ON ROTATIONAL GAS FLOWS*

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Introduction. The main body of the science of aerodynamics is based on the classical theory of frictionless, incompressible, irrotational fluids. Recently airplanes have attained such high velocities that this fluid model has proved to be too restricted and interest has centered on the irrotational motion of frictionless, compressible fluids. By the term "compressible fluid" one generally means a fluid for which the density ρ and pressure p are connected by the isentropic relation $p\rho^{-\gamma}=$const. However, the student of aerodynamics is frequently interested in supersonic phenomena and because of the possible occurrence of shock waves, such flows cannot be described, in general, by isentropic, irrotational flows. Accordingly, it becomes necessary to study the motion of gases under less restricted conditions.

Let us call a fluid barotropic when there is a unique functional relationship between the pressure and the density of the fluid. The most important examples are the incompressible fluid where the same constant density belongs to each pressure and the isentropic fluid where the relation $p\rho^{-\gamma}=$const. holds. The dynamics of frictionless barotropic fluids is based on a theorem due to Lagrange. *If a fluid particle is irrotational at one moment, it will remain so for all subsequent time.* One can generally assume in aerodynamics that the air starts from rest. The dynamics of the flow can then be summed up in the single statement that the motion is irrotational. It follows that the velocity distribution admits a potential, and the comparative mathematical simplicity of the dynamics of frictionless barotropic fluids follows from this fact.

Classical fluid dynamics deals almost exclusively with the theory of frictionless barotropic fluids. To find an example of frictionless non-barotropic fluids, we turn to the theory of the propagation of waves. When Newton developed his theory of sound-waves, he assumed that the motion of air was isothermal. Later his theory was superseded by a better one which assumes isentropic motion. Thus, both theories assumed that the transmitting medium was barotropic. The mathematical theory of one-dimensional *large disturbances*, a much more difficult problem, was developed first by Riemann, who again *assumed* that the flow is isentropic. However, the isentropic theory of shock-waves turns out to be fallacious because it can be shown to violate the law of conservation of energy. *When shock-waves are considered the fluid model must be extended to include non-barotropic fluids.*

In this connection, let us draw attention to the thermodynamical aspect of the general theory of compressible fluids. In the case of a three-dimensional flow there are six unknowns: three velocity components, pressure, density and temperature. The laws of conservation of matter and momentum together with the equation of state yield only five equations. To get the missing sixth equation the law of conservation of energy, i.e., the first law of thermodynamics, must be used. Flows will be isentropic only when as a consequence of these laws the entropy turns out to be a constant.

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Although the theory of one-dimensional shock-waves requires a non-barotropic fluid model, this fluid model is a very special one. Even if there is an increase of entropy across shock-waves, the flow remains isentropic *between* shock-waves. Moreover, a one-dimensional fluid motion is always irrotational But when one turns to two or three-dimensional shock-waves, the situation becomes quite different. In this case Hadamard[1] was the first to point out in 1903, that *vortices are generated suddenly by shock-waves* and, in general, the flow becomes non-barotropic after shock-waves.

Hadamard determined the sudden change of circulation across a shock-wave but was not interested in the circulation variations occurring in the fluid behind shock-waves. A general circulation theorem for frictionless barotropic fluids was established by Bjerknes[2] in 1900, for the purposes of his dynamical theory of meteorology. The motion of air masses originating from non-homogeneous conditions is clearly a phenomenon requiring a non-barotropic fluid model.

Crocco,[3] in 1937, again took up the question of the motion of frictionless fluids behind shock-waves. By restricting himself to the steady state he discovered a very useful theorem. Recently, this theorem was generalized by the author of the present paper.[4]

So far, we have spoken only about frictionless fluids. There are problems with respect to the flow of gases where viscosity cannot be neglected. We mention, for instance, the boundary layer theory and the behavior of a gas within shock-waves. It appears probable that when considering the viscous flow of gases, the conductivity of the gas cannot be neglected in general. Variations in viscosity might have importance also. No general theorems are available for such flows and we shall have to be content with presenting the fundamental differential equations governing these phenomena. Any investigation with respect to the flow of gases must be based on these equations. While in the case of frictionless flows some general consequences of the fundamental equations are available, in the case of viscous flows we must start the investigation of each problem by examining the fundamental equations anew.

