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LIFTING-LINE THEORY FOR A WING IN NON-UNIFORM FLOW*

BY

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1. Introduction. Prandtl's theory of the lifting line gave the answer to most of the questions in the aerodynamic design of airplane wings. Thus the three-dimensional wing theory became a standard tool of airplane designers. One restriction involved in the conventional wing theory is the uniformity of the undisturbed flow in which the wing is placed. Now there are many important cases which do not satisfy this condition. For instance, in the case of a wing spanning an open jet wind tunnel, the velocity of the air stream has a maximum at the center of the jet and drops to zero outside of the jet. Another example is the problem of the influence of the propeller slip-stream on the characteristics of the wing. Here the higher velocity of the propeller slip-stream makes the application of the Prandtl wing theory difficult. Such cases led several authors to investigate the problem of a wing in non-uniform flow. Some investigators found a satisfactory solution of the problem for the case of "stepwise" velocity distribution. In this case the flow in regions of uniform velocity can be determined by using Prandtl's concepts with additional continuity conditions at the boundaries between such regions. On the other hand, the problem of a continuously varying velocity field seems to need an appropriate treatment. K. Bausch[1] has tried to modify the Prandtl theory for the case of small inhomogeneity in the air stream; however, besides the restriction of slight deviation from uniform flow, his method encounters a further difficulty in estimating the error introduced by the approximations. The seriousness of this difficulty becomes evident when one tries to compare the results of Bausch with that of F. Vandrey.[2] Vandrey considers the problem with variable velocity as the limiting case of a wing in a stepwise velocity field, and his result seems to differ from that of Bausch. Recently R. P. Isaacs[3] has investigated the same problem, but the authors have not yet had the opportunity to study his work.

It seems to the authors that a general and more satisfactory solution for the flow of a wing in a non-uniform stream can be obtained by studying the three-dimensional problem anew in this generalized case, introducing the modifications of Prandtl's fundamental concepts. The first fundamental concept is the following: the span of

* Received September 27, 1944.

[1] K. Bausch, *Auftriebsverteilung und daraus abgeleitete Grössen für Tragflügel in schwach inhomogenen Strömungen*, Luftfahrtforschung, **16**, 129–134 (1939).

[2] F. Vandrey, *Beitrag zur Theorie des Tragflügels in schwach inhomogener Parallelströmung*, Zeitschrift f. angew. Math. u. Mech. **20**, 148–152 (1940).

[3] R. P. Isaacs, *Airfoil theory for flows of variable velocity*, abstract in Bulletin of the American Mathematical Society, **50**, 186 (1944).

the wing is sufficiently large compared with the chord so that the variation of the velocities in the spanwise direction is small when compared with the variation of the velocities in a plane normal to the span; then the flow at each sectional plane perpendicular to the span can be considered as a two-dimensional flow around an airfoil. The only additional feature for the flow in this sectional plane is the modification of the geometrical angle of attack, as defined by the undisturbed flow, on account of the so-called induced velocity. The second fundamental concept of Prandtl is the replacement of the wing by a lifting line having the same distribution of lifting forces along the span as the wing. This concept, with the additional assumption that the disturbance caused by the lifting line is small, i.e., that the wing is lightly loaded, makes the calculation of the induced velocity relatively simple. In this paper the authors will study the flow around a lightly loaded lifting line placed in a parallel stream whose velocity is perpendicular to the span (Fig. 1) and is assumed to vary in both directions normal to the flow. Due to the rather complicated character of the flow, the usual concept of the picturesque system of trailing vortices encountered in Prandtl's wing theory is not very useful here. A method, which is mathematically more convenient, has to be adopted. This method has already been used by the senior author[4] in explaining the similarity between Prandtl's wing theory and the theory of planning surfaces. After the general theory is formulated, the problem of minimum induced drag will be considered. Finally a general expression for calculating the induced drag of a wing in a stream of varying velocity will be presented.

