ON THEODORSEN'S METHOD OF CONFORMAL MAPPING OF NEARLY CIRCULAR REGIONS*

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1. Introduction. In determining the complex velocity potential of the two-dimensional flow around an airfoil, one is lead to the problem of finding the analytic function which maps the exterior of a circle conformally onto that of a "nearly circular" contour. T. Theodorsen developed a method for the practical computation of this mapping function, a method which was later elaborated on in a joint paper by Theodorsen and I. E. Garrick.[1] Theodorsen reduces the problem of determining the mapping function to the solution of a certain non-linear integral equation which then is solved by successive approximations. In both papers examples of wing sections of airplanes are calculated demonstrating the use of the process and the rapidity with which it converges. However, the validity of the method from a mathematical point of view, such as the proof of the convergence of the successive approximations, is not discussed. The present paper is an attempt to supply such a discussion. Simple conditions on the nearly circular contour (essentially involving the tangent angle and the curvature) are established which insure the convergence of the process. The absolute value of the difference between the mapping function and the successive approximations is estimated. These estimates serve both to prove the convergence and to appraise the accuracy of the approximation. The analogous problem for the derivative of the mapping function is treated. (The derivative of the mapping function enters in the computation of the velocity and pressure distribution on the surface of the wing.) Finally, conditions are discussed under which the map of the circle by means of the successive approximations is star-shaped.

Although Theodorsen's method is of particular importance in the theory of airfoils, it represents the solution of a general problem in conformal mapping. For this reason all results of the present paper are derived for the "standard" case where the *interior* of a circle about the origin is mapped onto the *interior* of the nearly circular contour containing the origin under preservation of the positive line element at the origin. However, all results obtained remain the same for the mapping function of the *exteriors* and for a different normalization of the mapping function (see §3).

Sections 2–8 contain the actual results and proofs of the paper. To simplify the presentation some auxiliary results used in the text are listed in §9.

2. Theodorsen's integral equation and the successive approximations. Let C be a simple closed curve represented in polar co-ordinates by the equation $\rho=\rho(\theta)$ $(0\leqq\theta\leqq 2\pi)$, where $\rho(\theta)$ is absolutely continuous[2] and for some ϵ $(0<\epsilon<1)$,

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[1] T. Theodorsen, *Theory of wing sections of arbitrary shape*, NACA Tech. Rep. No. **411** (1931); T. Theodorsen and I. E. Garrick, *General potential theory of arbitrary wing sections*, NACA Tech. Rep. No. **452** (1934).

[2] A function $g(\theta)$ is absolutely continuous in an interval if its derivative $g'(\theta)$ exists for all θ of this

$$\frac{a}{1+\epsilon} \leqq \rho(\theta) \leqq a(1+\epsilon), \tag{2.1}$$

a being a positive constant, and

$$\left| \frac{\rho'(\theta)}{\rho(\theta)} \right| \leqq \epsilon. \tag{2.2}$$

Any curve C satisfying these conditions will be called a *nearly circular contour.*

Let us suppose that the function $w=f(z)$ maps the circle $|z|<1$ conformally onto the interior of C, and that $f(0)=0, f'(0)>0$. The function

$$F(z) = \log \frac{f(z)}{z} = \log \left| \frac{f(z)}{z} \right| + i \arg \frac{f(z)}{z}, \tag{2.3}$$

which is defined as the real-valued $\log f'(0)$ when $z=0$, is single-valued and analytic for $|z|<1$ and continuous for $|z|\leqq 1$. For $z=e^{i\phi}$ we write $\arg [f(e^{i\phi})e^{-i\phi}] = \theta(\phi)-\phi$, and therefore

$$F(e^{i\phi}) = \log \rho[\theta(\phi)] + i(\theta(\phi) - \phi). \tag{2.4}$$

Hence

$$\theta(\phi) - \phi = -\frac{1}{2\pi}\int_0^\pi \{\log \rho[\theta(\phi+t)] - \log \rho[\theta(\phi-t)]\} \cot \frac{t}{2}\, dt. \tag{2.5}$$

(The term $\arg [f(z)/z]_{z=0} = \arg f'(0)$, which should be added to the integral on the right, is zero.) Thus the function $F(e^{i\phi})$ and hence $f(z)$ may be found by solving this integral equation for $\theta(\phi)$. The *existence* of a continuous solution of this integral equation is assured by Riemann's mapping theorem. This solution is also *unique* as is shown in §9(a). In order to *compute* the solution we follow Theodorsen and form the successive approximations

$$\left.\begin{aligned} &\theta_0(\phi) \equiv \phi, \\ &\theta_n(\phi) - \phi = -\frac{1}{2\pi}\int_0^\pi \{\log \rho[\theta_{n-1}(\phi+t)] - \log \rho[\theta_{n-1}(\phi-t)]\} \cot \frac{t}{2}\, dt, \\ &\qquad\qquad (n = 1, 2, \cdots). \end{aligned}\right\} \tag{2.6}$$

The functions $\theta_n(\phi)$ are continuous for $0 \leqq \phi \leqq 2\pi$; in fact, they are absolutely continuous and the squares of their first derivatives are integrable (for the proof of this, see §9(b)); $\theta_n(\phi)-\phi$ is a conjugate function of $\log \rho[\theta_{n-1}(\phi)]$.

We shall show that *the sequence $\theta_n(\phi)$ converges uniformly to $\theta(\phi)$ as $n\to\infty$*. Hence, also $\log \rho[\theta_n(\phi)]$ converges uniformly to $\log \rho[\theta(\phi)]$ as $n\to\infty$, so that the functions

$$F_0(e^{i\theta}) \equiv \log a, \quad F_n(e^{i\phi}) = \log \rho[\theta_{n-1}(\phi)] + i(\theta_n(\phi) - \phi) \quad (n \geqq 1), \tag{2.7}$$

may be used to compute $F(e^{i\phi})$ with any desired degree of accuracy.

Let $F_n(z)$ denote the function which is analytic for $|z|<1$ and assumes the boundary values $F_n(e^{i\phi})$ for $|z|=1$. By the principle of the maximum modulus the uniform

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interval except possibly for a set of Lebesgue measure zero and if $\int_a^b g'(\theta)d\theta = g(b)-g(a)$ for every a and b of this interval. In order to establish the convergence of Theodorsen's method under reasonably general conditions, we employ the integral of Lebesgue.

convergence of $F_n(e^{i\phi})$ to $F(e^{i\phi})$ implies that $F_n(z)$ converges to $F(z)$ uniformly for $|z| \leqq 1$, and thus the functions $f_n(z) = ze^{F_n(z)}$ converge uniformly for $|z| \leqq 1$ to the mapping function $f(z)$.

In order to prove the convergence of the functions $\theta_n(\phi)$ and $\theta_n'(\phi)$ we shall derive estimates for the differences $|\theta_n(\phi) - \theta(\phi)|$ and $|\theta_n'(\phi) - \theta'(\phi)|$ in terms of ϵ and n. These differences approach zero as $n \to \infty$, and will at the same time permit us to appraise the degree of accuracy of the nth approximation.

