THIN CYLINDRICAL SHELLS SUBJECTED TO CONCENTRATED LOADS*

BY

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Abstract. A single differential equation of the eighth order in the radial displacement is given for the equilibrium of an element of a cylindrical shell undergoing small displacements due to a laterally distributed external load. The radial deflection of thin cylindrical shells subjected to concentrated, equal and opposite forces, acting at the ends of a vertical diameter, is analyzed by the Fourier method. Applications of the solution of the problem of the infinitely long cylinder to the problems of a couple acting on an infinitely long cylinder in the direction of either the generatrix or the circumference are also discussed.

1. Introduction. The bending problem of an infinitely long cylinder loaded with concentrated, equal and opposite forces, acting at the ends of a vertical diameter, is discussed first. The equations of equilibrium of an element of a cylindrical shell undergoing small displacements due to a laterally distributed external load are reduced to a single differential equation of the eighth order in the radial displacement. In this equation the various terms are compared as to the order of magnitude and it is found that some of the terms are negligible.

The specified loading function is represented by a Fourier integral in the longitudinal direction, and by a Fourier series in the circumferential direction. The integral representation has the advantage that the boundary conditions are automatically taken care of, and no subsequent determination of Fourier coefficients is necessary. The Fourier coefficients and the undetermined function in the Fourier integral in this case are determined simply from the loading condition. The radial displacement is represented in a like manner with the aid of an undetermined function which is obtained by substituting both radial displacement and loading expressions in the differential equation. The definite integrals involved in the expression for radial deflection are evaluated by means of Cauchy's theorem of residues.

The problem of the inextensional deformation of cylindrical and spherical shells was treated in detail by Lord Rayleigh in his "Theory of sound." The assumption of this type of deformation underlies the solution of many problems of practical importance, such as the determination of stresses in thin cylindrical shells subjected to two equal and opposite forces acting at the ends of a diameter or to internal hydrostatic pressure. It is found that the results obtained in the case of inextensional deformations correspond only to a first approximation of the solution in this paper, and the stresses in the proximity of the points of application of the forces are not given with sufficient accuracy.

The expression for the radial deflection of a thin cylinder of finite length is obtained from the corresponding solution for an infinitely long cylinder by using the method of images. It is seen that the difference of these two radial deflections can be given by a correction factor included in the expression for a cylinder of finite length.

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The difference is believed to result from restraining the edges at the two ends of the finite cylinder. The results indicate that the radial deflection of an infinitely long cylinder has a very long wave length along the generatrix; however, the wave length decreases as the ratio of radius over thickness decreases. It is believed that the long wave length phenomenon is due to the elastic reaction along the circumference of the shell which can be explained by the radial deflection along the circumference.

Deflection curves of cylindrical shells with various lengths are calculated and the results show that the maximum radial deflection occurs at length over radius ratio $l/a \simeq 20$. The radial deflection of an infinitely long cylinder with the radius over thickness ratio $a/h = 100$, becomes zero at about $x/a = 15$ and then reverses its sign. The edges of the corresponding cylinder with finite length are so restrained that the negative deflection portion of the infinite cylinder is brought to zero at the edges of the cylinder with finite length. Hence, the maximum deflection of a cylinder with $l/a \simeq 20$ is greater than that of the corresponding infinitely long cylinder.

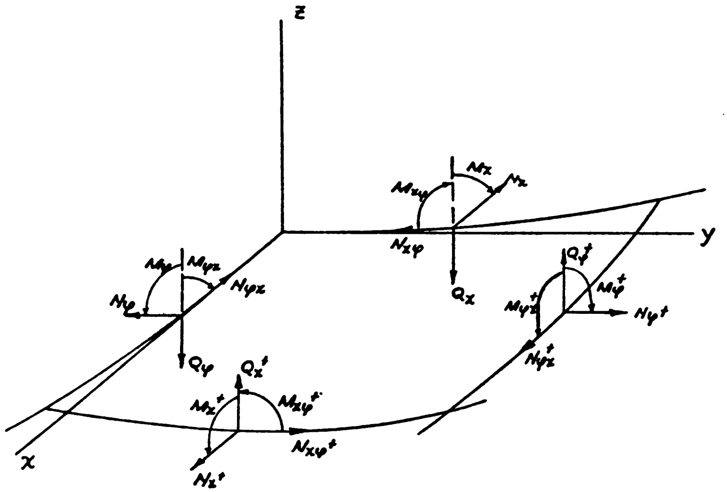

FIG. 1. Forces and moments on element of wall.

The problems of a couple acting on an infinitely long cylinder in the direction of either the generatrix or the circumference are also analyzed by using the corresponding solution for the radial deflection under a concentrated load. The action of the couple is equivalent to that of two equal and opposite forces acting at an infinitely small distance apart.

2. Fundamental equations. The fundamental equations of a cylindrical shell under the specified loading are obtained from considering the equilibrium of an element cut out by two diametrical sections and two cross sections perpendicular to the axis of the cylindrical shell as shown in Fig. 1.

In this discussion the usual assumptions are made; namely, that the material is isotropic and follows Hooke's law, the undeformed tube is cylindrical, the wall thickness is uniform and small compared to the radius, the deflections are small compared to this thickness so that second order strains can be neglected, and that straight lines in the cylinder wall and perpendicular to the middle surface remain straight after distortion.

The notation used for resultant forces and moments per unit length of wall section are indicated in Fig. 1. After simplification, the following equations of equilibrium are obtained:*

$$
\begin{aligned}
&a\frac{\partial N_x}{\partial x}+\frac{\partial N_{\phi x}}{\partial \phi}=0, && a\frac{\partial M_{x\phi}}{\partial x}-\frac{\partial M_{\phi}}{\partial \phi}+aQ_{\phi}=0,\\
&\frac{\partial N_\phi}{\partial \phi}+a\frac{\partial N_{x\phi}}{\partial x}-Q_\phi=0, && a\frac{\partial M_{x}}{\partial x}+\frac{\partial M_{\phi x}}{\partial \phi}-aQ_{x}=0,\\
&a\frac{\partial Q_x}{\partial x}+\frac{\partial Q_\phi}{\partial \phi}+N_\phi+qa=0, && (N_{x\phi}-N_{\phi x})a=0,
\end{aligned}
\tag{1}
$$

in which q is the normal pressure on the element.