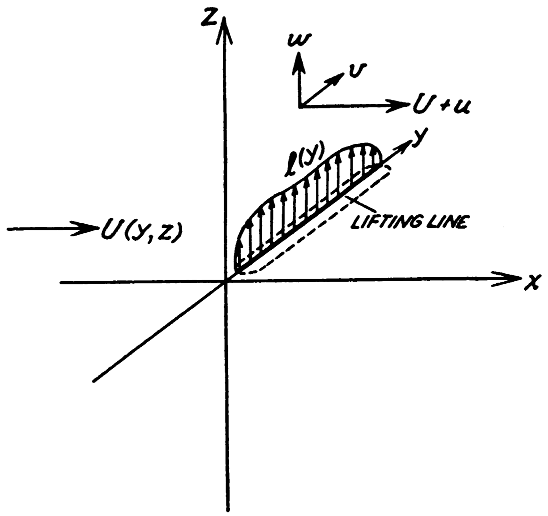

FIG. 1. Lifting line in a non-uniform flow.

Of course, the complete solution of the problem of a wing in a non-uniform stream requires a knowledge of the "section characteristic" or the two-dimensional properties of the airfoil sections of the wing. If the velocity of the main stream is varying only in the direction of the span, the required section characteristics are those of an airfoil in a two-dimensional uniform flow, and are common knowledge in applied aerodynamics. However, if the velocity of the main stream is also varying in a direction perpendicular to the span and to the velocity itself, the required section characteristics are those of an airfoil in a two-dimensional non-uniform flow. Such flow problems have not yet been studied extensively.[5]

2. General theory of a lifting line. Let the x-axis be parallel to the direction of the main flow, the y-axis coincide with the lifting line and the z-axis be normal to the

[4] Th. von Kármán, *Neue Darstellung der Tragflügeltheorie*, Zeitschrift f. angew. Math. u. Mech. **15**, 56–61 (1935).

[5] H. S. Tsien, *Symmetrical Joukowsky airfoils in shear flow*, Quart. Appl. Math. **1**, 130–148 (1943).

lifting line (Fig. 1). If p is the pressure, ρ the density, and v_1, v_2, v_3 the components of the velocity, the dynamical equations for the steady motion of an inviscid, incompressible fluid without external forces are

$$v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_1}{\partial y} + v_3 \frac{\partial v_1}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{1}$$

$$v_1 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y} + v_3 \frac{\partial v_2}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{2}$$

$$v_1 \frac{\partial v_3}{\partial x} + v_2 \frac{\partial v_3}{\partial y} + v_3 \frac{\partial v_3}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}. \tag{3}$$

The equation of continuity is

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0. \tag{4}$$

Equations (1) to (4) constitute a system of four simultaneous equations for the four unknowns v_1, v_2, v_3 and p.

For the particular problem of a lightly loaded lifting line, the velocity components can be expressed in the following forms:

$$v_1 = U + u, \quad (5); \qquad v_2 = v, \quad (6); \qquad v_3 = w. \quad (7)$$

Here u, v, w are the velocity components due to the presence of the lifting line and U is the main stream velocity assumed to be a function of y and z but independent of x. Since the lifting line is assumed to be lightly loaded, u, v and w are small compared with the main velocity U. By substituting Eqs. (5) to (7) into the dynamical equations and neglecting higher order terms, a set of linear equations for u, v and w is obtained. Thus

$$U \frac{\partial U}{\partial x} + v \frac{\partial U}{\partial y} + w \frac{\partial U}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{8}$$

$$U \frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad (9); \qquad U \frac{\partial w}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial z}. \quad (10)$$

Then the equation of continuity becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{11}$$

If Eqs. (8), (9) and (10) are differentiated with respect to x, y and z respectively and the results added, the sum can be simplified by using Eq. (11) and can, finally, be written in the form

$$\frac{1}{U^2}\frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{1}{U^2}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{U^2}\frac{\partial p}{\partial z}\right) = 0. \tag{12}$$