Remark. Theodorsen considers the case where the exterior of a circle $|\zeta| = R$ is mapped onto the exterior of a "nearly circular" closed curve Γ whereby the mapping function $\omega = g(\zeta)$ is so normalized that $\lim_{\zeta\to\infty} \omega/\zeta = 1$. This case is immediately reduced to the one considered above by means of the transformations $w = \omega^{-1}$ and $z = R/\zeta$. Let us suppose that Γ is represented by the equation $r = r(\Theta)$ $(0 \leqq \Theta \leqq 2\pi)$, where, for some positive b and $0 < \epsilon < 1$, $b(1+\epsilon)^{-1} \leqq r(\Theta) \leqq b(1+\epsilon)$ and $|r'(\Theta)/r(\Theta)| \leqq \epsilon$. Then the function $w = f(z) = 1/g(\zeta)$, where $\zeta = R/z$, maps the circle $|z| < 1$ onto the interior of the nearly circular contour C represented by the equation $\rho = \rho(\theta) = 1/r(\Theta)$, where $\theta = -\Theta$ and $\rho(\theta)$ satisfies the conditions (2.1) and (2.2) with $a = b^{-1}$. For $\zeta = Re^{i\psi}$, we write arg $[g(\zeta)/\zeta] = \Theta(\psi) - \psi$, where arg $[g(\zeta)/\zeta]$ is defined as 0 when $\zeta = \infty$. Then, for $\zeta = Re^{i\psi}$ and $z = e^{i\phi}$ where $\phi = -\psi$,

$$
\begin{aligned}
\log \frac{g(\zeta)}{\zeta} &= \log r[\Theta(\psi)] + i(\Theta(\psi) - \psi) - \log R \\
&= -\log \frac{f(z)}{z} - \log R \\
&= -\log \rho[\theta(\phi)] - i(\theta(\phi) - \phi) - \log R.
\end{aligned}
$$

Thus one can form the successive approximations $\Theta_n(\psi)$ for the function $\Theta(\psi)$ in the same manner as the $\theta_n(\phi)$ are formed for $\theta(\phi)$. Furthermore, $\Theta_n(\psi) = -\theta_n(\phi)$, $\psi = -\phi$ and $\Theta_n(\psi) - \Theta(\psi) = -(\theta_n(\phi) - \theta(\phi))$. Hence any bound obtained for $|\theta_n(\phi) - \theta(\phi)|$ is also a bound for $|\Theta_n(\psi) - \Theta(\psi)|$, and the same remark applies to the derivatives of these differences.

3. Statement of results. We shall prove the following estimates:

I. *If C is a nearly circular contour, and if $\theta_n(\phi)$ and $\theta(\phi) = \arg f(e^{i\phi})$ are defined by* (2.6) *and* (2.4), *respectively, then*

$$
|\theta_n(\phi) - \theta(\phi)| \leqq 2\left(\frac{\pi^2}{1-\epsilon^2}\right)^{1/4} \epsilon^{(n+2)/2}. \tag{3.1}
$$

The bound for $|\theta_n(\phi) - \theta(\phi)|$ obtained here approaches zero as $n \to \infty$ (since $0 < \epsilon < 1$) and is therefore sufficient to establish the convergence of the functions $\theta_n(\phi)$ to $\theta(\phi)$. However, a bound which *converges to zero more rapidly* can be found if a further assumption regarding C is made.

II. *If C is a nearly circular contour and if*

$$
\left| \frac{\rho'(\theta_2)}{\rho(\theta_2)} - \frac{\rho'(\theta_1)}{\rho(\theta_1)} \right| \leqq \epsilon |\theta_2 - \theta_1|, \tag{3.2}
$$

ϵ being the same as in (2.1), *then*

$$
|\theta_n(\phi) - \theta(\phi)| \leqq (2\pi A(n+1))^{1/2} \epsilon^{n+1}, \tag{3.3}
$$

where $A = 4^{\epsilon} e^{\epsilon^2}$.

The following result is obtained for the derivatives $\theta_n'(\phi)$.

III. *If C is a nearly circular contour, if* (3.2) *holds, and if* $p(\theta)=d[\rho'(\theta)/\rho(\theta)]/d\theta$ *satisfies the condition*

$$|p(\theta_2)-p(\theta_1)|\leqq \epsilon|\theta_2-\theta_1|, \tag{3.4}$$

ϵ being the same as in (2.1), *then*

$$|\theta_n'(\phi)-\theta'(\phi)|\leqq \sqrt{2\pi\sigma_n}\,(A(n+1))^{3/2}\epsilon^{n+1}, \tag{3.5}$$

where $A=4^{\epsilon}e^{\epsilon^2}$ *and*

$$\sigma_1=1+\epsilon,\qquad \sigma_n=(1+\epsilon)\prod_{k=2}^{n}(1+\epsilon^k\sqrt{2\pi A k}). \tag{3.6}$$

For all n,

$$\sigma_n\leqq(1+\epsilon)\exp\left[2\epsilon^2\sqrt{\pi A}\,(1-\epsilon)^{-3/2}\right], \tag{3.7}$$

so that σ_n is bounded if $0<\epsilon<1$.

Estimates for the difference $|F_n(z)-F(z)|$, $|z|\leqq 1$, may be obtained from those for $|\theta_n(\phi)-\theta(\phi)|$. For by (2.2),

$$|F_n(e^{i\phi})-F(e^{i\phi})|\leqq\left\{\epsilon^2(\theta_{n-1}(\phi)-\theta(\phi))^2+(\theta_n(\phi)-\theta(\phi))^2\right\}^{1/2},$$

and for $|z|\leqq 1$

$$|F_n(z)-F(z)|\leqq\max_{\phi}|F_n(e^{i\phi})-F(e^{i\phi})|.$$

Thus, for example, in case II we find by use of (3.3) that

$$|F_n(z)-F(z)|\leqq 2(A\pi(n+\tfrac{1}{2}))^{1/2}\epsilon^{n+1}.$$

Hence, if $0<\epsilon<1$, the successive approximations $F_n(z)$ converge uniformly to $F(z)=\log[f(z)/z]$ when $|z|\leqq 1$. An analogous statement applies to the derivatives $\partial[F_n(re^{i\phi})]/\partial\phi$ and $\partial[F(re^{i\phi})]/\partial\phi$.

To prove the three theorems I, II, and III, we shall first derive bounds for the square means

$$\left.\begin{aligned}M_n^2=\frac{1}{2\pi}\int_0^{2\pi}(\theta_n(\phi)-\theta(\phi))^2d\phi,\qquad M_n'^2=\frac{1}{2\pi}\int_0^{2\pi}(\theta_n'(\phi)-\theta'(\phi))^2d\phi,\\ M_n''^2=\frac{1}{2\pi}\int_0^{2\pi}(\theta_n''(\phi)-\theta''(\phi))^2d\phi.\end{aligned}\right\} \tag{3.8}$$

The above results will then be obtained by use of the inequalities (see §4(c))

$$|\theta_n(\phi)-\theta(\phi)|\leqq(2\pi M_nM_n')^{1/2},\qquad |\theta_n'(\phi)-\theta'(\phi)|\leqq(2\pi M_n'M_n'')^{1/2}.$$

The functions $f_n(z)=ze^{F_n(z)}$ map the circle $|z|=1$ onto closed curves C_n. Since the functions $f_n(z)$ are to be used as approximations to the mapping function $f(z)$, it is essential to know that the C_n are simple closed curves. This will certainly be the case if the C_n are *star-shaped* with respect to the origin. (A closed curve is star-shaped with respect to the origin if every ray from the origin intersects the curve in exactly one point.) Knowing that C_n is star-shaped has the additional advantage[3] that $\theta_n(\phi)$

[3] Cf. Theodorsen and Garrick, l.c., pp. 184–185.

is then an increasing function of ϕ and therefore possesses a unique inverse function $\phi=\phi_n(\theta)$. This permits us to form immediately the inverse $z=e^{i\phi_n(\theta)}$ of the mapping function $w=f(z)$ for w on C_n. We examine therefore the question when the C_n are star-shaped, and obtain the following result:

IV. *If C is a nearly circular contour and if the condition* (3.2) *is satisfied, then the curve C_1 is star-shaped with respect to the origin if $\epsilon\leqq(2\log 2)^{-1}$, C_2 if $\epsilon\leqq 0.34$, C_3 if $\epsilon\leqq 0.31$, and C_4 if $\epsilon=0.3$. For $n\geqq 4$ all C_n are star-shaped if $\epsilon\leqq 0.295$.*

This result is derived by examining the values of ϵ for which $|\theta_n'(\phi)-1|\leqq 1$, so that $\theta_n'(\phi)\geqq 0$ and $\theta_n(\phi)$ is therefore monotone increasing. For large values of n ($n\geqq 4$) a more favorable estimate for ϵ may be obtained by making use of (3.5) and of a lower bound for $\theta'(\phi)$ which is given in §9(d).

