_1 is a cylindrical function of order 1 which must satisfy the conditions

$$Z_1(2ia^{1/2}c^{1/2}) = 0, \qquad \pm\, \sigma\beta a^{1/2}iZ_0(2ia^{1/2}c^{1/2}) = F, \tag{4.5}$$

$$\pm\, \sigma\beta a^{1/2}iZ_0[2ia^{1/2}(c \pm L)^{1/2}] = F + W. \tag{4.5'}$$

The second equation in (4.5) and Eq. (4.5′) are obtained by use of the formula[10] $Z_1'(x)=Z_0(x)-x^{-1}Z_1(x)$. We put

$$Z_1(2ia^{1/2}v^{1/2}) = AiJ_1(2ia^{1/2}v^{1/2}) + BH_1^{(1)}(2ia^{1/2}v^{1/2}), \tag{4.6}$$

where J_1 and $H_1^{(1)}$ are the Bessel and the Hankel functions of the first kind and first order.[10] Equations (4.5), (4.5′) then give the following conditions for A, B and c:

[9] See for instance R. Courant-D. Hilbert, *Methoden der mathematischen Physik*, vol. I, 2nd Edition, Julius Springer, Berlin, 1931, pp. 119, 120, 133; or Riemann-Weber, *Die Differential- und Integralgleichungen der Mechanik und Physik*, vol. I, 7th Edition, F. Vieweg, Braunschweig, 1925, pp. 426–428 or E. Hellinger-O. Toeplitz, *Enc. d. math. Wiss.*, vol. II. 3, pp. 1459, 1460.

[10] E. Jahnke-F. Emde, *Funktionentafeln mit Formeln und Kurven*, 3rd Edition, B. G. Teubner, Leipzig, 1938, pp. 128, 144–147, 224, 237, 242.

$$AiJ_1(2ia^{1/2}c^{1/2}) + BH_1^{(1)}(2ia^{1/2}c^{1/2}) = 0, \tag{4.7'}$$

$$-AJ_0(2ia^{1/2}c^{1/2}) + BiH_0^{(1)}(2ia^{1/2}c^{1/2}) = \pm F/(\sigma\beta a^{1/2}), \tag{4.7''}$$

$$-AJ_0[2i(ac \pm aL)^{1/2}] + BiH_0^{(1)}[2i(ac \pm aL)^{1/2}] = \pm (F + W)/(\sigma\beta a^{1/2}). \tag{4.8}$$

We discuss first the case $c=0$, i.e., $v(x)=x$ (the lower sign in (4.3) has no meaning here, since v must be $\geqq 0$). The free end of the cantilever is represented by $v=x=0$. Therefore, since $H_1^{(1)}(0)=\infty$, the constant B must be zero. Since $J_1(0)=0$ and $J_0(0)=1$, Eqs. (4.7′), (4.7″) require that $A=-F/(\sigma\beta a^{1/2})$, and Eq. (4.8) gives

$$J_0(2ia^{1/2}L^{1/2}) = 1 + (W/F).$$

If L and W are not related by this equation, the constant c must be distinct from zero. The determinant of the coefficients of (4.7′), (4.7″), considered as equations in A and B is, by a known relation of Bessel functions,[10]

$$J_0(z)H_1^{(1)}(z) - H_0^{(1)}(z)J_1(z) = -\pi^{-1}a^{-1/2}c^{-1/2}, \tag{4.9}$$

where $z=2ia^{1/2}c^{1/2}$. Since this cannot be zero it is seen that there are no solutions if $F=0$, which agrees with the general result of §3. If $F\neq 0$, the solutions of (4.7′), (4.7″) are

$$A = \pm \pi(\sigma\beta)^{-1}c^{1/2}FH_1^{(1)}(2ia^{1/2}c^{1/2}), \qquad B = \mp \pi(\sigma\beta)^{-1}c^{1/2}FiJ_1(2ia^{1/2}c^{1/2}). \tag{4.10}$$

Substitution from these into (4.8) gives a relation for W,[11]

$$\pi a^{1/2}c^{1/2}[J_1(z)H_0^{(1)}(\zeta) - H_1^{(1)}(z)J_0(\zeta)] = 1 + (W/F), \tag{4.11}$$

where $z=2i(ac)^{1/2}$, $\zeta=2i(ac\pm aL)^{1/2}$.