If Q_x and Q_ϕ are eliminated from Eqs. (1) and the relations

$$N_{x\phi}=N_{\phi x};\qquad M_{x\phi}=-M_{\phi x}$$

are used, the six equations in (1) can be reduced to the following three:

$$
\begin{aligned}
&a\frac{\partial N_x}{\partial x}+\frac{\partial N_{x\phi}}{\partial \phi}=0,\\
&\frac{\partial N_\phi}{\partial \phi}+a\frac{\partial N_{x\phi}}{\partial x}+\frac{\partial M_{x\phi}}{\partial x}-\frac{1}{a}\frac{\partial M_\phi}{\partial \phi}=0,\\
&-\frac{2}{a}\frac{\partial^2 M_{x\phi}}{\partial\phi\partial x}+\frac{\partial^2 M_x}{\partial x^2}+\frac{1}{a^2}\frac{\partial^2 M_\phi}{\partial \phi^2}+\frac{N_\phi}{a}+q=0.
\end{aligned}
\tag{2}
$$

The relation between the resultant forces and moments and the strains of the middle surface will be taken the same as in the case of a flat plate:

$$
\begin{aligned}
&N_x=\frac{Eh}{1-\nu^2}(\epsilon_x+\nu\epsilon_\phi), && N_\phi=\frac{Eh}{1-\nu^2}(\epsilon_\phi+\nu\epsilon_x), && N_{x\phi}=N_{\phi x}=\frac{\gamma Eh}{2(1+\nu)},\\
&M_x=-D(X_x+\nu X_\phi), && M_\phi=-D(X_\phi+\nu X_x), && M_{x\phi}=-M_{\phi x}=D(1-\nu)X_{x\phi},
\end{aligned}
$$

where $D=Eh^3/12(1-\nu^2)$ is the flexural rigidity of the shell and h is the thickness.

Resolving the displacement at an arbitrary point in the middle surface during deformation into three components—u along the generator, v along the tangent to the circular section, and w along the normal to the surface drawn inwards—one finds that the extensional strains and changes of curvature in the middle surface are

$$
\begin{aligned}
&\epsilon_x=\frac{\partial u}{\partial x}, && \epsilon_\phi=\frac{1}{a}\frac{\partial v}{\partial \phi}-\frac{w}{a}, && \gamma_{x\phi}=\frac{\partial v}{\partial x}+\frac{\partial u}{a\partial \phi},\\
&X_x=\frac{\partial^2 w}{\partial x^2}, && X_\phi=\frac{1}{a^2}\frac{\partial^2 w}{\partial \phi^2}+\frac{1}{a^2}\frac{\partial v}{\partial \phi}, && X_{x\phi}=\frac{1}{a}\frac{\partial^2 w}{\partial x\partial \phi}+\frac{1}{a}\frac{\partial v}{\partial x}.
\end{aligned}
$$

Hence, Eqs. (2) can be put int the form of three equations with three unknowns u, v, w:

$$\frac{\partial^2 u}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial^2 v}{\partial s\partial x}-\frac{\nu}{a}\frac{\partial w}{\partial x}+\frac{1-\nu}{2}\frac{\partial^2 u}{\partial s^2}=0,$$

* See Ref. 5, p. 440.

$$\frac{\partial^2 v}{\partial s^2}+\frac{1+\nu}{2}\frac{\partial^2 u}{\partial s\partial x}+\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2}-\frac{1}{a}\frac{\partial w}{\partial s}+\frac{h^2}{12a}\left(\frac{\partial^3 w}{\partial x^2\partial s}+\frac{\partial^3 w}{\partial s^3}\right)+\frac{h^2}{12a^2}\left((1-\nu)\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial s^2}\right)=0, \tag{3}$$

$$\frac{h^2}{12}\nabla^4 w-\frac{1}{a}\left(\frac{\partial v}{\partial s}-\frac{w}{a}+\nu\frac{\partial u}{\partial x}\right)+\frac{h^2}{12a}\left((2-\nu)\frac{\partial^3 v}{\partial x^2\partial s}+\frac{\partial^3 v}{\partial s^2}\right)-\frac{1-\nu^2}{Eh}q=0,$$

where

$$s=a\phi.$$

In the problem under investigation, the quantities u and v are of the order of magnitude of $\sqrt{hw/a}$. Consequently the last term and the third term in the second and the third equations, respectively, in (3) can be neglected safely.

In order to solve the simultaneous equations (3), one can apply first the operation $\partial^2/\partial x^2$, and then $\partial^2/\partial s^2$ to the first Eq. (3). Solving in each case for the term containing v, and substituting these expressions in the equation obtained by applying $\partial^2/\partial x\partial s$ to the second Eq. (3), one obtains an equation from which v has been eliminated:

$$a\nabla^4 u=\nu\frac{\partial^3 w}{\partial x^3}-\frac{\partial^3 w}{\partial x\partial s^2}+\frac{1+\nu}{1-\nu}\frac{h^2}{12}\left(\frac{\partial^5 w}{\partial x^3\partial s^2}+\frac{\partial^5 w}{\partial x\partial s^4}\right). \tag{4}$$