Lagrange's theorem plays a fundamental role in our concepts about fluid dynamics. Its validity is restricted, however. The art of aeronautics is now at a point where we have to extend our fluid model and thus modify some of our basic concepts. We must accept for instance the fact that vortices can be generated in the midst of a frictionless fluid. Whether this extended fluid model will be able to account for all the phenomena which we may wish to consider, only the future can tell

I. THE FUNDAMENTAL EQUATIONS

1. Continuity equation. From the law of conservation of matter it can be proved that

$$\operatorname{div} \rho \mathbf{q} = -\frac{\partial \rho}{\partial t},$$

[1] J. Hadamard, *Sur les tourbillons produit par les ondes de choc*, Note III, in *Leçons sur la propagation des ondes*, A. Hermann, Paris, 1903, p. 362.

[2] V. Bjerknes, *Das dynamische Princip der Circulationsbewegungen in der Atmosphäre*, Meteorologische Zeitschrift, **17**, 97–106 (1900).

[3] L. Crocco, *Eine neue Stromfunktion für die Erforschung der Bewegung der Gase mit Rotation*, Zeitschrift f. Ang. Math. und Mech., **17**, 1–7 (1937).

[4] A. Vazsonyi, *On two-dimensional rotational gas flows*, Bull. of the American Mathematical Society, **50**, 188 (1944).

where $\mathbf{q}$ is the velocity vector. Another useful form of the continuity equation is given by

$$\operatorname{div} \mathbf{q} = -\frac{1}{\rho}\frac{d\rho}{dt}\cdot \tag{C}$$

2. The Navier-Stokes equation. From the law of conservation of momentum it can be proved that the equation of motion is given by

$$\frac{d\mathbf{q}}{dt} = -\frac{1}{\rho}\operatorname{grad} p + \frac{\mu}{\rho}\Delta\mathbf{q} + \frac{\mu}{3\rho}\operatorname{grad}\operatorname{div}\mathbf{q}, \tag{M}$$

where it is assumed that the viscosity μ is constant.

It will be useful to derive certain other forms of this equation. The specific enthalpy h of a gas is defined by

$$h = U + p\rho^{-1}, \tag{2.1}$$

where U denotes the specific internal energy. The specific entropy s is defined by

$$Tds = dU + pd(\rho^{-1}) \tag{2.2}$$

where T denotes the absolute temperature. From Eqs. (2.1) and (2.2) it follows that

$$Tds = dh - \rho^{-1}dp. \tag{2.3}$$

Using vector notation and considering only spacial variations, we may then write

$$T \operatorname{grad} s = \operatorname{grad} h - \rho^{-1}\operatorname{grad} p. \tag{2.3'}$$

From this last equation and the equation of motion we find that

$$\frac{d\mathbf{q}}{dt} = T\operatorname{grad} s - \operatorname{grad} h + \frac{\mu}{\rho}\Delta\mathbf{q} + \frac{\mu}{3\rho}\operatorname{grad}\operatorname{div}\mathbf{q}. \tag{M'}$$

Another useful form of the equation of motion can be obtained by using the stagnation enthalpy

$$h_0 = h + \tfrac{1}{2}q^2 \tag{2.4}$$

and the identity

$$\frac{d\mathbf{q}}{dt} = \frac{\partial\mathbf{q}}{\partial t} + \operatorname{grad}(\tfrac{1}{2}q^2) - \mathbf{q}\times\boldsymbol{\omega} \qquad (\boldsymbol{\omega} = \operatorname{curl}\mathbf{q}) \tag{2.5}$$

together with the equation of motion (M'). Thus one obtains

$$\frac{\partial\mathbf{q}}{\partial t} - \mathbf{q}\times\boldsymbol{\omega} = -\operatorname{grad} h_0 + T\operatorname{grad} s + \frac{\mu}{\rho}\Delta\mathbf{q} + \frac{\mu}{3\rho}\operatorname{grad}\operatorname{div}\mathbf{q}. \tag{M''}$$

A fourth useful form of the equation of motion can be obtained by introducing the rate of change of the stagnation enthalpy. Differentiating Eq. (2.4) with respect to t, we obtain

$$\frac{dh_0}{dt} = \frac{dh}{dt} + \frac{1}{2}\frac{dq^2}{dt}\cdot \tag{2.6}$$

From Eq. (2.3) it then follows that

$$\frac{dh_0}{dt} = T\frac{ds}{dt} + \frac{1}{\rho}\frac{dp}{dt} + \mathbf{q}\cdot\frac{d\mathbf{q}}{dt} = T\frac{ds}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{1}{\rho}\mathbf{q}\cdot\text{grad}\, p + \mathbf{q}\cdot\frac{d\mathbf{q}}{dt}. \tag{2.7}$$