This is now an equation for the pressure p only and can be used conveniently as the starting point of the solution. If the pressure of the undisturbed main flow is chosen as the reference pressure and set equal to zero, one of the boundary conditions to be satisfied by p is

$$p = 0, \quad \text{for} \quad |x| \to \infty, \qquad |y| \to \infty, \qquad \text{or} \quad |z| \to \infty. \tag{13}$$

The condition at the lifting line, or y-axis, is that the lifting force is represented by a suction force on the "upper surface" of the lifting line and a pressure force of equal magnitude on the "lower surface" (Fig. 2). Hence the pressure p must satisfy the following expressions

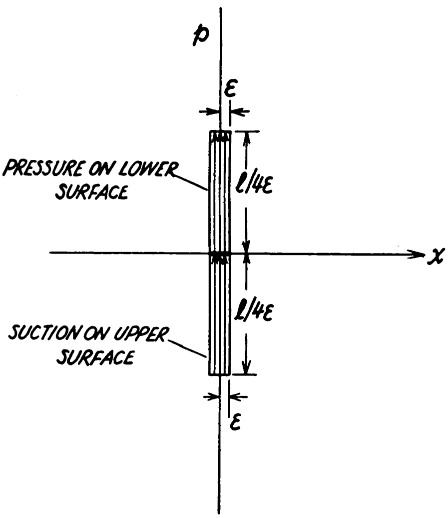

FIG. 2. Representation of lift as pressure forces acting on the two "surfaces" of the lifting line.

$$\int_{-\epsilon}^{\epsilon} p dx = -\tfrac{1}{2} l(y), \quad \text{for} \quad z = +0, \tag{14}$$

and

$$\int_{-\epsilon}^{\epsilon} p dx = \tfrac{1}{2} l(y), \quad \text{for} \quad z = -0, \tag{15}$$

where $l(y)$ is the lift per unit length of the lifting line at the point y. Furthermore, on account of the symmetry of the flow,

$$p = 0 \quad \text{for} \quad z = 0, \qquad |x| > \epsilon. \tag{16}$$

To solve Eq. (12) together with the boundary conditions given by Eqs. (13) to (16), the Fourier integral theorem can be used to build up the solution of the problem from the elementary solutions of Eq. (12) of the form

$$P(y, z, \lambda) \cos \lambda x.$$

The equation to be satisfied by P is

$$U^2 \frac{\partial}{\partial y}\left(\frac{1}{U^2} \frac{\partial P}{\partial y}\right) + U^2 \frac{\partial}{\partial z}\left(\frac{1}{U^2} \frac{\partial P}{\partial z}\right) - \lambda^2 P = 0. \tag{17}$$

To determine P uniquely, it is convenient to impose the following conditions

$$P = 0, \quad \text{for} \quad |y| \to \infty, \quad |z| \to \infty, \tag{18}$$

$$P = -\tfrac{1}{2} l(y) \quad \text{for} \quad z = +0, \tag{19}$$

$$P = \tfrac{1}{2} l(y) \quad \text{for} \quad z = -0. \tag{20}$$

The required solution for p can then be written as

$$p = \frac{1}{\pi} \int_0^\infty \cos \lambda x P(y, z, \lambda) d\lambda. \tag{21}$$

By substituting Eq. (21) into Eqs. (9) and (10), the "induced velocities" v and w are obtained;

$$v(x, y, z) = v(0, y, z) - \frac{1}{\rho U} \frac{1}{\pi} \int_0^\infty \frac{\sin \lambda x}{\lambda} \frac{\partial}{\partial y} P(y, z, \lambda) d\lambda, \tag{22}$$

$$w(x, y, z) = w(0, y, z) - \frac{1}{\rho U} \frac{1}{\pi} \int_0^\infty \frac{\sin \lambda x}{\lambda} \frac{\partial}{\partial z} P(y, z, \lambda) d\lambda. \tag{23}$$