Similarly, applying $\partial^2/\partial x^2$ and $\partial^2/\partial s^2$ to the second Eq. (3) and solving for the terms containing u, and substituting in the first Eq. (3) after applying $\partial^2/\partial x\partial s$ to it, one obtains an equation from which u has been eliminated:

$$a\nabla^4 v=(2+\nu)\frac{\partial^3 w}{\partial x^2\partial s}+\frac{\partial^3 w}{\partial s^3}-\frac{h^2}{12}\left(\frac{2}{1-\nu}\frac{\partial^5 w}{\partial x^4\partial s}+\frac{3-\nu}{1-\nu}\frac{\partial^5 w}{\partial x^2\partial s^3}+\frac{\partial^5 w}{\partial s^5}\right). \tag{5}$$

Applying $\partial/\partial x$ to Eq. (4) and $\partial/\partial s$ to Eq. (5) and substituting these two equations into the third Eq. (3), after applying ∇^4 to it, one obtains an equation from which both u and v are absent:

$$\nabla^8 w+\frac{12(1-\nu^2)}{a^2h^2}\frac{\partial^4 w}{\partial x^4}+\frac{1}{a^2}\left(\frac{\partial^6 w}{\partial s^6}+(2+\nu)\frac{\partial^6 w}{\partial x^4\partial s^2}+(3+\nu)\frac{\partial^6 w}{\partial x^2\partial s^4}\right)-\frac{1}{D}\nabla^4 q=0. \tag{6}$$

It is evident that the third term in Eq. (6) is neligible in comparison with the other terms. Equation (6) is reduced to

$$\nabla^8 w+\frac{12(1-\nu^2)}{a^2h^2}\frac{\partial^4 w}{\partial x^4}-\frac{1}{D}\nabla^4 q=0. \tag{6a}$$

Equation (6a) differs from the differential equation of the flat plate only by the second term. The flat plate equation can be obtained from equation (6a) by the substitution of $a=\infty$. Consequently, this second term represents the effect of curvature in the problem of the cylindrical shell.

3. Infinitely long cylinder loaded with two equal and opposite forces. The above equation will now be applied to an infinitely long thin cylinder loaded, as shown in

Fig. 2, by two equal and opposite compressive forces P acting at the ends of a vertical diameter.

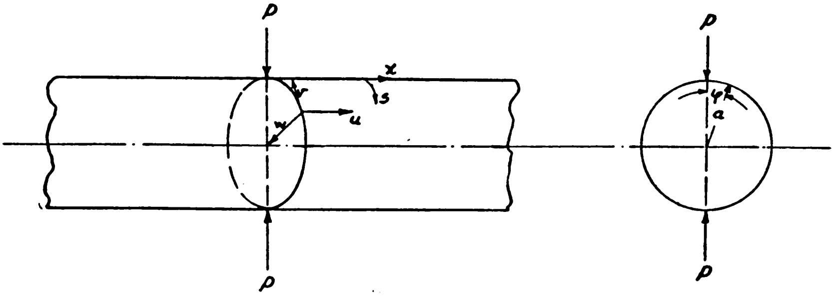

FIG. 2. Loads and components of displacements of an infinitely long cylinder.

The difficulties of integrating Eq. (6a) for this type of loading can be circumvented by replacing the concentrated force P by a distributed load q expressed as function of the longitudinal and circumferential coordinates, and applied to a small area which subsequently is reduced to an infinitesimal. This is made possible by representing the function in the longitudinal direction, by a Fourier integral and in the circumferential direction by a Fourier series. Since q is an even function of both x and s, it can be expressed by

$$q(x, s) = \left[\frac{q_0}{2} + \sum_{n=2,4\cdots}^{\infty} q_n \cos\frac{ns}{a}\right]\int_0^{\infty} f(\lambda)\cos\frac{x\lambda}{a}\,d\lambda. \tag{7}$$

The displacement w can be expanded in a similar manner in terms of a function $w(\lambda)$ as yet undetermined:

$$w = \sum_{n=0,2\cdots}^{\infty} \cos\frac{ns}{a}\int_0^{\infty} w(\lambda)\cos\frac{\lambda x}{a}\,d\lambda. \tag{8}$$

It can be shown that the above expression for w satisfies the following requirements: at the point where the load is applied, the deflection and moment are continuous, and the slope of the deflection curve vanishes. Furthermore, the deflection vanishes at infinity. Substituting Eqs. (7) and (8) in the differential equation (6a) one obtains the following relations. For $n=0$,

$$\int_0^{\infty}\left\{w(\lambda)\left[\left(\frac{\lambda}{a}\right)^8 + \frac{Eh}{a^2D}\left(\frac{\lambda}{a}\right)^4\right] - \frac{q_0}{2D}f(\lambda)\left(\frac{\lambda}{a}\right)^4\right\}\cos\frac{\lambda x}{a}\,d\lambda \equiv 0;$$

therefore,

$$w(\lambda) = \frac{(q_0/2D)f(\lambda)}{(\lambda/a)^4 + Eh/a^2D}\cdot$$

Similarly for $n=2, 4, \cdots$,

$$w(\lambda) = \frac{(q_nf(\lambda)/D)\left[(\lambda/a)^2 + (n/a)^2\right]^2}{\left[(\lambda/a)^2 + (n/a)^2\right]^4 + (Eh/a^2D)(\lambda/D)^4}\cdot$$

Hence, the solution of Eq. (6a) is

$$w = \frac{1}{2D}\int_0^\infty \frac{q_0 f(\lambda)}{(\lambda/a)^4 + (Eh/a^2D)} \cos\frac{\lambda x}{a}\, d\lambda$$
$$+ \frac{1}{D}\sum_{n=2,4\cdots}^{\infty} \cos\frac{ns}{a}\int_0^\infty \frac{q_n f(\lambda)\left[(\lambda/a)^2 + (n/a)^2\right]^2}{\left[(\lambda/a)^2 + (n/a)^2\right]^4 + (Eh/a^2D)(\lambda/a)^4} \cos\frac{\lambda x}{a}\, d\lambda. \quad (9)$$

It is next desired to find q_n and $f(\lambda)$. In order to accomplish this the functions q_n and $f(\lambda)$ must be determined from the loading condition. This is shown in Fig. 3. Since the cylinder is loaded symmetrically with respect to the generatrix and with respect to the circle passing through the origin, only the positive direction need be considered.