Using the continuity equation (C′), we may write this as follows:

$$\frac{dh_0}{dt} = T\frac{ds}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial t} + \mathbf{q}\cdot\left(\frac{1}{\rho}\text{grad}\, p + \frac{d\mathbf{q}}{dt}\right). \tag{2.8}$$

Introducing the last expression into the equation of motion (M′), we finally obtain

$$\frac{dh_0}{dt} = T\frac{ds}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{\mu}{\rho}\mathbf{q}\cdot(\Delta\mathbf{q} + \tfrac{1}{3}\,\text{grad div}\,\mathbf{q}). \tag{M'''}$$

3. The energy equation. From the law of conservation of energy, it can be shown[5] that

$$\frac{dU}{dt} + p\frac{d(\rho^{-1})}{dt} = \frac{1}{\rho}\phi + Q. \tag{E}$$

The first term on the left-hand side accounts for the rate of change of internal energy; the second term stands for the work required to compress the fluid. The first term on the right-hand side represents the heat generated by the viscous forces. The dissipation function ϕ is defined by

$$\phi = \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 - \frac{2}{3}(\text{div}\,\mathbf{q})^2\right]. \tag{3.1}$$

Finally, the last term on the right-hand side accounts for the heat added to the fluid per unit of time per unit of mass. For instance, when all the heat transfer is due to the conductivity of the fluid (no external heat sources, no radiation), Q is given by

$$Q = \rho^{-1}\,\text{div}\,(k\,\text{grad}\,T). \tag{3.2}$$

The energy equation can be simplified by introducing the entropy from Eq. (2. 3); thus we have

$$T\frac{ds}{dt} = Q + \frac{1}{\rho}\phi. \tag{E'}$$

The interpretation of this last equation is particularly simple. *The right-hand side represents the total heat increase of the fluid, while the left-hand side gives the corresponding product of the temperature by the entropy change.* By combining the equation of motion (M''') with the energy equation (E'), we obtain the following important relation,

$$\frac{dh_0}{dt} = \frac{1}{\rho}\frac{\partial p}{\partial t} + Q + \frac{1}{\rho}\phi + \frac{\mu}{\rho}\mathbf{q}\cdot(\Delta\mathbf{q} + \tfrac{1}{3}\,\text{grad div}\,\mathbf{q}). \tag{E''}$$

In the literature, the energy equation is frequently given in this last form.

[5] See J. Ackeret, *Handbuch der Physik*, vol. 7, Berlin 1927, chap. 5, p. 293, or H. Lamb, l.c. pp. 575 and 637. In the energy equation it is *not* assumed that μ is constant.

II. VORTEX THEOREMS FOR FRICTIONLESS FLUIDS

4. The circulation theorems. The circulation is defined by

$$\Gamma = \oint \mathbf{q}\cdot d\mathbf{l}. \tag{4.1}$$

where the integration is to be taken along a closed curve formed by fluid particles. It is easy to show that the rate of change of the circulation is given by

$$\frac{d\Gamma}{dt} = \oint \frac{d\mathbf{q}}{dt}\cdot d\mathbf{l}. \tag{4.2}$$

Let us combine the last relation with the equation of motion (M). If the viscous terms are omitted, we have

$$\frac{d\Gamma}{dt} = -\oint \frac{1}{\rho}\,\mathrm{grad}\ p\cdot d\mathbf{l} = -\oint \frac{dp}{\rho}. \tag{4.3}$$

By means of the identity

$$0 = \oint d(\rho^{-1}p) = \oint p\,d(\rho^{-1}) + \oint \rho^{-1}dp, \tag{4.4}$$

Eq. (4.3) can be transformed into

$$\frac{d\Gamma}{dt} = -\oint p(\mathrm{grad}\ \rho^{-1})\cdot d\mathbf{l} = -\oint p\,d(\rho^{-1}). \tag{4.3'}$$

Instead of using Eq. (M), we can use Eq. (M′) and thus obtain

$$\frac{d\Gamma}{dt} = \oint T(\mathrm{grad}\ s)\cdot d\mathbf{l} = \oint T ds. \tag{4.3''}$$

