A CYLINDER COOLING PROBLEM*

BY

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1. Introduction. The linear cooling problem for non-homogeneous solids has been investigated extensively by Rust,[1] Churchill,[2] Carslaw,[3] Mersman,[4] and others. It is the purpose of this paper to obtain a solution for the corresponding cylindrical problem. The method used is that of the Laplace Transform.

2. The Problem. Let us consider an infinitely long circular cylinder of radius a and initial temperature T_0, instantaneously immersed in an infinite medium initially at zero temperature. Let the heat conductivities and diffusivities of the cylinder and external medium be respectively K_ν and h_ν^2 $(\nu=1, 2)$. Then if r is the distance from the axis of the cylinder and t is the time, the following differential system is satisfied[5] by the temperature functions $T_\nu(r, t)$:

$$h_1^2\left\{\frac{\partial^2 T_1}{\partial r^2}+\frac{1}{r}\frac{\partial T_1}{\partial r}\right\}=\frac{\partial T_1}{\partial t} \qquad 0 \leqq r < a, \quad t > 0, \tag{1}$$

$$h_2^2\left\{\frac{\partial^2 T_2}{\partial r^2}+\frac{1}{r}\frac{\partial T_2}{\partial r}\right\}=\frac{\partial T_2}{\partial t} \qquad r > a, \quad t > 0, \tag{2}$$

$$\lim_{r\to a-} T_1 = \lim_{r\to a+} T_2 \qquad t > 0, \tag{3}$$

$$\lim_{r\to a-} K_1\frac{\partial T_1}{\partial r} = \lim_{r\to a+} K_2\frac{\partial T_2}{\partial r} \qquad t > 0, \tag{4}$$

$$\lim_{t\to 0} T_1 = T_0 \qquad 0 \leqq r < a, \tag{5}$$

$$\lim_{t\to 0} T_2 = 0 \qquad r > a. \tag{6}$$

3. Solution. Let the Laplace transform of $T_\nu(r, t)$ be $T_\nu^*(r, s)$, i.e.,

$$T_\nu^*(r, s) = \int_0^\infty e^{-st}T_\nu(r, t)dt \qquad s > 0.$$

Applying this transform to (1)–(6), we obtain the corresponding set of ordinary differential equations containing s as a parameter;

* Received June 18, 1945.

[1] W. M. Rust, Jr., *Integral equations and the cooling problem*, Amer. J. Math. **54**, 190–212 (1932).

[2] R. V. Churchill, *A heat conduction problem*, Philos. Mag. (7), **31**, 81–87 (1941).

[3] H. S. Carslaw, *A simple application of the Laplace transformation*, Philos. Mag. (7), **30**, 414–417 (1940).

[4] W. A. Mersman, *Heat conduction in an infinite composite solid*, Bull. Amer. Math. Soc. **47**, 956–964 (1941).

[5] H. S. Carslaw, *Theory of heat conduction*, Macmillan New York, ed. 2, 1921, Chapter I.

$$h_1^2\left\{\frac{d^2T_1^*}{dr^2}+\frac{1}{r}\frac{dT_1^*}{dr}\right\} = -T_0+sT_1^* \qquad 0 \leqq r < a, \tag{1*}$$

$$h_2^2\left\{\frac{d^2T_2^*}{dr^2}+\frac{1}{r}\frac{dT_2^*}{dr}\right\} = sT_2^* \qquad r > a, \tag{2*}$$

$$\lim_{r\to a-} T_1^* = \lim_{r\to a+} T_2^*, \tag{3*}$$

$$\lim_{r\to a-} K_1\frac{dT_1^*}{dr} = \lim_{r\to a+} K_2\frac{dT_2^*}{dr}\cdot \tag{4*}$$

The solution[6] of this system is

$$T_1^*(r, s) = \frac{T_0}{s}\left[1+\frac{K_0'(\alpha_2\sqrt{s})\cdot I_0(r\sqrt{s}/h_1)}{D(\sqrt{s})}\right], \tag{7}$$

$$T_2^*(r, s) = \frac{T_0}{s}\,\frac{\alpha_3 I_0'(\alpha_1\sqrt{s})\cdot K_0(r\sqrt{s}/h_2)}{D(\sqrt{s})}, \tag{8}$$

where

$$\alpha_1 = a/h_1, \qquad \alpha_2 = a/h_2, \qquad \alpha_3 = K_1h_2/K_2h_1, \tag{9}$$

$$D(x) = \alpha_3 I_0'(\alpha_1 x)K_0(\alpha_2 x) - I_0(\alpha_1 x)K_0'(\alpha_2 x). \tag{10}$$

The functions $T_\nu(r, t)$ may now be obtained by use of the complex inversion formula[7]

$$T_\nu(r, t) = \lim_{\lambda\to\infty}\frac{1}{2\pi i}\int_{c-i\lambda}^{c+i\lambda} e^{zt}T_\nu^*(r, z)dz, \qquad c > 0. \tag{11}$$

In order to reduce these contour integrals to real integrals we must first establish the following lemma.