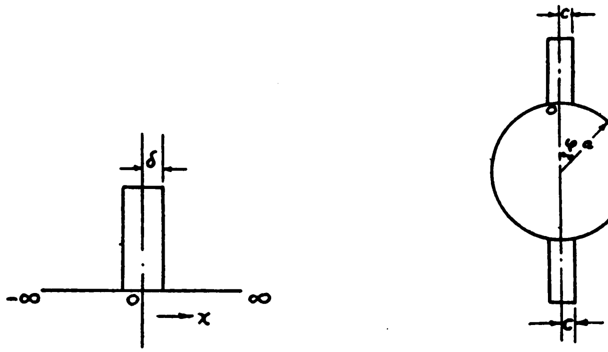

FIG. 3. Loading of the cylinder.

From Eq. (7),

$$q\left(\frac{x}{a}\right) = \int_0^\infty f(\lambda)\cos\frac{\lambda x}{a}\, d\lambda, \qquad f(\lambda) = \frac{1}{\pi}\int_{-\infty}^\infty q\left(\frac{x}{a}\right)\cos\frac{\lambda x}{a}\, d\left(\frac{x}{a}\right),$$

and

$$q\left(\frac{x}{a}\right) = 1 \quad \text{when} \quad -\delta \leqq x \leqq \delta, \qquad q\left(\frac{x}{a}\right) = 0 \quad \text{when} \quad x > \delta \text{ and } x < -\delta.$$

Therefore

$$f(\lambda) = \frac{2}{\pi}\int_0^\delta \cos\frac{\lambda x}{a}\, d\left(\frac{x}{a}\right) = \frac{2}{\pi\lambda}\sin\lambda\frac{\delta}{a}\cdot$$

Similarly q_n can be determined from the expansion of the loading function along the circumference in a Fourier series. With

$$q_0 = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} q(z)dz,$$

where $z = s/a$, and if $z = \pi/2$, $s = \pi a/2$, one obtains

$$q_0 = \frac{2}{\pi a}\int_{-c}^{c} q ds = \frac{4c}{\pi a} q, \qquad q_n = \frac{2}{\pi a}\int_{-c}^{c} q\cos n\frac{s}{a}\, ds = \frac{4q}{\pi n}\sin n\frac{c}{a},$$

where c is as shown in Fig. 3. Substituting q_n and $f(\lambda)$ in Eq. (9), one finds that

$$w = \frac{1}{2D}\int_0^\infty \frac{(8qc/\pi^2 a\lambda)\sin\lambda\delta/a}{(\lambda/a)^4 + Eh/a^2D}\cos\frac{\lambda x}{a}d\lambda$$
$$+\frac{1}{D}\sum_{n=2,4\cdots}^{\infty}\cos\frac{ns}{a}\int_0^\infty \frac{(8q/\pi^2 n\lambda)\sin nc/a\sin\lambda\delta/a[(\lambda/a)^2+(n/a)^2]}{[(\lambda/a)^2+(n/a)^2]^4+Eh/a^2D(\lambda/a)^4}\cos\frac{\lambda x}{a}d\lambda.$$

Next, the case of a concentrated load applied at the origin may be considered. Such a load can be obtained by making the lengths 2δ and $2c$ of the loaded portion infinitely small. Substituting

$$P = 4qc\delta, \qquad \sin\frac{\lambda\delta}{a}\approx\frac{\lambda\delta}{a}, \qquad \sin\frac{nc}{a}\approx\frac{nc}{a}$$

in the above equation, one obtains

$$w = \frac{Pa^2}{D\pi^2}\int_0^\infty \frac{\cos\lambda(x/a)d\lambda}{\lambda^4+J^2}$$
$$+\frac{2Pa^2}{\pi^2 D}\sum_{n=2,4\cdots}^{\infty}\cos\frac{ns}{a}\int_0^\infty\frac{[\lambda^2+n^2]^2\cos(\lambda x/a)d\lambda}{[\lambda^2+n^2]^4+J^2\lambda^4}, \tag{10}$$

where

$$J^2 = \frac{Eha^2}{D} = 12(1-\nu^2)\left(\frac{a}{h}\right)^2.$$

In order to evaluate the definite integrals in Eq. (10) Cauchy's theorem of residues will be applied. Let us consider the integral

$$\int_0^\infty \frac{\cos\lambda(x/a)d\lambda}{\lambda^4+J^2},$$

where the characteristic equation $\lambda^4+J^2=0$ has four complex roots

$$\lambda = J^{1/2}(-1)^{1/4}.$$

Cauchy's theorem yields

$$\int_0^\infty\frac{\cos\lambda(x/a)d\lambda}{\lambda^4+J^2} = \frac{\pi}{2\sqrt{2}}J^{-3/2}e^{-\sqrt{J/2}(x/a)}\left(\cos\sqrt{\frac{J}{2}}\frac{x}{a}+\sin\sqrt{\frac{J}{2}}\frac{x}{a}\right). \tag{11}$$