III). *$u(x)$ is proportional to $[v(x)]^n$*. This includes a circular cross section ($n=1$), a rectangular cross section of constant width ($n=0$), a rectangular cross section with the height proportional to the width ($n=1$), an elliptic cross section with axes proportional to each other ($n=1$). By a suitable choice of $u_1(t)$ and $v_1(t)$ we may reduce the problem to the case $u=v^n$. The first two equations in (3.2′) then give

$$x = \int_0^v v^{n+1}[C^2 + 2a(n+2)^{-1}(2n+3)^{-1}v^{2n+3}]^{-1/2}dv, \tag{4.12}$$

where C is a constant. The last equation of (3.2′) and Eq. (3.3′) give

$$\sigma\beta C = F/(n+2), \qquad W = [F^2 + (n+2)(2n+3)^{-1}2\gamma\sigma\alpha\beta v_{x=L}^{2n+3}]^{1/2} - F. \tag{4.13}$$

If $n=-1$ the cross sections have a constant area; this case gives elementary expressions for $u(x)$ and $v(x)$ but the width at the free end is infinite. When $n=1$, the in-

[11] By (4.3) the possible range of L is $(0, +\infty)$ or $(0, c)$. The left hand side of (4.11) is within this range a function of L which increases continuously from 1 to $+\infty$, as may be shown by the theory of Bessel functions and by (4.7′), (4.9), (4.10). Therefore for any assigned values of F, W, a, c there exists, in each case (4.3), one and only one value of L satisfying (4.11). Problems of this type in which the length L is not assigned but a given amount of material is to be distributed into a cantilever of uniform strength under the action of F and W or W alone have occurred in some biological fields (cf. C. Holtermann, *Schwendener's Vorlesungen über mechanische Probleme der Botanik*, Leipzig, 1909, pp. 18, 19 and O. Fischer, *Enc. d. math. Wiss.*, IV.8, p. 119).

tegral in (4.12) is hyperelliptic; when $n=0$, it is elliptic. If $F=0$, Eqs. (4.12) and (4.13) give

$$v = a[2(2n+3)(n+2)]^{-1}x^2. \tag{4.14}$$

IV. More general cases. Instead of u and v we introduce new variables $\omega = uv^2$, $\tau = 1/v$; ω is directly proportional to the section modulus, and τ inversely proportional to the radius of gyration of the cross section. From (3.2) we obtain

$$Fx + \alpha\gamma \int_0^x (x-\xi)\tau(\xi)\omega(\xi)d\xi = \sigma\beta\omega(x). \tag{4.15}$$

This is a Volterra integral equation in $\omega(x)$. From it, or directly from (3.2′), (3.3′) we have

$$\omega''(x) = a\tau(x)\omega(x), \quad \omega(0) = 0, \quad \sigma\beta\omega'(0) = F, \quad \sigma\beta\omega'(L) = F + W. \tag{4.16}$$

Since most of the so called special functions satisfy linear differential equations of second order, the first equation of (4.16) suggests the possibility of using such functions. The following are some results which may be easily checked. The constants p, q, s, a must satisfy the last three equations in (4.16).

IVa). $\tau(x) = p - qe^{2x}$, $\omega(x) = Z_m(ne^x)$, where Z_m = a cylindrical function (Bessel, Hankel, etc.), $m^2 = ap$, $n^2 = aq$.

IVb). $\tau(x) = p - q(\cosh x)^{-2}$, $\omega(x) = K_n^{(m)}(\tanh x)$, where $K_n^{(m)}$ = an associate Legendre function ($P_n^{(m)}$, $Q_n^{(m)}$), $m^2 = ap$, $n(n+1) = aq$.

IVc). $\tau(x) = p - q\cos x$, $\omega(x)$ = a function of an elliptic cylinder.[12]

IVd). $\tau(x) = (p - qx + x^2)/(4ax^2)$, $\omega(x)$ = a confluent hypergeometric function.[12]

IVe). $\tau(x) = (p - qx^s)/x^2$, $\omega(x) = x^{1/2}Z_m(nx^{s/2})$, where Z_m = a cylindrical function,[10] $m^2s^2 = 1 + 4ap$, $n^2s^2 = 4aq$. If $p = 0$, in order that v be finite s must be <2.

If the function $v(x) = 1/\tau(x)$ is assigned by means of any one of previous expressions for τ, the function $u = \omega\tau^2$ is determined by the corresponding expression for $\omega(x)$. In the case of a rectangular cross section, $v(x)$ represents the height and $u(x)$ the width.

5. The deflection curve. The curvature of the geometric axis of a beam of constant strength in bending is[1,2,4] $1/r = h/v(x)$ where $h = \sigma/(Ev_m)$, E being the modulus of elasticity and v_m the value of $v_1(t)$ at the point of maximum stress (cf. §2). We note that this equation is a form of the well-known relation $\sigma = Ey/r$. For small deflections the usual approximation is $1/r = d^2y/dx^2$. Thus

$$y(x) = -\int_0^x \varphi(x)dx, \qquad \varphi(x) = h\int_x^L [v(x)]^{-1}dx, \tag{5.1}$$

since

$$y(0) = 0, \qquad (dy/dx)_{x=L} = 0. \tag{5.2}$$

A simple formula for the deflection at the free end is obtained through integration of (5.1) by parts. Setting $-y(L) = Y$, we have

$$Y = h\left\{x\int_x^L [v(x)]^{-1}dx\right\}_{x=0}^{x=L} + h\int_0^L [v(x)]^{-1}xdx = h\int_0^L [v(x)]^{-1}xdx. \tag{5.3}$$

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[12] E. T. Whittaker-G. N. Watson, *Modern analysis*, 4th Edition, Cambridge University Press, 1935, pp. 337, 405.