Sometimes it is preferable to transform the line integrals into surface integrals with the aid of Stokes' theorem. Thus it follows from the last two equations that

$$\frac{d\Gamma}{dt} = -\oint\!\!\oint [\mathrm{grad}\ \rho^{-1}) \times \mathrm{grad}\ p]dA, \tag{4.5}$$

$$\frac{d\Gamma}{dt} = \oint\!\!\oint [(\mathrm{grad}\ T) \times \mathrm{grad}\ s]dA. \tag{4.5 }$$

We now come to the interpretation of the equations for $d\Gamma/dt$. When the fluid is barotropic, the right-hand side is zero in all of these equations, and the theorem of Lord Kelvin is then obtained. *The circulation along a closed "fluid line" in a barotropic fluid is constant for all time.* In particular, when the circulation is zero at a certain instant, it will remain so for all subsequent time. By applying Kelvin's theorem to an indefinitely small closed line, Lagrange's theorem is obtained.

In the case of non-barotropic fluids, the situation is quite different. The right-hand sides in the equations are not zero in general and Kelvin's theorem does not hold. Bjerknes[2] gave a simple geometrical interpretation of Eq. (4.5). Let us draw equidistant members of the families of surfaces $p=$const. and $\rho^{-1}=$const. and so obtain a series of tubes bounded by these surfaces. The theorem of Bjerknes states that the *rate of change of circulation per unit of time along a fluid line C is proportional to the number of tubes surrounded by C.* (In the case of a barotropic fluid the surfaces $p=$const. and $\rho^{-1}=$const. are identical.)

A very similar interpretation can be given to Eq. (4.5′) by considering tubes formed by the families of surfaces $T=$const. and $s=$const. In the case of a barotropic fluid, these two families of surfaces are identical, unless the flow is isentropic in which case the surfaces $s=$const. are no longer defined. Bjerknes' theorem, in this modified form, will be useful in a later part of this paper.

5. Theorems with respect to the rotation. Helmholtz's theorem. With the aid of Eqs. (2.5) and (C′) it can be easily proved that

$$\operatorname{curl}\frac{d\mathbf{q}}{dt}=\frac{d(\rho^{-1}\boldsymbol{\omega})}{dt}-(\boldsymbol{\omega}\nabla)\cdot\mathbf{q}. \tag{5.1}$$

Applying the operator curl to both sides of the equation of motion (M) or (M′), we obtain in the frictionless case

$$\frac{d(\rho^{-1}\boldsymbol{\omega})}{dt}-(\boldsymbol{\omega}\nabla)\cdot\mathbf{q}=-\operatorname{grad}\rho^{-1}\times\operatorname{grad}p, \tag{5.2}$$

or

$$\frac{d(\rho^{-1}\boldsymbol{\omega})}{dt}-(\boldsymbol{\omega}\nabla)\cdot\mathbf{q}=\operatorname{grad}T\times\operatorname{grad}s. \tag{5.2′}$$

In the case of barotropic fluids, the right-hand side equals zero. (For two-dimensional flows the second term on the left-hand side equals zero because $\boldsymbol{\omega}$ is everywhere normal to $\mathbf{q}$.) Thus, for barotropic fluids,

$$\frac{d(\rho^{-1}\boldsymbol{\omega})}{dt}=(\rho^{-1}\boldsymbol{\omega}\nabla)\cdot\mathbf{q}. \tag{5.2″}$$

A geometrical interpretation of the last equation led Helmholtz to the discovery of his famous vortex theorems. *Vortex lines are material lines. The product of the cross-sectional area and of the vorticity* ω *of a vortex filament is constant both in space and time.** (Holmholtz unnecessarily restricted his investigations to incompressible fluids.) In the case of non-barotropic fluids, (5.2″) must be replaced by the more general Eq. (5.2) and the Helmholtz vortex theorems do not hold any more. Friedman[6] derived certain theorems for non-barotropic fluids which are somewhat analogous to the Helmholtz theorems.

6. The theorem of Crocco and its generalization. In the case of steady, frictionless flows the equation of motion (M″) simplifies to the important relation

$$\mathbf{q}\times\boldsymbol{\omega}=\operatorname{grad}h_0-T\operatorname{grad}s. \tag{6.1}$$

We will see later that for a very important type of flow h_0 is constant throughout the field. In this case Eq. (6.1) reduces to

* A line which at each point is tangent to the vorticity vector $\boldsymbol{\omega}$, at this point, is called a vortex line. An infinitely thin tube formed by vortex lines is called a vortex filament.