The rational function in the integrand of second definite integral in Eq. (10) can be expressed in the form of partial fractions,

$$\frac{(\lambda^2+n^2)^2}{[\lambda^2+n^2]^4+J^2\lambda^4} = \frac{1}{2}\left\{\frac{1}{(\lambda^2+n^2)^2+iJ\lambda^2}+\frac{1}{(\lambda^2+n^2)^2-iJ\lambda^2}\right\}$$
$$= \frac{1}{\alpha_1^2-\alpha_2^2}\frac{1}{\lambda^2-\alpha_1^2}+\frac{1}{\alpha_2^2-\alpha_1^2}\frac{1}{\lambda^2-\alpha_2^2}$$
$$+\frac{1}{\alpha_3^2-\alpha_4^2}\frac{1}{\lambda^2-\alpha_3^2}+\frac{1}{\alpha_4^2-\alpha_3^2}\frac{1}{\lambda^2-\alpha_4^2}, \tag{12}$$

where $\pm\alpha_1$, $\pm\alpha_2$, $\pm\alpha_3$ and $\pm\alpha_4$ are the roots of the denominator,

$$\begin{aligned}
\pm\alpha_1 &= \pm\alpha_4^* = \pm A \pm iB \\
&= \pm\frac{1}{\sqrt{2}}\left[\sqrt{(-n^2+\eta)^2+\left(-\frac{J}{2}+\phi\right)^2}-(n^2-\eta)\right]^{\frac{1}{2}} \\
&\quad \pm\frac{i}{\sqrt{2}}\left[\sqrt{(-n^2+\eta)^2+\left(-\frac{J}{2}+\phi\right)^2}+(n^2-\eta)\right]^{\frac{1}{2}}, \\
\pm\alpha_2 &= \pm\alpha_3^* = \pm C \mp iG \\
&= \pm\frac{1}{\sqrt{2}}\left[\sqrt{(n^2+\eta)^2+\left(\frac{J}{2}+\phi\right)^2}-(n^2+\eta)\right]^{\frac{1}{2}} \\
&\quad \pm\frac{i}{\sqrt{2}}\left[\sqrt{(n^2+\eta)^2+\left(\frac{J}{2}+\phi\right)}+(n^2+\eta)\right]^{\frac{1}{2}},
\end{aligned} \tag{13}$$

where the asterisk denotes the complex conjugate, and

$$\phi = \sqrt{\tfrac{1}{2}(R^2+\tfrac{1}{4}J^2)}, \qquad \eta = \sqrt{\tfrac{1}{2}(R_2-\tfrac{1}{4}J^2)}, \qquad R_2 = n^2J\sqrt{1+(J/4n^2)^2}. \tag{13a}$$

Hence

$$\begin{aligned}
I &= \int_{-\infty}^{\infty}\frac{(\lambda^2+n^2)^2e^{i\lambda x/a}d\lambda}{(\lambda^2+n^2)^4+J^2\lambda^4} \\
&= \frac{2\pi i}{8R_2}\left\{\frac{\alpha_2}{\alpha_1\alpha_2}(\eta-i\phi)e^{ix\alpha_1/a}-\frac{\alpha_1}{\alpha_1\alpha_2}(\eta-i\phi)e^{ix\alpha_2/a}\right. \\
&\quad \left.-\frac{\alpha_4}{\alpha_3\alpha_4}(\eta+i\phi)e^{ix\alpha_3/a}+\frac{\alpha_3}{\alpha_3\alpha_4}(\eta+i\phi)e^{ix\alpha_4/a}\right\}.
\end{aligned} \tag{14}$$

Since

$$\alpha_1\alpha_2 = -n^2 = \alpha_3\alpha_4,$$

Eq. (14) can be simplified as follows. Now

$$\begin{aligned}
&\int_0^{\infty}\frac{(\lambda^2+n^2)^2\cos x\lambda/a\,d\lambda}{(\lambda^2+n^2)^4+J^2\lambda^4} \\
&\quad = \frac{\pi}{4R_2n^2}\left\{\left[(\phi C+\eta G)\cos\frac{Ax}{a}+(\phi G-\eta C)\sin\frac{Ax}{a}\right]e^{-Bx/a}\right. \\
&\qquad \left.+\left[(\phi A-\eta B)\cos\frac{Cx}{a}+(\eta A+\phi B)\sin\frac{Cx}{a}\right]e^{-Gx/a}\right\}.
\end{aligned} \tag{15}$$

Simplifying the integrals (15) and (11) in Eq. (10), one obtains

$$\begin{aligned}
\frac{w/h}{P/Eh^2} &= \frac{\sqrt[4]{3(1-\nu^2)}}{2\pi}\left(\frac{a}{h}\right)^{1/2}\left(\cos\sqrt{\frac{J}{2}}\frac{x}{a}+\sin\sqrt{\frac{J}{2}}\frac{x}{a}\right)e^{-\sqrt{J/2}(x/a)} \\
&\quad + \frac{6(1-\nu^2)}{\pi}\left(\frac{a}{h}\right)^2\sum_{n=2,4\cdots}^{\infty}\frac{\cos ns/a}{R_2n^2}\left\{\left[(\phi C+\eta G)\cos\frac{Ax}{a}\right.\right.
\end{aligned}$$

$$+ (\phi G - \eta C) \sin \frac{Ax}{a}\Bigg] e^{-Bx/a} + \Bigg[(\phi A - \eta B) \cos \frac{Cx}{a}$$

$$+ (\eta A + \phi B) \sin \frac{Cx}{a}\Bigg] e^{-Gx/a} \Bigg\} . \tag{16}$$

It is seen that the first term of the above expression is very small as compared to the second term, and therefore can be neglected without appreciable error. For a