Lemma. *$D(z)$ does not vanish for* $|\arg z| \leqq \frac{1}{2}\pi$.

Proof. We choose two numbers A and B, arbitrary except that

$$0 < A < 1 < B. \tag{12}$$

Then since α_ν is positive ($\nu=1, 2, 3$), it will be sufficient to show that, when $|\arg z| \leqq \frac{1}{2}\pi$, $D(z)$ does not vanish for any values of α_ν such that $A \leqq \alpha_\nu \leqq B$ ($\nu=1, 2, 3$). The proof now follows in four parts.

i) There is a number $R_1>0$, such that $D(z)$ does not vanish for

$$A \leqq \alpha_\nu \leqq B, \qquad |\arg z| \leqq \tfrac{1}{2}\pi, \qquad |z| < R_1.$$

This is true because we may use the ordinary series expansions of the Bessel function[8] to write

$$D(z) = \frac{1}{\alpha_2 z} + (\alpha_2 - \alpha_1\cdot\alpha_3)\cdot\frac{z}{2}\cdot\log\frac{\alpha_2 z}{2} + B(z),$$

[6] G. N. Watson, *Theory of Bessel functions*, Cambridge University Press, Cambridge, ed. 2, 1944, p. 79.

[7] D. V. Widder, *The Laplace transformation*, Princeton University Press, Princeton; Oxford University Press, London H. Milford, 1941, p. 66.

[8] G. N. Watson, loc. cit., pp. 77, 80.

where $B(z)$ is bounded in the neighborhood of $z=0$. The result is then evident.

ii) There is a number $R_2>0$, such that $D(z)$ does not vanish for

$$A \leqq \alpha_\nu \leqq B, \qquad |\arg z| \leqq \tfrac{1}{2}\pi, \qquad |z| > R_2.$$

To see this we use the well-known asymptotic formulae for these Bessel functions[9] to write

$$D(z) = \frac{e^{(\alpha_1-\alpha_2)z}}{2z\sqrt{\alpha_1\alpha_2}}\left\{(\alpha_3 + 1 + \epsilon_1) - i(\alpha_3 - 1 + \epsilon_2)e^{-2\alpha_1 z}\right\},$$

where $\epsilon_k \to 0$ as $z\to\infty$ for $k=1, 2$, uniformly in the α_ν providing $A \leqq \alpha_\nu \leqq B$ ($\nu=1, 2, 3$). Clearly, $D(z)$ can vanish only if

$$e^{2\alpha_1 z} = \frac{\alpha_3 - 1 + \epsilon_2}{\alpha_3 + 1 + \epsilon_1}\, i.$$

But for sufficiently large $|z|$, say $|z|>R_2$, the right member is less than unity in absolute value. Hence for $|z|>R_2$, this relation cannot hold with $|\arg z| \leqq \frac{1}{2}\pi$.

iii) $D(z)$ does not vanish for $|\arg z| = \frac{1}{2}\pi$. To see this, we let $z=e^{\frac{1}{2}i\pi}y$ (y real). Then[10]

$$I_0(e^{\frac{1}{2}i\pi}y) = J_0(y), \tag{13}$$

$$K_0(e^{\frac{1}{2}i\pi}y) = -\tfrac{1}{2}i\pi[J_0(y) - iY_0(y)]. \tag{14}$$

Hence

$$\begin{aligned} D(ye^{\frac{1}{2}i\pi}) = &\left\{\alpha_3 J_0'(\alpha_1 y)Y_0(\alpha_2 y) - J_0(\alpha_1 y)Y_0'(\alpha_2 y)\right\} \\ &+ i\left\{\alpha_3 J_0'(\alpha_1 y)J_0(\alpha_2 y) - J_0(\alpha_1 y)J_0'(\alpha_2 y)\right\}. \end{aligned}$$

Therefore $D(ye^{\frac{1}{2}i\pi})$ can vanish only if

$$\alpha_3 J_0'(\alpha_1 y)Y_0(\alpha_2 y) - J_0(\alpha_1 y)Y_0'(\alpha_2 y) = \alpha_3 J_0'(\alpha_1 y)J_0(\alpha_2 y) - J_0(\alpha_1 y)J_0'(\alpha_2 y) = 0.$$