Because the integrals are odd functions of x, the following relations hold for velocities far ahead of the lifting line and far behind the lifting line:

$$\tfrac{1}{2}[v(-\infty, y, z) + v(\infty, y, z)] = v(0, y, z), \quad \tfrac{1}{2}[w(-\infty, y, z) + w(\infty, y, z)] = w(0, y, z).$$

However, it is evident that the induced velocities far ahead of the lifting lines must be zero. Hence

$$v(0, y, z) = \tfrac{1}{2} v(\infty, y, z), \quad (24); \qquad w(0, y, z) = \tfrac{1}{2} w(\infty, y, z). \quad (25)$$

The induced velocities v and w at the lifting line are then one-half of those far downstream. This is in accordance with the usual wing theory based upon the concept of trailing vortices.

One meets an apparent difficulty if the x component of the induced velocity is calculated; integration of Eq. (8) with respect to x furnishes the x-component of the induced velocity:

$$u = -\frac{1}{\rho U} p - \frac{1}{U} \frac{\partial U}{\partial y} \int_{-\infty}^{x} v dx - \frac{1}{U} \frac{\partial U}{\partial z} \int_{-\infty}^{x} w dx. \tag{26}$$

Since p tends to zero, v and w tend to finite quantities as x tends to infinity, and u increases indefinitely as x tends to infinity. This is in contradiction to the assumption of small disturbances introduced at the beginning of the present investigation. However, it is believed that this difficulty does not prevent the application of the theory to practical cases, since the apparent large value of the u component is due to the distortion of the variable main stream by the induced cross flow and the infinite value for $x \to \infty$ is due to the linearization of the differential equations. Some further remarks on this point are given in Section 4.

3. Conditions far downstream. For the application of the lifting-line theory to the wing problem, the quantity of primary interest is the z component of the induced velocity at the lifting line. The simple relations given by Eqs. (24) and (25) suggest a possible simplification of the calculation by considering conditions far downstream, or the "Trefftz plane" according to the terminology of the conventional wing theory. To abbreviate the notation, we let

$$\left.\begin{aligned} v_0 &= v(0, y, z), & w_0 &= w(0, y, z), \\ v_1 &= v(\infty, y, z), & w_1 &= w(\infty, y, z). \end{aligned}\right\} \tag{27}$$

Then, according to (24) and (25), $v_0 = \frac{1}{2} v_1$, $w_0 = \frac{1}{2} w_1$. Therefore, Eqs. (22) and (23) give

$$v_1 = -\frac{1}{\rho U} \lim_{x \to \infty} \frac{1}{\pi} \int_0^\infty \frac{\sin \lambda x}{\lambda} \frac{\partial}{\partial y} P(y, z, \lambda) d\lambda,$$

$$w_1 = -\frac{1}{\rho U} \lim_{x \to \infty} \frac{2}{\pi} \int_0^\infty \frac{\sin \lambda x}{\lambda} \frac{\partial}{\partial z} P(y, z, \lambda) d\lambda.$$

Let us consider $P(y, z, \lambda)$ as a regular function of λ; then

$$P(y, z, \lambda) = P(y, z, 0) + \lambda \left[\frac{\partial P}{\partial \lambda} \right]_{\lambda=0} + \cdots .$$

By using the variable $t = \lambda x$, the expressions for v_1 and w_1 can be rewritten,

$$v_1 = -\frac{1}{\rho U} \lim_{x \to \infty} \frac{2}{\pi} \int_0^\infty \frac{\sin t}{t} \frac{\partial}{\partial y} \left[P(y, z, 0) + \frac{t}{x} \left(\frac{\partial P}{\partial \lambda} \right)_{\lambda=0} + \cdots \right] dt,$$

$$w_1 = -\frac{1}{\rho U} \lim_{x \to \infty} \frac{2}{\pi} \int_0^\infty \frac{\sin t}{t} \frac{\partial}{\partial z} \left[P(y, z, 0) + \frac{t}{x} \left(\frac{\partial P}{\partial \lambda} \right)_{\lambda=0} + \cdots \right] dt.$$