4. Proof of I. (a) Estimate of M_n. Let $F(e^{i\phi})$ and $F_n(e^{i\phi})$ be the functions in (2.4) and (2.7). Because of the representations of $\theta(\phi)-\phi$ and $\theta_n(\phi)-\phi$ by means of the integrals (2.5) and (2.6), respectively, we have

$$\int_0^{2\pi}(\theta(\phi)-\phi)d\phi=0,\qquad \int_0^{2\pi}(\theta_n(\phi)-\phi)d\phi=0. \tag{4.1}$$

We now apply the following well known theorem:[4] *If the function $g(\phi)$ is real-valued, periodic (period 2π), and $(g(\phi))^2$ is integrable (in the sense of Lebesgue) $0\leqq\phi\leqq 2\pi$, and if $\bar g(\phi)$ is a conjugate function of $g(\phi)$ (then surely existing), then*

$$\frac{1}{2\pi}\int_0^{2\pi}[\bar g(\phi)]^2d\phi+\alpha^2=\frac{1}{2\pi}\int_0^{2\pi}[g(\phi)]^2d\phi+\beta^2, \tag{4.2}$$

where

$$\alpha=\frac{1}{2\pi}\int_0^{2\pi}g(\phi)d\phi,\qquad \beta=\frac{1}{2\pi}\int_0^{2\pi}\bar g(\phi)d\phi.$$

Applying this with $g(\phi)+i\bar g(\phi)=F_n(e^{i\phi})-F(e^{i\phi})$ and observing that $\beta=0$ (because of (4.1)), we obtain

$$M_n^2=\frac{1}{2\pi}\int_0^{2\pi}(\theta_n(\phi)-\theta(\phi))^2d\phi\leqq\frac{1}{2\pi}\int_0^{2\pi}\{\log\rho[\theta_{n-1}(\phi)]-\log\rho[\theta(\phi)]\}^2d\phi. \tag{4.3}$$

By hypothesis (2.2),

$$|\log\rho[\theta_{n-1}(\phi)]-\log\rho[\theta(\phi)]|\leqq\epsilon|\theta_{n-1}(\phi)-\theta(\phi)|,$$

and therefore

$$M_n^2\leqq\epsilon^2\frac{1}{2\pi}\int_0^{2\pi}(\theta_{n-1}(\phi)-\theta(\phi))^2d\phi\leqq\epsilon^2M_{n-1}^2,$$

or

$$M_n\leqq\epsilon M_{n-1},\qquad M_n\leqq M_0\epsilon^n.$$

For $n=0$ we obtain from (4.2) by use of (2.1),

$$M_0^2=\frac{1}{2\pi}\int_0^{2\pi}(\theta(\phi)-\phi)^2d\phi\leqq\frac{1}{2\pi}\int_0^{2\pi}\left(\log\frac{\rho[\theta(\phi)]}{a}\right)^2d\phi\leqq\epsilon^2.$$

[4] See, for example, A. Zygmund, *Trigonometric series*, Warsaw, 1935, p. 76 (Eq. (4)).

Thus we have proved that *if* $\rho(\theta)$ *satisfies hypotheses* (2.1) *and* (2.2),

$$M_n \leqq \epsilon^{n+1}. \tag{4.4}$$

(b). ESTIMATE OF M_n'. It follows from §9(b) and §9(c) that $F_n(e^{i\phi})$ and $F(e^{i\phi})$ are absolutely continuous and that $\{d[F_n(e^{i\phi})]/d\phi]\}^2$ and $\{d[F(e^{i\phi})]/d\phi\}^2$ are integrable. Furthermore, because of the absolute continuity of $F_n(e^{i\phi})-F(e^{i\phi})$, the imaginary part of the derivative $d\{F_n(e^{i\phi})-F(e^{i\phi})\}/d\phi$ is a conjugate function of the real part. Finally,

$$\int_0^{2\pi}\frac{d}{d\phi}F(e^{i\phi})d\phi = [F(e^{i\phi})]_{\phi=0}^{\phi=2\pi} = 0, \qquad \int_0^{2\pi}\frac{d}{d\phi}F_n(e^{i\phi})d\phi = 0.$$

Hence, applying (4.2) with $g(\phi)+i\bar{g}(\phi)=d[F_n(e^{i\phi})-F(e^{i\phi})]/d\phi$, we obtain[5]

$$\begin{aligned} M_n'^2 &= \frac{1}{2\pi}\int_0^{2\pi}(\theta_n'(\phi)-\theta'(\phi))^2d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{\rho'}{\rho}[\theta_{n-1}(\phi)]\theta_{n-1}'(\phi)-\frac{\rho'}{\rho}[\theta(\phi)]\theta'(\phi)\right\}^2 d\phi. \end{aligned} \tag{4.5}$$

By (2.2) we have (omitting the argument ϕ in the integrands)

$$M_n'^2 \leqq 2\epsilon^2\frac{1}{2\pi}\int_0^{2\pi}(\theta_{n-1}'^2+\theta'^2)d\phi. \tag{4.6}$$

As is shown in §9(c),

$$\frac{1}{2\pi}\int_0^{2\pi}\theta'^2d\phi \leqq \frac{1}{1-\epsilon^2}\cdot \tag{4.7}$$

Furthermore, applying (4.2) with $g(\phi)+i\bar{g}(\phi)=d[F_n(e^{i\phi})]/d\phi$, we obtain by (2.2),

$$\frac{1}{2\pi}\int_0^{2\pi}(\theta_n'-1)^2d\phi = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\rho'}{\rho}[\theta_{n-1}]\theta_{n-1}'\right)^2d\phi \leqq \epsilon^2\frac{1}{2\pi}\int_0^{2\pi}\theta_{n-1}'^2d\phi,$$

or

$$\frac{1}{2\pi}\left\{\int_0^{2\pi}\theta_n'^2d\phi - 2\int_0^{2\pi}\theta_n'd\phi + 2\pi\right\} \leqq \epsilon^2\frac{1}{2\pi}\int_0^{2\pi}\theta_{n-1}'^2d\phi.$$

Since $\int_0^{2\pi}\theta_n'd\phi = 2\pi$, we find that

$$\frac{1}{2\pi}\int_0^{2\pi}\theta_n'^2d\phi - 1 \leqq \epsilon^2\frac{1}{2\pi}\int_0^{2\pi}\theta_{n-1}'^2d\phi.$$

If we set

$$m_n^2 = \frac{1}{2\pi}\int_0^{2\pi}\theta_n'^2d\phi,$$

we have $m_n^2 \leqq 1+m_{n-1}^2\epsilon^2$, and therefore

$$m_n^2 \leqq 1+\epsilon^2+\epsilon^4+\cdots+\epsilon^{2n}m_0^2.$$

Since

[5] The notation $\frac{\rho'}{\rho}[\theta]$ or $(\rho'/\rho)[\theta]$ means $\rho'(\theta)/\rho(\theta)$.

$$m_0^2 = \frac{1}{2\pi}\int_0^{2\pi} d\phi = 1,$$

we obtain

$$m_n^2 \leqq (1 - \epsilon^2)^{-1}. \tag{4.8}$$

Thus by (4.6), (4.7), and (4.8),

$$M_n'^2 \leqq 4\epsilon^2/(1 - \epsilon^2). \tag{4.9}$$

(c) ESTIMATE OF $|\theta_n(\phi) - \theta(\phi)|$. To complete the proof we now apply the following theorem: *If $g(\phi)$ is a real-valued, absolutely continuous and periodic function (period 2π) and if $(g'(\phi))^2$ is integrable, then for any ϕ_0,*

$$[g(\phi)]^2 - [g(\phi_0)]^2 \leqq 2\pi M M', \tag{4.10}$$

where

$$M^2 = \frac{1}{2\pi}\int_0^{2\pi} [g(\phi)]^2 d\phi, \qquad M'^2 = \frac{1}{2\pi}\int_0^{2\pi} [g'(\phi)]^2 d\phi.$$

The factor 2π is the "best possible" constant; it cannot be replaced by a smaller one.[6]

Let $g(\phi) = \theta_n(\phi) - \theta(\phi)$. Since then $\int_0^{2\pi} g(\theta) d\phi = 0$, there exists a value ϕ_0 such that $g(\phi_0) = 0$. Hence

$$|\theta_n(\phi) - \theta(\phi)| \leqq (2\pi M_n M_n')^{1/2}.$$

Using (4.4) and (4.9), we find (3.1).