But this is impossible since it would imply either the existence of a common root for at least two of these Bessel functions,[11] or the vanishing of the Wronskian

$$W[J_0(\alpha_2 y), Y_0(\alpha_2 y)] = \frac{2}{\pi\alpha_2 y}.$$

iv) We consider now the integral (see Fig. 1)

$$f(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{2\pi i}\int_C \frac{D'(z)}{D(z)}\, dz.$$

From *i*), *ii*), and *iii*) it follows that $D(z)$ does not vanish on C for all α_ν such that $A \leqq \alpha_\nu \leqq B$ ($\nu=1, 2, 3$). Now these Bessel functions are all analytic except possibly at $z=0$. Hence $f(\alpha_1, \alpha_2, \alpha_3)$ is continuous.

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[9] G. N. Watson, loc. cit., pp. 202, 203.

[10] G. N. Watson, loc. cit., pp. 77, 78.

[11] It is a well-known result that these Bessel functions have no common roots. See G. N. Watson, loc. cit., pp. 479, 480, 481.

Since $D(z)$ has no singularities inside C, $f(\alpha_1, \alpha_2, \alpha_3)$ gives the number of zeros of $D(z)$ inside C. It can therefore take on only integral values; but this implies that $f(\alpha_1, \alpha_2, \alpha_3)$ is constant.

Finally

$$f(1, 1, 1) = \frac{1}{2\pi i}\int_C \frac{W'[I_0(z), K_0(z)]}{W[I_0(z), K_0(z)]}\,dz = 0.$$

Hence $f(\alpha_1, \alpha_2, \alpha_3) \equiv 0$, for all α_ν satisfying the relation $A \leqq \alpha_\nu \leqq B$. Therefore $D(z)$ has no roots inside C. Since the radii R_3 and R_4 (see Fig. 1) are arbitrary, except that

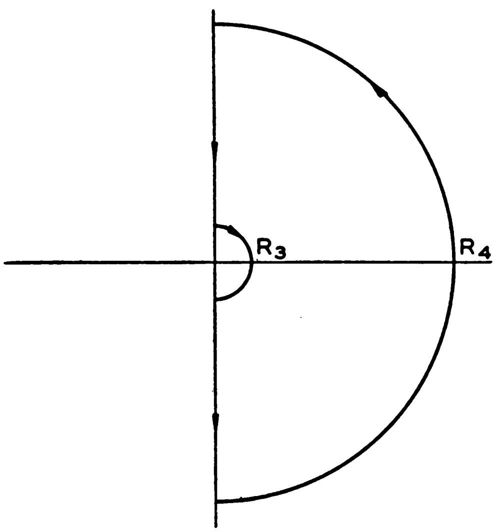

FIG. 1. The contour C, consisting of the circular arcs $|z| = R_3$ and $|z| = R_4$ and the line segments on the imaginary axis joining them. The only restriction is that $R_3 < R_1$ and $R_4 > R_2$.

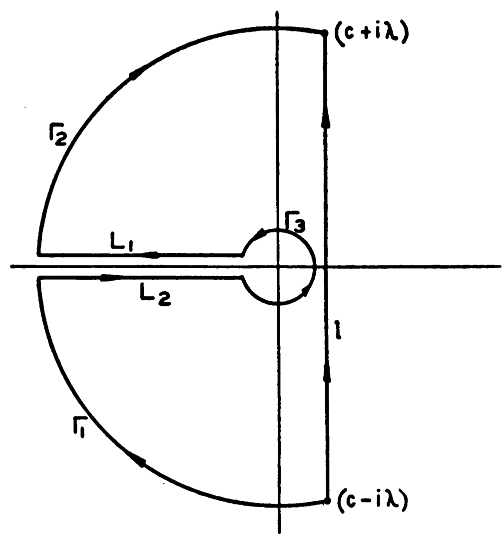

FIG. 2. The contours l, Γ_1, Γ_2, Γ_3, L_1 and L_2. The radius of Γ_3 is ρ.

$R_3 < R_1$ and $R_4 > R_2$, it follows that $D(z)$ has no zeros in the entire right-half plane, which concludes the proof of the lemma.