At the limit, only the first terms of the integrands are significant, and furthermore

$$\frac{2}{\pi} \int_0^\infty \frac{\sin t}{t} dt = 1.$$

Hence

$$v_1 = -\frac{1}{\rho U} \frac{\partial}{\partial y} P(y, z, 0), \qquad (28); \qquad w_1 = -\frac{1}{\rho U} \frac{\partial}{\partial z} P(y, z, 0). \qquad (29)$$

Equations (28) and (29) simplify the problem of calculating the induced velocities at the Trefftz plane considerably. In fact, by introducing a "potential function" ϕ defined by the relation

$$\phi(y, z) = -P(y, z, 0), \qquad (30)$$

the problem can be formulated as follows: the differential equation to be satisfied by ϕ can be deduced from Eq. (17) by setting $\lambda = 0$; thus

$$\frac{\partial}{\partial y} \left(\frac{1}{U^2} \frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{1}{U^2} \frac{\partial \phi}{\partial z} \right) = 0. \qquad (31)$$

The boundary conditions to be satisfied by ϕ are

$$\phi = 0 \quad \text{for} \quad |y| \to \infty, \quad |z| \to \infty, \qquad (32)$$

$$\phi = l(y)/2 \quad \text{for} \quad z = +0, \qquad (33)$$

$$\phi = -l(y)/2 \quad \text{for} \quad z = -0. \qquad (34)$$

Then

$$v_1 = \frac{1}{\rho U} \frac{\partial \phi}{\partial y}, \qquad (35); \qquad w_1 = \frac{1}{\rho U} \frac{\partial \phi}{\partial z}. \qquad (36)$$

By substituting Eqs. (35) and (36) into Eq. (31), one has

$$\frac{\partial}{\partial y} \left(\frac{v_1}{U} \right) + \frac{\partial}{\partial z} \left(\frac{w_1}{U} \right) = 0. \qquad (37)$$

This equation has a very simple physical meaning. Since v_1 and w_1 are considered to be small quantities, the ratios v_1/U and w_1/U are the angles of inclination, β and γ, of the stream lines with respect to the zx and xy planes. Consider parallel planes perpendicular to the x-axis and dx apart (Fig. 3). If the width of the stream tube at the section x is δ_y, then at the section $x + dx$, the width of the stream tube is

$\delta_y[1+dx\ \partial\beta/\partial y]$. If the height of the stream tube at the section x is δ_z, then at the section $x+dx$, the height of the stream tube is $\delta_z[1+dx\ \partial\gamma/\partial z]$. The total increase in the cross-sectional area of the stream tube from x to $x+dx$ is then approximately

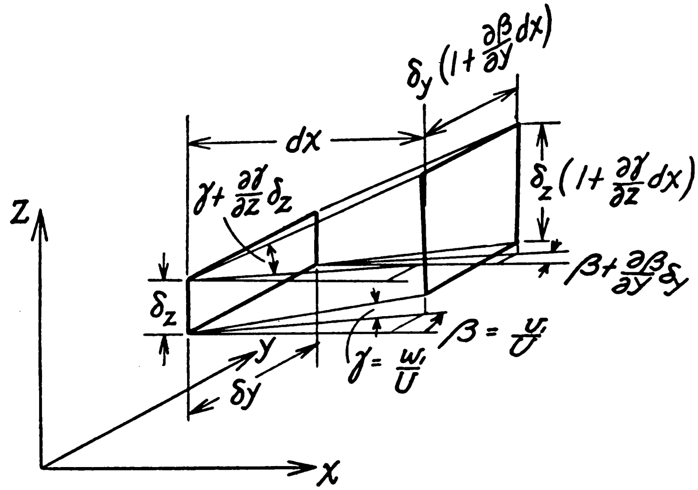

FIG. 3. Stream tube far downstream from the lifting line.