5. Proof of II. (a) ESTIMATE OF M_n'. Under the present hypotheses an estimate for M_n' sharper than (4.9) may be obtained. *We shall prove that if $\rho(\theta)$ satisfies* (2.1), (2.2) *and* (3.2), *then*

$$M_n' \leqq A(n + 1)\epsilon^{n+1}, \qquad (A = 4^\epsilon e^{\epsilon^2}). \tag{5.1}$$

Using the relation (4.5) we obtain for $n \geqq 1$,

$$M_n' = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left[\left(\frac{\rho'}{\rho}[\theta_{n-1}] - \frac{\rho'}{\rho}[\theta]\right)\theta' + \frac{\rho'}{\rho}[\theta_{n-1}](\theta_{n-1}' - \theta')\right]^2 d\phi\right\}^{1/2}$$

$$\leqq \left\{\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\rho'}{\rho}[\theta_{n-1}] - \frac{\rho'}{\rho}[\theta]\right)^2 \theta'^2 d\phi\right\}^{1/2}$$

$$+ \left\{\frac{1}{2\pi}\int_0^{2\pi}(\theta_{n-1}' - \theta')^2\left(\frac{\rho'}{\rho}[\theta_{n-1}]\right)^2 d\phi\right\}^{1/2},$$

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[6] To prove (4.10), we note first that for $0 \leqq \phi \leqq 2\pi$, $0 \leqq \phi_0 \leqq 2\pi$,

$$g^2(\phi) - g^2(\phi_0) = 2\int_{\phi_0}^{\phi} g(t)g'(t)dt = 2\int_{\phi_0}^{\phi-2\pi} g(t)g'(t)dt. \tag{*}$$

Since

$$\left|\int_{\phi_0}^{\phi} |gg'|\,dt\right| + \left|\int_{\phi_0}^{\phi-2\pi} |gg'|\,dt\right| = \int_{\phi-2\pi}^{\phi} |gg'|\,dt = \int_0^{2\pi} |gg'|\,dt,$$

one of the two integrals in (*) does not exceed $\frac{1}{2}\int_0^{2\pi}|gg'|dt$. Hence, by the inequality of Schwarz,

$$[g(\phi)]^2 - [g(\phi_0)]^2 \leqq \int_0^{2\pi} |gg'|\,dt \leqq 2\pi M M'.$$

Applying (4.10) with $g(\phi) = \cos^n \phi\,(\phi_0 = \frac{1}{2}\pi)$ and letting $n \to \infty$, we see that the constant 2π cannot be replaced by a smaller one.

by Minkowski's inequality. Under the present hypotheses we have by §9(d),

$$0 < \theta'(\phi) \leqq A = 4^{\epsilon} e^{\epsilon^2}.$$

Hence, by (3.2) and (2.2),

$$M_n' \leqq \epsilon A \left\{ \frac{1}{2\pi} \int_0^{2\pi} (\theta_{n-1} - \theta)^2 d\phi \right\}^{1/2} + \epsilon M_{n-1}' = \epsilon(AM_{n-1} + M_{n-1}'),$$

and, therefore, by (4.4),

$$M_n' \leqq \epsilon(A\epsilon^n + M_{n-1}'), \qquad (n \geqq 1). \tag{5.2}$$

For $n = 0$, we have, using (4.2) and (2.2),

$$M_0'^2 = \frac{1}{2\pi} \int_0^{2\pi} (\theta' - 1)^2 d\phi = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\rho'}{\rho} [\theta]\theta' \right)^2 d\phi \leqq \epsilon^2 \frac{A}{2\pi} \int_0^{2\pi} \theta' d\phi = \epsilon^2 A.$$

This inequality proves (5.1) for $n = 0$. For $n \geqq 1$, (5.1) is easily seen to be true by induction. Assuming that it holds for some $n \geqq 0$, we obtain by use of (5.2),

$$M_{n+1} \leqq \epsilon(A\epsilon^{n+1} + A(n+1)\epsilon^{n+1}) = A(n+2)\epsilon^{n+2},$$

i.e., (5.1) is also true for $n+1$.

(b). ESTIMATE OF $|\theta_n(\phi) - \theta(\phi)|$. Applying (4.10), (4.4) and (5.1), we find

$$|\theta_n(\phi) - \theta(\phi)| \leqq (2\pi M_n M_n')^{1/2} \leqq (2\pi A(n+1))^{1/2} \epsilon^{n+1}.$$

6. Proof of III. (a). SOME PROPERTIES OF THE FUNCTIONS $F(e^{i\phi})$ AND $F_n(e^{i\phi})$. Because of the hypothesis (3.4), $F(e^{i\phi})$ has a continuous second derivative for[7] $0 \leqq \phi \leqq 2\pi$. The same is true for all $F_n(e^{i\phi})$ as is shown in §9(e). Differentiating $F(e^{i\phi})$ and $F_n(e^{i\phi})$ twice with respect to ϕ, we obtain

$$\frac{dF}{d\phi} = \frac{\rho'}{\rho} [\theta]\theta' + i(\theta' - 1), \qquad \frac{d^2F}{d\phi^2} = p[\theta]\theta'^2 + \frac{\rho'}{\rho} [\theta]\theta'' + i\theta'',$$

$$\frac{dF_n}{d\phi} = \frac{\rho'}{\rho} [\theta_{n-1}]\theta_{n-1}' + i(\theta_n' - 1), \qquad \frac{d^2F_n}{d\phi^2} = p[\theta_{n-1}]\theta_{n-1}'^2 + \frac{\rho'}{\rho} [\theta_{n-1}]\theta_{n-1}'' + i\theta_n''.$$

The present proof is similar to that of (II) and we estimate first

$$M_n'' = \left\{ \frac{1}{2\pi} \int_0^{2\pi} (\theta_n'' - \theta'')^2 d\phi \right\}^{1/2}.$$

We prove that, for $n \geqq 1$,

$$M_n'' \leqq A^2(n+1)^2 \sigma_n \epsilon^{n+1}, \tag{6.1}$$

where $A = 4^{\epsilon} e^{\epsilon^2}$ *and* σ_n *is defined in* (3.6).

(b). PROOF OF THE INEQUALITY (6.1). Since

$$\int_0^{2\pi} \frac{d^2}{d\phi^2} (F_n(e^{i\phi}) - F(e^{i\phi})) d\phi = 0,$$

[7] See, for example, S. E. Warschawski, *On the higher derivatives at the boundary in conformal mapping*, Trans. Amer. Math. Soc. **38**, 326 (Theorem III), (1935).

we have, applying (4.2),

$$(M''_{n+1})^2 = \frac{1}{2\pi}\int_0^{2\pi} \Big\{(p[\theta_n] - p[\theta])\theta'^2 + p[\theta_n](\theta_n'^2 - \theta'^2) + \left(\frac{\rho'}{\rho}[\theta_n] - \frac{\rho'}{\rho}[\theta]\right)\theta'' + \frac{\rho'}{\rho}[\theta_n](\theta_n'' - \theta'')\Big\}^2 d\phi. \tag{6.2}$$

Because of (3.2),

$$| p(\theta) | \leqq \epsilon. \tag{6.3}$$

Using (9.3), (3.4), (6.3), (3.2), and (2.2), we find that

$$(M''_{n+1})^2 \leqq \frac{\epsilon^2}{2\pi}\int_0^{2\pi} \{A^2 | \theta_n - \theta | + | \theta_n'^2 - \theta'^2 | + | \theta'' | | \theta_n - \theta | + | \theta_n'' - \theta'' | \}^2 d\phi.$$

If M_n and M_n' are defined as in (3.8), we have by Minkowski's inequality:

$$M''_{n+1} \leqq \epsilon\left[A^2 M_n + \left\{\frac{1}{2\pi}\int_0^{2\pi} (\theta_n'^2 - \theta'^2)^2 d\phi\right\}^{1/2} + \left\{\frac{1}{2\pi}\int_0^{2\pi} (\theta_n'')^2(\theta_n - \theta)^2 d\phi\right\}^{1/2} + M_n''\right]. \tag{6.4}$$

Since by (9.4),

$$M_0'' = \left\{\frac{1}{2\pi}\int_0^{2\pi} (\theta'')^2 d\phi\right\}^{1/2} \leqq A^{3/2}\sqrt{2}\,\epsilon, \tag{6.5}$$

and by (3.3),

$$| \theta_n(\phi) - \theta(\phi) | \leqq (2\pi A(n+1))^{1/2}\epsilon^{n+1},$$

we have

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} (\theta_n - \theta)^2\theta''^2 d\phi\right\}^{1/2} \leqq (2\pi A(n+1))^{1/2}A^{3/2}\sqrt{2}\,\epsilon^{n+2} = 2A^2\epsilon^{n+2}\sqrt{\pi(n+1)}. \tag{6.6}$$