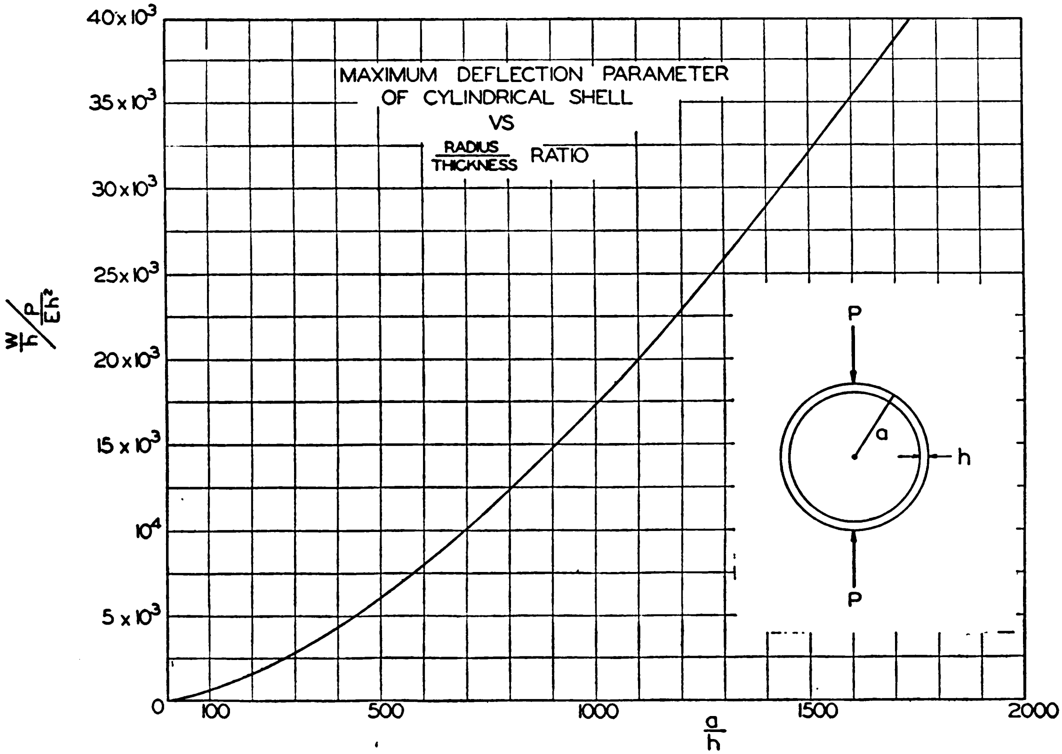

FIG. 4

certain value of the a/h ratio, G is found to be very large as compared to B. The terms containing $e^{-Gx/a}$ can then be completely neglected, provided that x/a is not near zero.

In the case when $x/a = 0$ Eq. (16) can be simplified as follows:

$$\left[\frac{w/h}{P/Eh^2}\right]_{x/a=0} = \frac{3\sqrt{2}(1-\nu^2)}{\pi}\left(\frac{a}{h}\right)^2 \sum_{n=2,4\cdots}^{\infty} \frac{\cos ns/a}{n^3} \frac{\sqrt{1+\Xi}}{\Xi}, \tag{17}$$

where

$$\Xi^2 = 1 + \frac{3(1-\nu^2)a^2}{4n^4h^2}.$$

The left side of (17) has a maximum at $s = 0$. Figure 4 shows the variation of this maximum with the ratio a/h. Figure 5 shows the variation of w along the generatrix through a point of loading, and Fig. 6 shows the projection of lines of constant w on the plane through the axis of the cylinders and perpendicular to the line of action of the two forces P.

4. A cylinder of finite length loaded with two equal and opposite forces. The expression for the radial deflection in a thin cylinder of finite length can be obtained from Eq. (16) by using the method of images.* If one imagines the cylinder of finite length prolonged in both the positive and the negative x-directions, and loaded with a series of forces, P, of alternating sense, applied along the generatrix ($s/a=0$) at a distance l from one another (see Fig. 7), then the deflections of the infinite cylinder

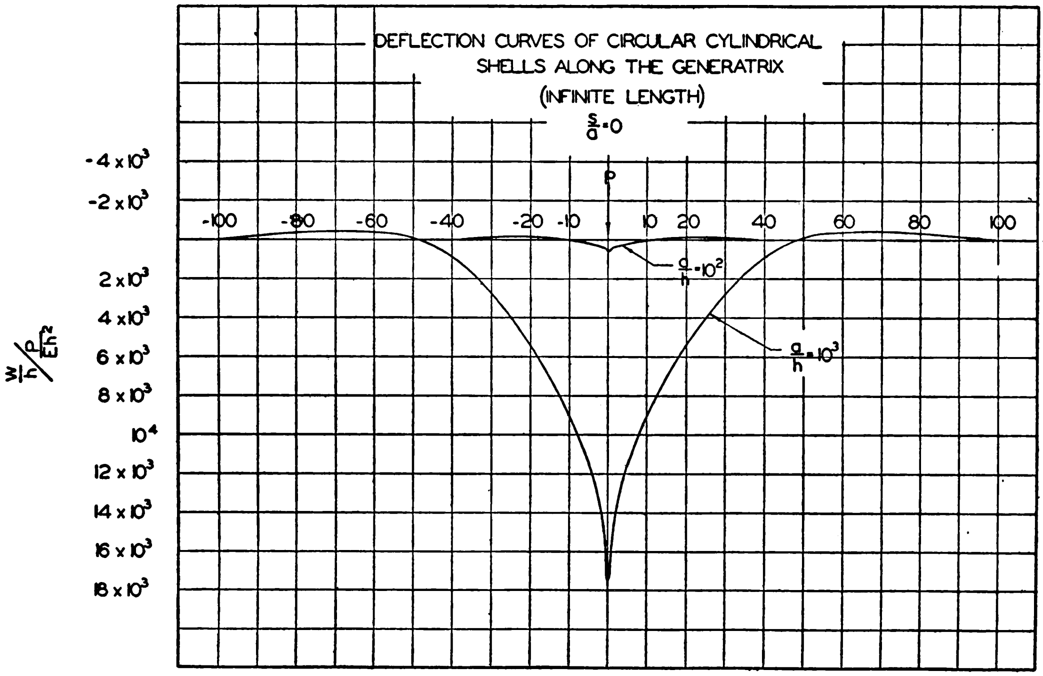

FIG. 5

are evidently equal to zero at a distance $l/2$ from the applied loads P. Hence one may consider the given cylinder of length l and radius a as a portion of the infinitely long cylinder loaded as shown in Fig. 7. From Eq. (16) one finds that the deflection of any point, β, (at a distance ζ from the s-axis) on the shell due to the load P acting at the center is

$$w_a = \frac{Pa^2}{2\pi D} \sum_{n=2,4\cdots}^{\infty} \frac{\cos ns/a}{R_2 n} \left\{ \left[(\phi C + \eta G) \cos A \frac{\zeta}{a} + (\phi G - \eta C) \sin A \frac{\zeta}{a} \right] e^{-B\zeta/a} \right.$$
$$\left. + \left[(\phi A - \eta B) \cos C \frac{\zeta}{a} + (\eta A + \phi B) \sin C \frac{\zeta}{a} \right] e^{-G\zeta/a} \right\}. \tag{18a}$$