[6] A. A. Friedmann, *Über Wirbelbewegung in einer kompressiblem Flüssigkeit,* Zeitschrift f. Ang. Math. und Mech., **4,** 102–107 (1924).

$$\mathbf{q} \times \boldsymbol{\omega} = -T \operatorname{grad} s. \tag{6.1'}$$

This last relation was discovered by Crocco.[3] When both h_0 and s are constant the right-hand side of Eq. (6.1) is zero and so the motion must be irrotational. *The importance of Eq.* (6.1) *lies in the fact that it relates the rotation of the fluid to the rates of change of h_0 and s.*

III. ADIABATIC, STEADY, FRICTIONLESS FLOWS

7. General relations. For the flows considered in this chapter the energy equations (E′) and (E″) reduce to

$$\frac{\partial s}{\partial \sigma} = 0, \quad (7.1) \quad ; \quad \frac{\partial h_0}{\partial \sigma} = 0, \tag{7.2}$$

where $\partial/\partial\sigma$ indicates differentiation along a streamline. Accordingly, both *the entropy and the stagnation enthalpy are constant along each streamline* (but they might vary from one streamline to another). Because of its great importance, we shall write out the integral of Eq. (7.2) in detail for a perfect gas with constant specific heats. One obtains

$$h_0 = \frac{1}{2} q^2 + h = \frac{1}{2} q^2 + c_p T = \frac{1}{2} q^2 + \frac{c_p}{R} \frac{p}{\rho} = \text{const. along a streamline}, \tag{7.2}$$

where the equation of state

$$p/\rho = RT \tag{7.3}$$

is used.

The modified Bjerknes theorems simplify somewhat for the flows considered in this chapter, because the lines of constant entropy coincide with the streamlines. Similarly the generalized Crocco theorem [Eq. (6.1)] simplifies, because the streamlines coincide with both the lines of constant entropy and the lines of constant stagnation enthalpy.

An example illustrating these theorems will be useful.* Consider the discharge of a perfect gas from a container. We assume that the gas is originally in equilibrium, that is, that the pressure p_0 is constant, but do not assume that the temperature T_0 is constant. In order to use Bjerknes' theorem (in its modified form) we construct the net formed by the lines of constant entropy and the lines of constant temperature. At the beginning of the experiment the pressure is constant and these lines coincide. Thus it follows from Bjerknes' theorem that $d\Gamma/dt = 0$. However, at a subsequent instant, the lines become distinct and so the motion becomes rotational. Let us proceed now to determine the rotation. In order to use the generalized Crocco theorem we consider only steady state flow (infinite container). According to our energy theorem, the entropy and the stagnation enthalpy (and consequently the stagnation temperature) are constant along each streamline. Furthermore, since p_0 is a constant, it follows from the thermodynamical relation (2.3′) that

$$\operatorname{grad} h_0 = T_0 \operatorname{grad} s. \tag{7.4}$$

Thus from Eq. (6.1)

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* The author is indebted to Professor H. W. Emmons of Harvard University for this example.

$$\mathbf{q} \times \boldsymbol{\omega} = \left(1 - \frac{T}{T_0}\right) \text{grad } h_0 \tag{7.5}$$

or, after simplifications,

$$\mathbf{q} \times \boldsymbol{\omega} = \tfrac{1}{2}q^2 \cdot \text{grad } (\ln T_0). \tag{7.5'}$$

We observe again that although the flow originates from a resting gas, the motion is rotational in general.

8. Two-dimensional flow. The continuity equation shows that in this case there exists a stream function such that

$$u = \rho^{-1}\partial\psi/\partial y, \qquad v = -\rho^{-1}\partial\psi/\partial x. \tag{8.1}$$

From the definition of the rotation, it follows that the stream function must satisfy the following equation

$$\partial(\rho^{-1}\psi_x)/\partial x + \partial(\rho^{-1}\psi_y)/\partial y = -\omega \qquad (= -\partial v/\partial x + \partial u/\partial y). \tag{8.2}$$

In order to determine ω we use Eq. (6.1). Because s and h_0 are constant along a streamline,

$$q\omega = T\frac{\partial s}{\partial n} - \frac{\partial h_0}{\partial n}, \tag{8.3}$$

where $\partial/\partial n$ indicates differentiation normal to a streamline. Both the entropy and the stagnation enthalpy are functions of ψ alone. By using the relation

$$\frac{\partial}{\partial n} = q\rho\frac{\partial}{\partial\psi}, \tag{8.4}$$

we find from Eq. (8.3),

$$\omega = \rho\left(T\frac{\partial s}{\partial\psi} - \frac{\partial h_0}{\partial\psi}\right). \tag{8.3'}$$