Next, applying the theorem of §4(c) with $g(\phi) = \theta_n'(\phi) - \theta'(\phi)$, we obtain

$$(\theta_n' - \theta')^2 \leqq 2\pi M_n' M_n'',$$

and taking the square root and using (9.3), we have

$$| \theta_n' + \theta' | \leqq 2A + \sqrt{2\pi M_n' M_n''}.$$

Hence

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} (\theta_n'^2 - \theta'^2)^2 d\phi\right\}^{1/2} \leqq M_n'(2A + \sqrt{2\pi M_n' M_n''}) = 2AM_n' + M_n'\sqrt{2\pi M_n' M_n''}.$$

Applying the inequality[8] $M_n' \leqq \sqrt{M_n M_n''}$ to the factor M_n' of the square root we find that

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[8] If we set $g(\phi) = \theta_n(\phi) - \theta(\phi)$, we have by integration by parts

$$\int_0^{2\pi} (g'(\phi))^2 d\varphi = \left| [g(\phi)g'(\phi)]_0^{2\pi} - \int_0^{2\pi} g(\phi)g''(\phi)d\phi \right| \leqq \int_0^{2\pi} | g(\phi)g''(\phi)| \, d\phi \leqq \left\{ \int_0^{2\pi} [g(\phi)]^2 d\phi \cdot \int_0^{2\pi} (g''(\phi))^2 d\phi \right\}^{1/2}.$$

This proves the inequality of the text.

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}(\theta_n'^2-\theta'^2)^2 d\phi\right\}^{1/2} \leqq 2AM_n' + M_n''\sqrt{2\pi M_n M_n'}. \tag{6.7}$$

Thus we obtain from (6.4) using (4.4), (6.7), (6.6) and (5.1),

$$M_{n+1}'' \leqq \epsilon\{A^2\epsilon^{n+1} + 2A^2(n+1)\epsilon^{n+1} + 2A^2\epsilon^{n+2}\sqrt{\pi(n+1)} + (1+\epsilon^{n+1}\sqrt{2\pi A(n+1)})M_n''\},$$

and therefore

$$M_{n+1}'' \leqq A^2\epsilon^{n+2}\left\{1 + 2(n+1) + 2\epsilon\sqrt{\pi(n+1)} + (1+\epsilon^{n+1}\sqrt{2\pi A(n+1)})\frac{M_n''}{A^2\epsilon^{n+1}}\right\}. \tag{6.8}$$

Assuming now that (6.1) is true for some $n \geqq 2$, we see from this inequality that (6.1) also holds for $n+1$. For, if we substitute in (6.8) for M_n'' the right-hand side of (6.1), we find that

$$M_{n+1}'' \leqq A^2\epsilon^{n+2}\{1 + 2(n+1) + 2\epsilon\sqrt{\pi(n+1)} + (1+\epsilon^{n+1}\sqrt{2\pi A(n+1)})(n+1)^2\sigma_n\}.$$

For $n \geqq 2$,

$$1 + 2(n+1) + 2\epsilon\sqrt{\pi(n+1)} < [1+2(n+1)](1+\epsilon) < [1+2(n+1)]\sigma_{n+1},$$

and therefore

$$M_{n+1}'' \leqq A^2\epsilon^{n+2}\sigma_{n+1}(1+2(n+1)+(n+1)^2) = A^2(n+2)^2\sigma_{n+1}\epsilon^{n+2}.$$

To complete the induction we show that (6.1) holds for $n=1$ and $n=2$. From (6.2) with $n=0$, we find that

$$M_1' = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left[(p(\phi)-p[\theta(\phi)])\theta'^2 + p(\phi)(1-\theta'^2) - \frac{\rho'}{\rho}[\theta(\phi)]\theta''\right]^2 d\phi\right\}^{1/2}$$

$$\leqq \epsilon\left\{A^2M_0 + \left(\frac{1}{2\pi}\int_0^{2\pi}(1-\theta'^2)^2 d\phi\right)^{1/2} + M_0''\right\},$$

by Minkowski's inequality and (3.4), (6.3), and (2.2). Applying (4.4), (9.3), (5.1), and observing that by (9.4) $M_0'' \leqq A^2\epsilon(1+\epsilon)$, we find that

$$M_1'' \leqq \epsilon^2(A^2 + (1+A)A + A^2(1+\epsilon)) = A^2\epsilon^2\left(2 + \frac{1+A}{A} + \epsilon\right).$$

Since $A \geqq 1$, $(1+A)/A \leqq 2$, and therefore

$$M_1'' \leqq A^2\epsilon^2(4+\epsilon) < 4A^2\epsilon^2(1+\epsilon). \tag{6.9}$$

To prove (6.1) for $n=2$, we apply (6.8) with $n=1$ and M_1'' replaced by $A^2\epsilon^2(4+\epsilon)$ (see (6.9)), to obtain

$$M_2'' \leqq A^2\epsilon^3[5 + 2\epsilon\sqrt{2\pi} + (1+2\epsilon^2\sqrt{\pi A})(4+\epsilon)].$$

Since $2\sqrt{2\pi} < 6$ and $1+2\epsilon^2\sqrt{\pi A} > 1$,

$$5 + 2\epsilon\sqrt{2\pi} + (1+2\epsilon^2\sqrt{\pi A})(4+\epsilon) < (5+6\epsilon+4+\epsilon)(1+2\epsilon^2\sqrt{\pi A})$$
$$< 9(1+\epsilon)(1+2\epsilon^2\sqrt{\pi A}) = 3^2\sigma_2,$$

and, therefore

$$M_2''\leqq 3^2\sigma_2 A^2\epsilon^3.$$

(c). ESTIMATE OF $|\theta_n'(\phi)-\theta'(\phi)|$. Applying the theorem of §4(c) with $g(\phi)=\theta_n'(\phi)-\theta'(\phi)$, we obtain from (5.1) and (6.1)

$$|\theta_n'(\phi)-\theta'(\phi)|\leqq\sqrt{2\pi\sigma_n}\,(A(n+1))^{3/2}\epsilon^{n+1}.$$

(d). PROOF OF (3.7). To estimate σ_n we first note that

$$\prod_{k=2}^{n}(1+\epsilon^k\sqrt{2\pi A k})\leqq\exp\left[\sqrt{2\pi A}\sum_{k=2}^{n}\epsilon^k\sqrt{k}\right].$$

Now

$$\sum_{k=2}^{n}\epsilon^k\sqrt{k}=\epsilon\sum_{k=2}^{n}\epsilon^{(k-1)/2}(\sqrt{k}\,\epsilon^{(k-1)/2})\leqq\epsilon\left\{\sum_{k=2}^{n}\epsilon^{k-1}\sum_{k=2}^{n}k\epsilon^{k-1}\right\}^{1/2},$$

by the inequality of Schwarz. Hence

$$\sum_{k=2}^{n}\epsilon^k\sqrt{k}\leqq\epsilon\left\{\frac{\epsilon}{1-\epsilon}\cdot\left(\frac{1}{(1-\epsilon)^2}-1\right)\right\}^{1/2}<\frac{\epsilon^2\sqrt{2}}{(1-\epsilon)^{3/2}}\cdot$$

We find therefore that $\sigma_n<(1+\epsilon)\exp\left[2\sqrt{\pi A}\,\epsilon^2(1-\epsilon)^{-3/2}\right]$.