$$\delta_y\delta_z\left(\frac{\partial\beta}{\partial y}+\frac{\partial\gamma}{\partial z}\right)dx.$$

Now at the Trefftz plane, the flow field can be considered as settled into a uniform condition; i.e., the pressure is constant in the x-direction. Hence, the velocity of the flow along any stream tube is constant. Then the cross-sectional area of the stream tube must be also constant. Therefore,

$$\frac{\partial\beta}{\partial y}+\frac{\partial\gamma}{\partial z}=0,$$

which is simply Eq. (37). From this point of view, Eq. (37) is really the equation of continuity, simplified under the conditions prevailing at the Trefftz plane.

On the other hand, ϕ can be eliminated from Eqs. (35) and (36). The result is

$$\frac{\partial}{\partial z}(Uv_1)-\frac{\partial}{\partial y}(Uw_1)=0. \tag{38}$$

This equation can be considered as the modified vorticity equation. It actually holds for all values of x under the approximation assumed in the present investigation. This can be seen in the following way: since U is a function of y and z but independent of x, Eqs. (9) and (10) can be written in the form

$$\frac{\partial}{\partial x}Uv=-\frac{1}{\rho}\frac{\partial p}{\partial y},\qquad \frac{\partial}{\partial x}Uw=-\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

By differentiating the first equation with respect to z and the second equation with respect to y and then subtracting, the result is

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial z}(Uv) - \frac{\partial}{\partial y}(Uw)\right] = 0.$$

Thus

$$\frac{\partial}{\partial z}(Uv) - \frac{\partial}{\partial y}(Uw) = \text{a function of } y \text{ and } z.$$

But for points far upstream, or for $x = -\infty$, v and w vanish; therefore the function of y and z on the right of above equation must be identically zero. Hence for all values of x,

$$\frac{\partial}{\partial z}(Uv) - \frac{\partial}{\partial y}(Uw) = 0. \tag{39}$$

It should be noted here that Eqs. (37), (38) and (39) are obtained without any reference to the lifting line and hence they are true for more general cases. However, the complete determination of v_1 and w_1 requires a knowledge of the relation between the induced velocities and the lift on the wing. This relation depends upon the type of lift distribution. For the particular case of a lifting line, this relation is supplied by Eqs. (33) and (34).

Equation (37) can be identically satisfied by introducing the "stream function" ψ defined by

$$v_1 = U\frac{\partial\psi}{\partial z}, \qquad w_1 = -U\frac{\partial\psi}{\partial y}. \tag{40}$$

Then Eq. (38) gives the differential equation for ψ:

$$\frac{\partial}{\partial y}\left(U^2\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(U^2\frac{\partial\psi}{\partial z}\right) = 0. \tag{41}$$

Both Eq. (31) and Eq. (41) reduce to the Laplace equation for the conventional wing theory when U is a constant.

4. Minimum induced drag. The induced downwash angle at the lifting line is equal to w_0/U or $\frac{1}{2}w_1/U$, according to Eq. (25). Therefore, Eq. (36) gives the downwash angle at the lifting line as $[1/2\rho U^2](\partial\phi/\partial z)_{z=0}$, and the induced drag D_i can then be expressed as

$$D_i = -\frac{1}{2\rho}\int [\phi(y, +0) - \phi(y, -0)]\frac{1}{U^2}\left(\frac{\partial\phi}{\partial z}\right)_{z=0} dy = \frac{1}{2\rho}\int_C \frac{\phi}{U^2}\frac{\partial\phi}{\partial z}ds. \tag{42}$$

The first integral is evaluated across the span of the lifting line. The second integral is calculated along a contour following the upper and lower "surface" of the horizontal strip shown in Fig. 4. Since $\phi \to 0$ for points far from the lifting line, the contour integral can be transformed into an area integral by Green's theorem, and

$$D_i = \frac{1}{2\rho}\iint \left\{\frac{\partial}{\partial y}\left(\frac{1}{U^2}\phi\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{U^2}\phi\frac{\partial\phi}{\partial z}\right)\right\} dydz. \tag{43}$$