The deflection produced by two adjacent forces a distance l apart is

* This method was used by A. Nádai, Z. angew. Math. Mech. **2**, 1 (1922), and by M. T. Huber, Z. angew. Math. Mech. **6**, 228 (1926).

$$w_b = -\frac{Pa^2}{2\pi D}\sum_{n=2,4\cdots}^{\infty}\frac{\cos ns/a}{R_2 n}\left\{\left[(\phi C+\eta G)\cos A\frac{l-\zeta}{a}\right.\right.$$

$$\left.+(\phi G-\eta C)\sin A\frac{l-\zeta}{a}\right]e^{-B(l-\zeta)/a}+\left[(\phi C+\eta G)\cos A\frac{l+\zeta}{a}\right.$$

$$\left.\left.+(\phi G-\eta C)\sin A\frac{l+\zeta}{a}\right]e^{-B(l+b)/a}\right\}. \tag{18b}$$

Since the terms containing $e^{-G(l+\zeta)/a}$ are all small compared to the other terms, they

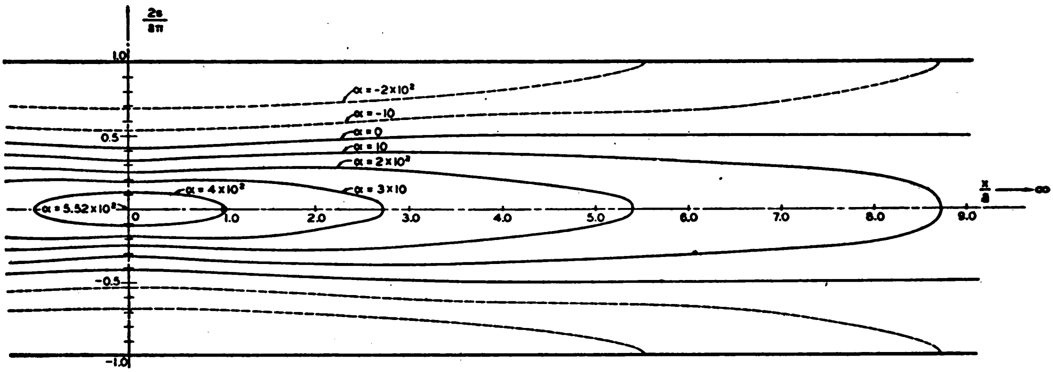

FIG. 6

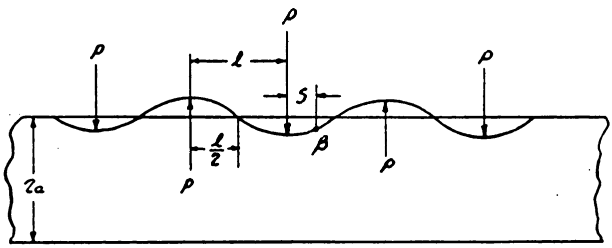

FIG. 7. Series of equidistant opposite forces acting on an infinitely long cylinder.

can be neglected without causing appreciable error. One obtains similarly w_c, w_d, $\cdots$. The total radial deflection at any point β is given by the sum

$$w = w_a + w_b + w_c + \cdots$$

$$= \frac{Pa^2}{2\pi D}\sum_{n=2,4\cdots}^{\infty}\frac{\cos ns/a}{R_2 n}\left\{\left[(\phi A-\eta B)\cos C\frac{\zeta}{a}+(\eta A+\phi B)\sin C\frac{\zeta}{a}\right]e^{-G\zeta/a}\right.$$

$$+(\phi C+\eta G)\left[\cos A\frac{\zeta}{a}e^{-B\zeta/a}-2\cos A\frac{\zeta}{a}\cosh B\frac{\zeta}{a}\left(\cos A\frac{l}{a}e^{-Bl/a}\right.\right.$$

$$\left.-\cos\frac{2Al}{a}e^{-2Bl/a}+\cos 3A\frac{l}{a}e^{-3Bl/a}-\cdots\right)$$

$$- 2 \sin A \frac{\zeta}{a} \sinh B \frac{\zeta}{a}\left(\sin A \frac{l}{a} e^{-Bl/a} - \sin 2A \frac{l}{a} e^{-2Al/a} + \cdots \right)\Big]$$

$$+ (\phi G - \eta C)\Big[\sin A \frac{\zeta}{a} e^{-B\zeta/a} - 2 \cos A \frac{\zeta}{a} \cosh B \frac{\zeta}{a}\Big(\sin A \frac{l}{a} e^{-Bl/a}$$

$$- \sin 2A \frac{l}{a} e^{-2Bl/a} + \sin 3A \frac{l}{a} e^{-3Bl/a} - \cdots \Big)$$

$$+ 2 \sin A \frac{\zeta}{a} \sinh B \frac{\zeta}{a}\left(\cos A \frac{l}{a} e^{-Bl/a} - \cos 2A \frac{l}{a} e^{-2Bl/a} + \cdots \right)\Big]\Big\}. \tag{19}$$