7. An integral representation for $\theta_n'(\phi)$. We shall discuss now the conditions under which the images C_n of the unit circle by means of the functions $w=f_n(z)=ze^{F_n(z)}$ are *star-shaped.* For this purpose we shall first establish the following representation for $\theta_n'(\phi)$. *If C is a nearly circular contour and if the function $\rho(\theta)$ which represents C satisfies hypothesis* (3.2), *then the derivative $\theta_n'(\phi)$ of $\theta_n(\phi)$ is continuous and*

$$\theta_1'(\phi)-1=-\frac{1}{2\pi}\int_{\phi-\pi}^{\phi+\pi}\left\{\frac{\rho'}{\rho}(t)-\frac{\rho'}{\rho}(\phi)\right\}\cot\frac{t-\phi}{2}\,dt, \tag{7.1}$$

$$\begin{aligned}\theta_n'(\phi)-1=&-\frac{1}{2\pi}\int_{\phi-\pi}^{\phi+\pi}\left\{\frac{\rho'}{\rho}[\theta_{n-1}(t)]-\frac{\rho'}{\rho}[\theta_{n-1}(\phi)]\right\}\theta_{n-1}'(t)\cot\frac{t-\phi}{2}\,dt\\&-\frac{\rho'}{\rho}[\theta_{n-1}(\phi)]\frac{\rho'}{\rho}[\theta_{n-2}(\phi)]\theta_{n-2}'(\phi)\qquad(n\geqq 2).\end{aligned} \tag{7.2}$$

PROOF. The integrand of (7.1) is continuous in both the variables t, ϕ except possibly for $t=\phi$, and is bounded because of (3.2). Hence the integral (7.1) is a continuous function of ϕ. Since this integral represents the conjugate function of $(\rho'/\rho)[\phi]$ for which the integral over the interval $(0, 2\pi)$ is zero, it is equal to $\theta_1'(\phi)-1$, at least for almost all ϕ, and, because of the continuity, for all ϕ. This proves (7.1).

Let us suppose it were proved that $\theta_k'(\phi)$ is a continuous function when $k=1, 2, \cdots, n$ $(n\geqq 1)$. We then show that the formula (7.2) holds with n replaced by $n+1$, and that $\theta_{n+1}'(\phi)$ is continuous. This will then prove the representation (7.2) and the continuity of $\theta_n'(\phi)$ for all n.

Since $F_{n+1}(e^{i\phi})=\log\rho[\theta_n(\phi)]+i(\theta_{n+1}(\phi)-\phi)$ is absolutely continuous (see §9(b)) it follows that $\theta_{n+1}'(\phi)-1$ is conjugate to $(\rho'/\rho)[\theta_n(\phi)]\theta_n'(\phi)$, and we have, for almost all ϕ,

$$\theta_{n+1}'(\phi) - 1 = -\frac{1}{2\pi}\int_0^\pi \left\{\frac{\rho'}{\rho}[\theta_n(\tau)]\theta_n'(\tau)\right\}_{\tau=\phi-t}^{\tau=\phi+t} \cot\frac{t}{2}\,dt,$$

the integral being convergent in the sense that $\lim_{\delta\to 0}\int_\delta^\pi$ exists. We write

$$\theta_{n+1}'(\phi) - 1 = -\frac{1}{2\pi}\int_0^\pi \left\{\frac{\rho'}{\rho}[\theta_n(\phi+t)] - \frac{\rho'}{\rho}[\theta_n(\phi)]\right\}\theta_n'(\phi+t)\cot\frac{t}{2}\,dt$$
$$+\frac{1}{2\pi}\int_0^\pi \left\{\frac{\rho'}{\rho}[\theta_n(\phi-t)] - \frac{\rho'}{\rho}[\theta_n(\phi)]\right\}\theta_n'(\phi-t)\cot\frac{t}{2}\,dt$$
$$-\frac{\rho'}{\rho}[\theta_n(\phi)]\frac{1}{2\pi}\int_0^\pi \{\theta_n'(\phi+t) - \theta_n'(\phi-t)\}\cot\frac{t}{2}\,dt.$$

Because of (3.2) and the continuity of $\theta_n'(\phi)$, the first two integrals represent continuous functions of ϕ. The third integral (without the factor $-(\rho'/\rho)[\theta_n(\phi)]$) is equal to $(\rho'/\rho)[\theta_{n-1}'(\phi)]\theta_{n-1}'(\phi)$, since $\theta_n'(\phi)-1$ is conjugate to this function. Introducing the variable $\tau=\phi+t$ in the first integral and $\tau=\phi-t$ in the second, we obtain

$$\theta_{n+1}'(\phi) - 1 = -\frac{1}{2\pi}\int_{\phi-\pi}^{\phi+\pi} \left\{\frac{\rho'}{\rho}[\theta_n(\tau)] - \frac{\rho'}{\rho}[\theta_n(\phi)]\right\}\theta_n'(\tau)\cot\frac{t-\phi}{2}\,d\tau$$
$$-\frac{\rho'}{\rho}[\theta_n(\phi)]\frac{\rho'}{\rho}[\theta_{n-1}(\phi)]\theta_{n-1}'(\phi).$$

The right-hand side of this equation represents a continuous function of ϕ. Hence $\theta_{n+1}'(\phi)$ may be defined as a continuous function for all ϕ, and therefore $\theta_{n+1}(\phi)$ has a continuous derivative for all ϕ. This completes the proof.

8. Conditions under which C_n is star-shaped. Proof of IV. The curve C_n is star-shaped if $\theta_n'(\phi)\geqq 0$. By (7.1) and (3.2)

$$|\theta_1(\phi) - 1| \leqq \frac{1}{2\pi}\int_{\phi-\pi}^{\phi+\pi}\left|\frac{\rho'}{\rho}(t) - \frac{\rho'}{\rho}(\phi)\right|\cot\left|\frac{t-\phi}{2}\right|dt$$
$$\leqq \frac{\epsilon}{2\pi}\int_{\phi-\pi}^{\phi+\pi}(t-\phi)\cot\frac{t-\phi}{2}\,dt = 2\epsilon\log 2.$$

Thus, if $2\epsilon\log 2\leqq 1$ or $\epsilon\leqq(2\log 2)^{-1}$, then $\theta_1'(\phi)\geqq 0$ and C_1 is star-shaped.

Let us suppose it were proved that $\theta_n'(\phi)\geqq 0$ for some $n\geqq 1$, provided ϵ does not exceed some value $\epsilon_0<1$. Then we examine $\theta_{n+1}'(\phi)$. By (7.2), (3.2), and (2.2),

$$|\theta_{n+1}'(\phi) - 1| \leqq \frac{\epsilon}{2\pi}\int_{\phi-\pi}^{\phi+\pi}(\theta_n(t) - \theta_n(\phi))\theta_n(t)\cot\frac{t-\phi}{2}\,dt + \epsilon^2|\theta_{n-1}'(\phi)|. \quad (8.1)$$

It is to be noted that $\theta_n(t)-\theta_n(\phi)$ has the same sign as $t-\phi$ since $\theta_n'(t)\geqq 0$. We find by integration by parts that

$$m_n^2 = \frac{1}{2\pi}\int_{\phi-\pi}^{\phi+\pi}(\theta_n(t) - \theta_n(\phi))\theta_n'(t)\cot\frac{t-\phi}{2}\,dt = \frac{1}{2\pi}\int_{\phi-\pi}^{\phi+\pi}\left(\frac{\theta_n(t)-\theta_n(\phi)}{2\sin\frac{1}{2}(t-\phi)}\right)^2 dt$$
$$= \frac{1}{2\pi}\int_{\phi-\pi}^{\phi+\pi}\left(\frac{\theta_n(t) - t - [\theta_n(\phi)-\phi] + t - \phi}{2\sin\frac{1}{2}(t-\phi)}\right)^2 dt.$$

Hence by Minkowski's inequality,

$$m_n \leqq \left\{\frac{1}{2\pi}\int_{\phi-\pi}^{\phi+\pi}\left(\frac{\theta_n(t) - t - [\theta_n(\phi) - \phi]}{2 \sin \frac{1}{2}(t - \phi)}\right)^2 dt\right\}^{1/2} + \left\{\frac{1}{2\pi}\int_{\phi-\pi}^{\phi+\pi}\left(\frac{t - \phi}{2 \sin \frac{1}{2}(t - \phi)}\right)^2 dt\right\}^{1/2}.$$

Integrating by parts, we find that

$$\frac{1}{2\pi}\int_{\phi-\pi}^{\phi+\pi}\left(\frac{t - \phi}{2 \sin \frac{1}{2}(t - \phi)}\right)^2 dt = \frac{1}{2\pi}\int_{\phi-\pi}^{\phi+\pi}(t - \phi)\cot\frac{t - \phi}{2}\,dt = 2 \log 2 = c^2.$$

Furthermore, by the theorem of §9(f),

$$\frac{1}{2\pi}\int_{\phi-\pi}^{\phi+\pi}\left(\frac{\theta_n(t) - t - [\theta_n(\phi) - \phi]}{2 \sin \frac{1}{2}(t - \phi)}\right)^2 dt = \frac{1}{2\pi}\int_{\phi-\pi}^{\phi+\pi}\left(\frac{\log \rho[\theta_{n-1}(t)] - \log \rho[\theta_{n-1}(\phi)]}{2 \sin \frac{1}{2}(t - \phi)}\right)^2 dt.$$