This integral extends throughout the region outside of the lifting line. Since ϕ satisfies the differential equation (31), Eq. (43) reduces to

$$D_i = \frac{\rho}{2}\iint \left\{\left(\frac{1}{\rho U}\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{1}{\rho U}\frac{\partial\phi}{\partial z}\right)^2\right\} dydz. \tag{44}$$

Therefore, the induced drag is represented by the kinetic energy corresponding to the velocity components v_1 and w_1 at the Trefftz plane. It is seen that the u component of the velocity does not appear in the expression for the induced drag. This is due to the fact that the increase of u with increasing x does not represent a real acceleration

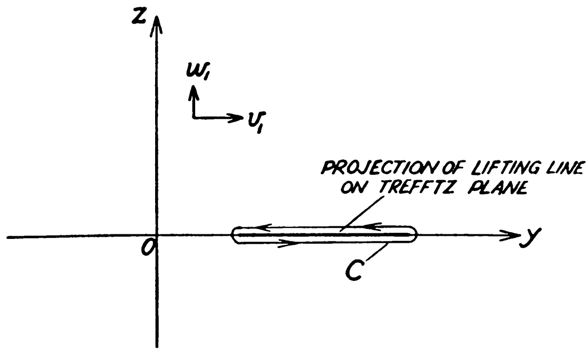

FIG. 4. Contour integration in the Trefftz plane.

of a fluid element in the x direction. Rather, it is due to the fact that the cross flow transports fluid elements from regions of lower main velocity to regions of higher main velocity and vice versa. This is in accordance with the modified continuity equation (37) which clearly indicates that the cross section of the individual stream tubes has a definite limiting value for $x \to \infty$, and therefore the velocity component in the direction of the stream tube tends to a finite value.

The problem of minimum induced drag requires the determination of the minimum of D_i as given by Eq. (44) together with the condition that the total lift L remains fixed. Thus

$$L = \int l dy = \int [\phi(y, +0) - \phi(y, -0)]dy = -\int_C \phi ds = \text{constant}. \tag{45}$$

By using the method of Lagrange's multiplier, the above problem can be reduced to that of finding the minimum of $D_i + K/\rho L$, where K is a constant. Hence,

$$\delta D_i + \frac{K}{\rho}\delta L = 0. \tag{46}$$

The variation of the induced drag can be obtained from Eq. (44),

$$\delta D_i = \frac{1}{\rho}\iint \left\{\frac{1}{U}\frac{\partial\phi}{\partial y}\frac{1}{U}\frac{\partial\delta\phi}{\partial y} + \frac{1}{U}\frac{\partial\phi}{\partial z}\frac{1}{U}\frac{\partial\delta\phi}{\partial z}\right\} dydz.$$

However, ϕ must satisfy the differential equation (31) ; thus

$$\delta D_i = \frac{1}{\rho}\iint \left\{\frac{\partial}{\partial y}\left(\frac{1}{U^2}\frac{\partial\phi}{\partial y}\delta\phi\right) + \frac{\partial}{\partial z}\left(\frac{1}{U^2}\frac{\partial\phi}{\partial z}\delta\phi\right)\right\} dydz = \frac{1}{\rho}\int_C \frac{1}{U^2}\frac{\partial\phi}{\partial z}\delta\phi ds.$$

On the other hand,

$$\delta L = -\int_C \delta\phi ds.$$

By substituting these results into Eq. (46), the condition of minimum induced drag is obtained in the form

$$\frac{1}{\rho}\int_C \left(\frac{1}{U^2}\frac{\partial\phi}{\partial z} - K\right)\delta\phi ds = 0. \tag{47}$$

The variation of $\delta\phi$ on the lifting line is arbitrary; therefore the minimum induced drag is given by the condition that the induced downwash angle must be constant along the span. If the main stream velocity U is constant, the above condition is reduced to the requirement of constant downwash. This is in agreement with the well-known result of Prandtl's wing theory.