It is seen from (5.3) that, if $v(x)$ tends to zero as kx^n with $n \geqq 2$ and k is constant, the deflection Y is infinite. This would occur, for instance, in the case corresponding to Eq. (4.14). Such a physically impossible conclusion may be explained by the fact that a large value of n implies a rapid variation of $v(x)$, i.e., a rapid change of the cross section, whereas the theory which was used is based on bending of beams of constant cross section.[13] More important still, the theory used in this paper neglects the shearing stresses in comparison with the bending stresses. Such a procedure is not permissible in the vicinity of the free end, and consequently it is understandable that the theoretical results for this part of the beam differ widely from reality.

6. More general loads. If $M(x)$ is the moment of the external load acting on the cantilever, we have instead of Eqs. (1.1''), (1.2')

$$M''(x) + \gamma A(x) = \sigma S''(x), \qquad \sigma S(0) = M(0), \qquad \sigma S'(0) = M'(0), \tag{6.1}$$

$$\sigma S'(L) = M'(L) + W. \tag{6.2}$$

For example, if the beam is acted upon by F and also by a load distributed uniformly along the axis of the beam of intensity T, we have $M(x) = Fx + \frac{1}{2}Tx^2$. If $v = \text{const.}$, we obtain by (6.1), (3.1), and (6.2),

$$u = [Fr \sinh (rx) + T \cosh (rx) - T]/(\alpha\gamma v) \tag{6.3}$$

$$\cosh (rL) + T(Fr)^{-1} \sinh (rL) = 1 + (W + TL)/F, \tag{6.4}$$

where $r = (a/v)^{1/2}$. If $T = 0$, Eqs. (6.3), (6.4) reduce to (4.1), (4.2). Eqs. (6.3), (6.4) may be easily generalized to the case $M(x) = \sum a_n x^n$.

7. Numerical examples. We consider a rectangular cross section of width 1 and height H. Then (cf. §3) $\alpha = H$, $v_m = H/2$, $\beta = H^2/6$, $a = 6\gamma(\sigma H)^{-1}$. Let $L = 10$ ft., $F = 9000$ lbs., $\sigma = 75000$ lbs./sq. ft., $\gamma = 150$ lbs./cu. ft., $E = 45 \times 10^7$ lbs./sq. ft. These values correspond to a certain type of concrete.

I). Cantilever of constant height (§4, Case I). We put $H = 1$ and assume the height $vH = v = 1.9$ ft. From (4.2) we obtain the weight $W = 3000$ lbs. We put $R = [6\gamma/(\sigma v)]^{1/2}$. Then $R = 0.0795$. From (4.1) we obtain the width

$$u(x) = FR(v\gamma)^{-1} \sinh (Rx) = 2.51 \sinh (0.0795x).$$

At the fixed end we then have $u = u(10) \approx 2.21$ ft. Equation (5.3) gives for the deflection $Y = \sigma L^2/Ev \approx 0.1$ in.

II). Cantilever with a linearly varying height (§4, Case II). Let the height at the fixed end be 2 ft. and at the free end 1/4 ft. In (4.3) we take $v(x) = c + x$. Since $Hc = 1/4$ ft., $H(c+10) = 2$ ft., we get $H = 7/40$, $c = 10/7$ ft. Eqs. (4.10) give $A = -69.1$, $B = 29.0$, and from Eqs. (4.4), (4.6) we obtain

$$u = v^{-3/2}[-69.1 iJ_1(i\xi) + 29.0 H_1^{(1)}(i\xi)], \quad \text{where} \quad \xi = 0.8(3v/7)^{1/2}. \tag{7.1}$$

At the fixed end $x = 10$, and we thus obtain $u \approx 2.2$ ft. At the free end, $v = c$ and by (4.4), (4.5) we have the general result $u = 0$. From (4.5') or (4.11) we get the weight $W \approx 4800$ lbs. From (5.3) the deflection is

$$Y = 2\sigma(EH)^{-1}[L - c \log_e (1 + Lc^{-1})] \approx 0.2 \text{ in.}$$

[13] A method which takes into account the variability of the cross section was worked out by J. Résal, *Résistance des matériaux*, Paris, 1898, pp. 393–405 for rectangular and double T cross sections of constant width.