We now transform the contour integrals of (11) into real integrals. Let us consider $T_1(r, t)$ first. According to the lemma just established, $D(\sqrt{z})$ does not vanish for $|\arg \sqrt{z}| \leqq \frac{1}{2}\pi$, i.e., for $|\arg z| \leqq \pi$. Hence the integrand in (11) is analytic for $|\arg z| \leqq \pi$, and we may (see Fig. 2) replace the integral along l by the sum of the integrals over Γ_1, Γ_2, Γ_3, L_1, and L_2. Using the asymptotic developments, we easily see that for large z,

$$\frac{K_0'(\alpha_2\sqrt{z})I_0(r\sqrt{z}/h_1)}{D(\sqrt{z})} = O\left\{\frac{1}{\sqrt{z}}\exp\left[\left(\frac{r}{h_1} - \alpha_1\right)\sqrt{z}\right]\right\}.$$

Therefore as $\lambda \to \infty$, the integrals over Γ_1 and Γ_2 vanish, since $t > 0$.

Near the origin, the term $K_0'(\alpha_2\sqrt{z})I_0(\alpha_1\sqrt{z})$ dominates the denominator, and hence

$$\lim_{z\to 0}\frac{K_0'(\alpha_2\sqrt{z})I_0(r\sqrt{z}/h_1)}{D(\sqrt{z})} = -1.$$

Therefore the integral over Γ_3 vanishes with ρ (see Fig. 2).

On L_1, we set $z=\sigma^2e^{i\pi}$, $\sigma>0$. Then, using (13) and (14), we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{L_1} e^{zt}T_1^*(r, z)dz=\frac{T_0}{\pi i}\int_{\sqrt{\rho}}^{\sqrt{R}}\frac{e^{-\sigma^2 t}}{\sigma}\\ \cdot\left\{1+\frac{J_0(r\sigma/h_1)Y_0'(\alpha_2\sigma)+iJ_0(r\sigma/h_1)J_0'(\alpha_2\sigma)}{[\alpha_3J_0'(\alpha_1\sigma)Y_0(\alpha_2\sigma)-J_0(\alpha_1\sigma)Y_0'(\alpha_2\sigma)]+i[\alpha_3J_0'(\alpha_1\sigma)J_0(\alpha_2\sigma)-J_0(\alpha_1\sigma)J_0'(\alpha_2\sigma)]}\right\}d\sigma.\end{aligned} \tag{15}$$

On L_2, we set $z=\sigma^2e^{-i\pi}$, $\sigma>0$, and obtain the conjugate of (15). Adding these and taking the limit as $\lambda\to\infty$ and $\rho\to 0$, we obtain finally,

$$T_1(r, t) = \frac{-4T_0\alpha_3}{\pi^2\alpha_2}\int_0^\infty\frac{e^{-\sigma^2 t}}{\sigma^2}\frac{J_0(r\sigma/h_1)J_0'(\alpha_1\sigma)}{\Delta(\sigma)}d\sigma, \tag{16}$$

where

$$\begin{aligned}\Delta(\sigma) = {} & [\alpha_3J_0'(\alpha_1\sigma)Y_0(\alpha_2\sigma) - J_0(\alpha_1\sigma)Y_0'(\alpha_2\sigma)]^2\\ & + [\alpha_3J_0'(\alpha_1\sigma)J_0(\alpha_2\sigma) - J_0(\alpha_1\sigma)J_0'(\alpha_2\sigma)]^2.\end{aligned} \tag{17}$$

Similarly

$$T_2(r, t)=\frac{2T_0\alpha_3}{\pi}\int_0^\infty\frac{e^{-\sigma^2 t}}{\sigma}\frac{J_0'(\alpha_1\sigma)[J_0(\alpha_1\sigma)C_1(\alpha_2\sigma, r\sigma/h_2)-\alpha_3J_0'(\alpha_1\sigma)C(\alpha_2\sigma, r\sigma/h_2)]}{\Delta(\sigma)}d\sigma, \tag{18}$$

where

$$C(x, y) = J_0(x)Y_0(y) - Y_0(x)J_0(y), \tag{19}$$

$$C_1(x, y) = J_0'(x)Y_0(y) - Y_0'(x)J_0(y). \tag{20}$$

These formulae constitute the solution of the differential system (1)–(6).

4. Remarks. Since the Laplace Transform method is essentially a formal one, any solution obtained in this manner must always be verified. In the present case this is easily done.[12]

It may also be shown, under certain conditions as to boundedness and continuity necessarily satisfied by any physical temperature distribution, that expressions (16) and (18) constitute the unique solution of the system (1)–(6).[13]

In conclusion, the author would like to express his thanks to Professor G. C. Evans for his help in the preparation of this paper.

[12] For an example of the method see H. S. Carslaw and J. C. Jaeger, *A problem in conduction of heat*, Proc. Cambridge Philos. Soc. **35**, 394–404 (1939).

[13] For an example of the method see W. M. Rust, Jr., loc. cit., p. 196.