For a perfect gas this reduces to

$$\omega = \frac{p}{R}\frac{\partial s}{\partial\psi} - \rho\frac{\partial h_0}{\partial\psi}. \tag{8.3''}$$

Noting that $\partial h_0/\partial\psi$ and $\partial s/\partial\psi$ are constant along any given streamline, one observes that the *rotation on each streamline is a linear combination of the density and the pressure.* If h_0 is constant, throughout the flow the rotation is proportional to the pressure.[3] If s is constant, throughout the flow the rotation is proportional to the density.[7] (The constant of proportionality is given by the rate of change of entropy or stagnation enthalpy normal to the streamline.)

It is of some interest to develop Eq. (7.5′) for the two-dimensional case. Here we find that

$$\omega = \frac{\rho q^2}{2}\frac{\partial(\ln T_0)}{\partial\psi}, \tag{8.5}$$

and the rotation is thus seen to be proportional to ρq^2 along each streamline.

Finally we mention that after rather lengthy computations the differential equation for ψ, Eq. (8.2), can be transformed into

[7] K. O. Friedrichs (and R. von Mises), *Fluid dynamics*, Brown University, Providence, R. I., 1941, p. 229.

$$\left(1-\frac{u^2}{a^2}\right)\psi_{xx}-\frac{2uv}{a^2}\psi_{xy}+\left(1-\frac{v^2}{a^2}\right)\psi_{yy}=\rho^2\left[\frac{\partial h_0}{\partial\psi}-\frac{k-1}{kR}\left(h_0+\frac{q^2}{2}\right)\frac{\partial s}{\partial\psi}\right]. \qquad (8.6)$$

When both the entropy and the stagnation enthalpy are constant throughout the field, the right-hand side of Eq. (8.6) becomes zero and one obtains the familiar equation of a steady isentropic irrotational flow.

9. Flow around an obstacle with shock-waves. Shock-waves can be included in our theory by admitting such discontinuities in the flow pattern as are compatible with the laws of conservation of matter, momentum and energy. Thus, the previous theory can be applied for flows between shock-waves For most purposes one can assume that the air comes from a homogeneous condition and in particular that h_0 and s are constant far ahead of the obstacle. It follows from Eqs. (7.1) and (7.2) that both h_0 and s are constant at least up to the first shock-wave, and then again along each streamline between consecutive shock-waves. *Hence $\omega=0$ on each streamline up to the first shock-wave. In particular, if a streamline is not intersected by a shock-wave, ω remains zero all along this streamline.* From the law of conservation of energy it can be deduced that h_0 must be continuous across a shock-wave and thus h_0 must be a constant throughout the field. Hence from Eq. (8.3″)

$$\omega=\frac{p}{R}\frac{\partial s}{\partial\psi}, \qquad (9.1)$$

and the rotation is proportional to the pressure along each streamline between shock-waves. Furthermore it is known that the entropy increases across a shock-wave and the increase depends on the magnitude of the shock. Hence $\partial s/\partial\psi$ is not zero in general after a shock-wave and the motion is rotational. *Generally speaking there is always a sudden increase of the rotation across shock-waves* (see Hadamard), *and then the rotation remains proportional to the pressure* (see Crocco).

10. Flow with axial symmetry. Let the x axis be the axis of symmetry of the flow. Then there is a stream function such that

$$u=r^{-1}\rho^{-1}\partial\psi/\partial r, \qquad v=-r^{-1}\rho^{-1}\partial\psi/\partial x, \qquad (10.1)$$

where

$$r=\sqrt{y^2+z^2} \qquad (10.2)$$

and v is the velocity component normal to the x axis.

Quite similarly to the two-dimensional case, it follows from Eq. (6.1) that, in the present case,

$$\omega=r\rho\left(T\frac{\partial s}{\partial\psi}-\frac{\partial h_0}{\partial\psi}\right). \qquad (10.3)$$

For a perfect gas, we have

$$\omega=\frac{rp}{R}\frac{\partial s}{\partial\psi}-r\rho\frac{\partial h_0}{\partial\psi}. \qquad (10.3')$$

When h_0 is a constant throughout the field, one recognizes in Eq. (10.3) a relation discovered by Crocco.[3]