By (2 2), the right-hand side of this equation is

$$\leqq \epsilon^2 \frac{1}{2\pi}\int_{\phi-\pi}^{\phi+\pi}\left(\frac{\theta_{n-1}(t) - \theta_{n-1}(\phi)}{2 \sin \frac{1}{2}(t - \phi)}\right)^2 dt = \epsilon^2 m_{n-1}^2.$$

Hence

$$m_n \leqq \epsilon m_{n-1} + c.$$

Since $m_0 = c$, we have

$$m_n \leqq c(1 + \epsilon + \epsilon^2 + \cdots + \epsilon^n) = c\,\frac{1 - \epsilon^{n+1}}{1 - \epsilon}\cdot$$

Hence, by (8.1),

$$| \theta'_{n+1}(\phi) - 1 | \leqq \epsilon m_n^2 + \epsilon^2 | \theta'_{n-1}(\phi) | \leqq 2\epsilon\left[\frac{1 - \epsilon^{n+1}}{1 - \epsilon}\right]^2 \log 2 + \epsilon^2 | \theta'_{n-1}(\phi) |. \qquad (8.2)$$

Applying (8.2) with $n = 1$, we find since $\theta'_0(t) = 1$ that,

$$| \theta'_2(\phi) - 1 | \leqq 2\epsilon(1 + \epsilon)^2 \log 2 + \epsilon^2, \qquad (8.3)$$

and this will be less than 1 if $\epsilon \leqq 0.34$.

For $n = 2$ we find, since $\theta'_1(\phi) \leqq 1 + 2\epsilon \log 2$ and $\theta'_1(\phi) > 0$ for $\epsilon < (2 \log 2)^{-1}$, that

$$| \theta'_3(\phi) - 1 | \leqq 2\epsilon(1 + \epsilon + \epsilon^2)^2 \log 2 + \epsilon(1 + 2\epsilon \log 2).$$

This expression will be less than 1, if $\epsilon \leqq 0.31$ ($< (2 \log 2)^{-1}$).

By (8.3), $|\theta'_2(\phi)| \leqq 1.7927$ if $\epsilon = 0.30$. Hence, applying (8.2) for $n \doteq 3$ and using this estimate for $\theta'_2(\phi)$, we find that $|\theta'_4(\phi) - 1| \leqq 1$, if $\epsilon \leqq 0.3$.

Assuming that, for some $n \geqq 1$, $0 < \theta'_{n-1}(\phi) \leqq 2$, we see from (8.2) that

$$| \theta'_{n+1}(\phi) - 1 | \leqq \frac{2 \log 2}{(1 - \epsilon)^2}\epsilon + 2\epsilon^2 < 1$$

if $\epsilon \leqq 0.295$. Since for $\epsilon \leqq 0.295$ this assumption is certainly satisfied for $n = 1$ and $n = 2$, it follows that for all n $|\theta'_{n+1}(\phi) - 1| < 1$ if $\epsilon \leqq 0.295$.

REMARK. For large values of n the bound for ϵ can be improved by use of Theorem IV and the left hand inequality in (9.3).

By Theorem IV, $|\theta_n'(\phi)-\theta'(\phi)| \leqq \sqrt{2\pi\sigma_n}(A(n+1))^{3/2}\epsilon^{n+1}$, and by (9.3), $\theta'(\phi) \geqq A^{-1}(1+\epsilon^2)^{-1/2}$. For any given fixed n, ϵ_0 can be chosen so that $\sqrt{2\pi\sigma_n}(A(n+1))^{3/2}\epsilon_0^{n+1} < A^{-1}(1+\epsilon_0^2)^{-1/2}$. Then, for all $\epsilon \leqq \epsilon_0$, $\theta_n'(\phi) \geqq \theta'(\phi) - A^{-1}(1+\epsilon^2)^{-1/2} \geqq 0$.

9. Auxiliary theorems. This section contains the proofs of some of the auxiliary results cited in the text.

(a). UNIQUENESS OF THE SOLUTION OF THEODORSEN'S INTEGRAL EQUATION. *If C is a nearly circular contour, the integral equation* (2.5) *has at most one continuous solution.*

Let us suppose that it had two such solutions, $\theta_1(\phi)$ and $\theta_2(\phi)$. Since

$$\int_0^{2\pi} (\theta_1(\phi) - \phi)d\phi = 0, \qquad \int_0^{2\pi} (\theta_2(\phi) - \phi)d\phi = 0,$$

it follows by use of the theorem cited in §4(a) that

$$M^2 = \frac{1}{2\pi}\int_0^{2\pi} (\theta_1(\phi) - \theta_2(\phi))^2 d\phi \leqq \frac{1}{2\pi}\int_0^{2\pi} \{\log \rho[\theta_1(\phi)] - \log \rho[\theta_2(\phi))]\}^2 d\phi.$$

By (2.2),

$$|\log \rho[\theta_1(\phi)] - \log \rho[\theta_2(\phi)]| \leqq \epsilon|\theta_1(\phi) - \theta_2(\phi)|,$$

so that we have $M^2 \leqq \epsilon^2 M^2$. Since $0<\epsilon<1$, $M=0$ and hence $\theta_1(\phi) \equiv \theta_2(\phi)$.

(b). A PROPERTY OF THE FUNCTIONS $\theta_n(\phi)$. *If C is a nearly circular contour, then the functions $\theta_n(\phi)$ defined by* (2.6) *are absolutely continuous, and $(\theta_n'(\phi))^2$ are integrable (in the sense of Lebesgue) for $0 \leqq \phi \leqq 2\pi$.*

This is clearly true when $n=0$. We suppose that this statement were proved for some $n \geqq 0$. Since $\log \rho(\theta)$ has bounded difference quotients (by (2.2)) and $\theta_n(\phi)$ is absolutely continuous, it follows that $\log \rho[\theta_n(\phi)]$ also is absolutely continuous. Furthermore, because of the inequality

$$\left(\frac{\rho'}{\rho}[\theta_n(\phi)]\theta_n'(\phi)\right)^2 \leqq \epsilon^2(\theta_n'(\phi))^2,$$

it follows that the integral

$$\int_0^{2\pi} \left(\frac{\rho'}{\rho}[\theta_n(\phi)]\theta_n'(\phi)\right)^2 d\phi$$

exists. Hence, the conjugate function of $\log \rho[\theta_n(\phi)]$, namely $\theta_{n+1}(\phi)-\phi$, exists and is absolutely continuous and the integral $\int_0^{2\pi}(\theta_{n+1}'(\phi)-1)^2 d\phi$ exists.[9]

(c). A PROPERTY OF $\theta(\phi)$. *If C is a nearly circular contour, then $\theta(\phi) = \arg f(e^{i\phi})$ (de-*

[9] We are using here the following theorem: if $g(\phi)$ is an absolutely continuous and periodic function (period 2π) for $0 \leqq \phi \leqq 2\pi$ and if $[g'(\phi)]^2$ is integrable, then the conjugate function $\bar{g}(\phi)$ is absolutely continuous and $(\bar{g}'(\phi))^2$ is integrable. (See, for example, W. Seidel, *Über die Ränderzuordnung bei konformen Abbildungen*, Math. Annalen **104**, 223 (1931).

fined by (2.4)) *is absolutely continuous and* $(\theta'(\phi))^2$ *is integrable* (*in the sense of Lebesgue*) *and*

$$\frac{1}{2\pi}\int_0^{2\pi} \theta'^2(\phi)d\phi \leqq \frac{1}{1-\epsilon^2}. \tag{9.1}$$

PROOF. Since the curve C is rectifiable, the function $F(e^{i\phi})$ is absolutely continuous.[10] Hence

$$\frac{d}{d\phi}F(e^{i\phi}) - i = \frac{\rho'}{\rho}[\theta(\phi)]\theta'(\phi) + i\theta'(\phi)$$

exists almost everywhere for $0 \leqq \phi \leqq 2\pi$, and is integrable. Furthermore, the function $\partial[F(z)]/\partial\phi - i = u(z) + iv(z)$, $z = re^{i\phi}$, may be represented by the Poisson Integral in the unit circle,

$$u(z) + iv(z) = \frac{1}{2\pi}\int_0^{2\pi}\{u(e^{it}) + iv(e^{it})\}\frac{1-r^2}{1+r^2-2r\cos(t-\phi)}dt. \tag{9.2}$$