5. Flow with velocity varying in the direction of span only. If the stream velocity varies only in the y direction, i.e., in the direction of the wing span, the calculation of induced velocity and induced drag can be simplified with the aid of characteristic functions connected with the differential equation for the potential function ϕ. In this case Eq. (31) becomes

$$\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - 2\frac{\frac{dU}{dy}}{U}\frac{\partial\phi}{\partial y} = 0. \tag{48}$$

To satisfy the boundary condition given by Eq. (32), ϕ is expressed by the following integral

$$\phi(y, z) = \int_0^\infty f(\lambda)e^{-\lambda z}Y_\lambda(y)d\lambda \tag{49}$$

for $z>0$. $f(\lambda)$ is an unknown function to be determined. For $z<0$,

$$\phi(y, z) = -\phi(y, -z). \tag{50}$$

By substituting Eq. (49) into Eq. (48), the differential equation for $Y_\lambda(y)$ is obtained,

$$\frac{d^2Y_\lambda}{dy^2} - 2\frac{\frac{dU}{dy}}{U}\frac{dY_\lambda}{dy} + \lambda^2 Y_\lambda = 0. \tag{51}$$

This equation will determine $Y_\lambda(y)$ uniquely if proper normalizing and boundary conditions are imposed.

At the span, the condition (33) must be satisfied. Thus

$$\frac{l(y)}{2} = \int_0^\infty f(\lambda)Y_\lambda(y)d\lambda. \tag{52}$$

This relation can be considered as the equation for determining $f(\lambda)$ with the given lift distribution $l(y)$. For example, in the case of constant stream velocity U or Prandtl's case, $Y_\lambda(y)$ is a trigonometric function and therefore $f(\lambda)$ can be determined easily by means of Fourier's inversion theorem. Equation (50) shows that with $f(\lambda)$ so determined, the condition (34) will be automatically satisfied.

The downwash velocity w_0 at the wing can then be easily calculated by using Eqs. (25), (36) and (49). The result is

$$w_0(y, 0) = -\frac{1}{2\rho U}\int_0^\infty \lambda f(\lambda)Y_\lambda(y)d\lambda. \tag{53}$$

The induced drag D_i is given by

$$D_i = -\int_{-\infty}^{\infty} l(y)\frac{w_0(y, 0)}{U}dy.$$

Therefore, in terms of $Y_\lambda(y)$, the following general expression for the induced drag is obtained:

$$D_i = \int_{-\infty}^{\infty}\frac{1}{\rho U^2}dy\int_0^\infty f(\lambda)Y_\lambda(y)d\lambda\int_0^\infty \eta f(\eta)Y_\eta(y)d\eta. \tag{54}$$

Thus the problem of calculating the induced drag with a given distribution of lift $l(y)$ is reduced to the problem of solving the integral equation (52) for $f(\lambda)$ and then evaluating the integral given by Eq. (54).

If the chord c, the geometrical angle of attack α and the slope k of the lift coefficient are given instead of the lift distribution $l(y)$, then

$$l(y) = \tfrac{1}{2}\rho U^2 ck\left\{\alpha + \frac{w_0(y, 0)}{U}\right\}. \tag{55}$$

Thus Eq. (52) is replaced by the following equation

$$\tfrac{1}{4}\rho U^2 ck\left\{\alpha - \frac{1}{2\rho U^2}\int_0^\infty \lambda f(\lambda)Y_\lambda(y)d\lambda\right\} = \int_0^\infty f(\lambda)Y_\lambda(y)d\lambda,$$

or

$$\tfrac{1}{4}\rho U^2 ck\alpha = \int_0^\infty\left(1 + \frac{ck}{8}\lambda\right)f(\lambda)Y_\lambda(y)d\lambda. \tag{56}$$

This is now the integral equation for $f(\lambda)$. When $f(\lambda)$ is determined, the induced drag D_i can be again calculated by using Eq. (54).