For almost all $\phi(0 \leqq \phi \leqq 2\pi)$,

$$\lim_{r\to 1} u(re^{i\phi}) = \frac{\rho'}{\rho}[\theta(\phi)]\theta'(\phi) = u(e^{i\phi}), \qquad \lim_{r\to 1} v(re^{i\phi}) = \theta'(\phi) = v(e^{i\phi}).$$

Since C is star-shaped, $\theta'(\phi) \geqq 0$, and we have by (2.2),

$$v(e^{i\phi}) \pm u(e^{i\phi}) \geqq \theta'(\phi)(1-\epsilon) \geqq 0.$$

Because of the representation (9.2) we conclude that $v(z)+u(z) \geqq 0$ and $v(z)-u(z) \geqq 0$ for $|z|<1$. Hence $v^2(z)-u^2(z) \geqq 0$ for $|z|<1$. Now

$$\frac{1}{2\pi}\int_0^{2\pi}(v^2(re^{i\phi}) - u^2(re^{i\phi}))d\phi = 1.$$

Hence, taking the limit as $r\to 1$, we obtain by Fatou's lemma

$$\frac{1}{2\pi}\int_0^{2\pi}\theta'^2(\phi)\left[1 - \left(\frac{\rho'}{\rho}[\theta(\phi)]\right)^2\right]d\phi \leqq 1,$$

and by (2.2),

$$\frac{1}{2\pi}\int_0^{2\pi}(\theta'(\phi))^2 d\phi \leqq \frac{1}{1-\epsilon^2}.$$

This proves that $(\theta'(\phi))^2$ is integrable and that (9.1) holds.

(d). AN ESTIMATE FOR $\theta'(\phi)$ AND $\theta''(\phi)$. *If C is a nearly circular contour and if in addition* (3.2) *is satisfied, then*

$$\frac{1}{A\sqrt{1+\epsilon^2}} \leqq \theta'(\phi) \leqq A = 4^{\epsilon}e^{\epsilon^2}, \tag{9.3}$$

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\{\theta''(\phi)\}^2 d\phi\right\}^{1/2} \leqq A^{3/2}\epsilon\min(1+\epsilon;\sqrt{2}). \tag{9.4}$$

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[10] This follows from a theorem of F. and M. Riesz, *Über die Randwerte einer analytischen Funktion*, Comptes Rendus du Quatrième Congrès des Mathématiciens Scandinaves à Stockholm (1916) pp. 27–44. See also, F. Riesz, Math. Zeitschrift, **18**, 95 (1923).

The proofs of these inequalities are contained in a paper to be published elsewhere.

(e). A PROPERTY OF THE FUNCTIONS $F_n(\phi)$. *If C is a nearly circular contour for which* (3.2) *and* (3.4) *are satisfied, then the functions $F_n(e^{i\phi})$ have continuous second derivatives which satisfy a Hölder condition with any fixed exponent α, $0<\alpha<1$.*

The proof may easily be given by induction. Since $\log \rho(\phi)$ and $\theta_1(\phi)-\phi$ are conjugate functions and since the second derivative of $\log \rho(\phi)$ satisfies the Lipschitz condition (3.4), it follows from a theorem of I. Privaloff,[11] that $\theta_1'(\phi)$ and $\theta_1''(\phi)$ exist and that $\theta_1''(\phi)$ satisfies a Hölder condition with any fixed exponent α, $0<\alpha<1$. Let us suppose now, that it had been shown that $\theta_n''(\phi)$ exists and satisfies a Hölder condition with any fixed exponent α, $0<\alpha<1$. Then $\log \rho[\theta_n(\phi)]$ has continuous first and second derivatives, $(\rho'/\rho)[\theta_n(\phi)]\theta_n'(\phi)$ and $p[\theta_n(\phi)]\theta_n'^2(\phi)+(\rho'/\rho)[\theta_n(\phi)]\theta_n''(\phi)$, respectively, and the latter satisfies a Hölder condition with any exponent α, $0<\alpha<1$, (because of (3.4) and (3.2)). Hence, again by Privaloff's theorem, the conjugate function $\theta_{n+1}(\phi)-\phi$ possesses a second derivative $\theta_{n+1}''(\phi)$ which satisfies such a Hölder condition. This completes the proof.

(f). A THEOREM ON CONJUGATE FUNCTIONS. *Let us suppose that $u(t)$ is a periodic function (period 2π) possessing a continuous derivative for $0\leqq t\leqq 2\pi$. Let $v(t)$ be conjugate to $u(t)$ and let us suppose that $v(t)$ also possesses a continuous derivative for $0\leqq t\leqq 2\pi$. Then for every θ,*

$$\int_0^{2\pi}\left(\frac{u(t)-u(\theta)}{\sin\frac{1}{2}(t-\theta)}\right)^2 dt=\int_0^{2\pi}\left(\frac{v(t)-v(\theta)}{\sin\frac{1}{2}(t-\theta)}\right)^2 dt.$$

PROOF. Let $G(z)=U(z)+iV(z)$ denote the function which is analytic for $|z|<1$ and assumes the boundary values $g(t)\equiv u(t)+iv(t)-(u(\theta)+iv(\theta))$ for $z=e^{it}$. Then the real part of $[G(z)]^2$ may be represented by the Poisson integral $(z=re^{i\phi})$

$$[U(z)]^2-[V(z)]^2=\frac{1}{2\pi}\int_0^{2\pi}\left\{(u(t)-u(\theta))^2-(v(t)-v(\theta))^2\right\}\frac{1-r^2}{1+r^2-2r\cos(t-\phi)}dt.$$

For $z=re^{i\theta}$,

$$\frac{[U(z)]^2-[V(z)]^2}{1-r^2}=\frac{1}{2\pi}\int_0^{2\pi}\frac{(u(t)-u(\theta))^2-(v(t)-v(\theta))^2}{(1-r)^2+4r\sin^2\frac{1}{2}(t-\theta)}dt.$$

As is easily seen, the limit of this integral as $r\to 1$ is

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{(u(t)-u(\theta))^2-(v(t)-v(\theta))^2}{4\sin^2\frac{1}{2}(t-\theta)}dt. \tag{9.5}$$

By the mean value theorem (since $U(e^{i\theta})=V(e^{i\theta})=0$)

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[11] I. Privaloff, *Sur les fonctions conjuguées*, Bull. Soc. Math. France **44**, 100–103 (1916); or Zygmund, l.c. p. 156. Privaloff's Theorem states: if $g(\phi)$ is periodic (period 2π) and satisfies a Hölder condition with the exponent α, $0<\alpha<1$, for all ϕ, then any conjugate function of $g(\phi)$ satisfies such a condition.

$$-\frac{[U(z)]^2 - [V(z)]^2}{1-r} = \frac{\partial}{\partial\rho}\left[U^2(\rho e^{i\theta}) - V^2(\rho e^{i\theta})\right]_{\rho=\bar{r}}$$

$$= 2\left\{U(\rho e^{i\theta})\frac{\partial}{\partial\rho}U(\rho e^{i\theta}) - V(\rho e^{i\theta})\frac{\partial}{\partial\rho}V(\rho e^{i\theta})\right\}_{\rho=\bar{r}}, \quad (r < \bar{r} < 1). \quad (9.5)$$

Since $g'(t)$ exists and is continuous,

$$\frac{\partial U}{\partial\rho} + i\frac{\partial V}{\partial\rho} = e^{i\theta}G'(\rho e^{i\theta}) \to -ig'(\theta)$$

as $\rho\to 1$. Thus $\lim_{\rho\to 1}\partial[U(\rho e^{i\theta})]/\partial\rho$ and $\lim_{\rho\to 1}\partial[V(\rho e^{i\theta})]/\partial\rho$ exist. Furthermore, $\lim_{\rho\to 1} U(\rho e^{i\theta}) = \lim_{\rho\to 1} V(\rho e^{i\theta}) = 0$. Hence, the limit as $r\to 1$ of (9.6) is zero and therefore the integral (9.5) is zero. This proves the theorem.