We sum the series in the above expression, obtaining

$$\sum_{m=1,3\cdots}^{\infty} e^{-mBl/a} \cos mA \frac{l}{a} = \frac{1}{2} \sum_{m=1,3\cdots}^{\infty} \left[e^{-m(B-iA)l/a} + e^{-m(B+iA)l/a}\right]$$

$$= \frac{1}{2}\left[\frac{\sinh (Bl/a) \cos (Al/a)}{\sinh^2 (Bl/a) \cos^2 (Al/a) + \cosh^2 (Bl/a) \sin^2 (Al/a)}\right],$$

$$\sum_{m=2,4\cdots}^{\infty} e^{-mBl/a} \cos mA \frac{l}{a} = \frac{e^{-lB/a}}{2} \frac{[\sin (Bl/a) \cos^2 (Al/a) - \cosh (Bl/a) \sin^2 (Al/a)]}{[\sin^2 (Al/a) \cosh^2 (Bl/a) + \cos^2 (Al/a) \sinh^2 (Bl/a)]},$$

$$\sum_{m=1,3\cdots}^{\infty} e^{-mBl/a} \sin mA \frac{l}{a} = \frac{1}{2i} \sum_{m=1,3\cdots}^{\infty} \left[e^{-m(B-iA)l/a} - e^{-m(B+iA)l/a}\right]$$

$$= \frac{1}{2}\left[\frac{\cosh (Bl/a) \sin (Al/a)}{\sinh^2 (Bl/a) \cos^2 (Al/a) + \cosh^2 (Bl/a) \sin^2 (Al/a)}\right],$$

$$\sum_{m=2,4\cdots}^{\infty} e^{-mBl/a} \sin mA \frac{l}{a} = \frac{e^{-Bl/a} \cos (Al/a) \sin (Al/a)[\sinh (Bl/a) + \cosh (Bl/a)]}{2[\sinh^2 (Bl/a) \cos^2 (Al/a) + \cosh^2 (Bl/a) \sin^2 (Al/a)]}.$$

Thus Eq. (19) is reduced to

$$\frac{w/h}{P/Eh^2} = \frac{6(1-\nu^2)}{\pi}\left(\frac{a}{h}\right)^2 \sum_{n=2,4\cdots}^{\infty} \frac{\cos ns/a}{R_2 n^2}\Big\{\Big[(\phi C + \eta G) \cos \frac{\zeta}{a} A$$

$$+ (\phi G - \eta C) \sin A \frac{\zeta}{a}\Big] e^{-B\zeta/a}$$

$$+ \left[(\phi A - \eta B) \cos C \frac{\zeta}{a} + (\eta A + \phi B) \sin C \frac{\zeta}{a}\right] e^{-G\zeta/a}$$

$$+ \frac{\sinh (Bl/a) \cos (Al/a) - e^{-Bl/a}[\sinh (Bl/a) \cos^2 (Al/a) - \cosh (Bl/a) \sin^2 (Al/a)]}{\sinh^2 (Bl/a) \cos^2 (Al/a) + \cosh^2 (Bl/a) \sin^2 (Al/a)}$$

$$\times \left[(\phi G - \eta C) \sin A \frac{\zeta}{a} \sinh B \frac{\zeta}{a} - (\phi C + \eta G) \cos A \frac{\zeta}{a} \cosh B \frac{\zeta}{a}\right]$$

$$- \frac{\cosh (Bl/a) \sin (Al/a) - e^{-Bl/a} \cos (Al/a) \sin (Al/a) [\sinh (Bl/a) + \cosh (Bl/a)]}{\sinh^2 (Bl/a) \cos^2 (Al/a) + \cosh^2 (Bl/a) \sin^2 (Al/a)}$$

$$\times \left[(\phi G - \eta C) \cos A \frac{\zeta}{a} \cosh B \frac{\zeta}{a} + (\phi C + \eta G) \sin A \frac{\zeta}{a} \sinh B \frac{\zeta}{a}\right]\Big\}. \tag{20}$$

It is obvious that the first two terms of Eq. (20) are equivalent to the solution of the infinitely long cylinder given by Eq. (16). The remaining terms are evidently the correction factors due to the restrained edges at the two ends of the cylinder of finite length. The radial deflection under the applied force can be obtained by putting $\zeta/a=0$,

$$
\begin{aligned}
\frac{w/h}{(P/Eh^2)} = \frac{6(1-\nu^2)}{\pi}\left(\frac{a}{h}\right)^2 \sum_{n=2,4\cdots}^{\infty} \frac{\cos ns/a}{R_2 n^2}\Big\{(\phi C+\eta G)+(\phi A-\eta B) \\
-(\phi C+\eta G)\,\frac{\sinh(Bl/a)\cos(Al/a)-e^{-Bl/a}[\sinh(Bl/a)\cos^2(Al/a)-\cosh(Bl/a)\sin^2(Al/a)]}{\sinh^2(Bl/a)\cos^2(Al/a)+\cosh^2(Bl/a)\sin^2(Al/a)} \\
-(\phi C-\eta C)\,\frac{\cosh(Bl/a)\sin(Al/a)-e^{-Bl/a}\cos(Al/a)\sin(Al/a)[\sinh(Bl/a)+\cosh(Bl/a)]}{\sinh^2(Bl/a)\cos^2(Al/a)+\cosh^2(Bl/a)\sin^2(Al/a)} \\
-(\phi A-\eta B)\,\frac{\sinh(Gl/a)\cos(Cl/a)-e^{-Gl/a}[\sinh(Gl/a)\cos^2(Cl/a)-\cosh(Gl/a)\sin^2(Cl/a)]}{\sinh^2(Gl/a)\cos^2(Cl/a)+\cosh^2(Gl/a)\sin^2(Cl/a)} \\
-(\eta A+\phi B)\,\frac{\cosh(Gl/a)\sin(Cl/a)-e^{-Gl/a}\cos(Cl/a)\sin(Cl/a)[\sinh(Gl/a)+\cosh(Dl/a)]}{\sinh^2(Gl/a)\cos^2(Cl/a)+\cosh^2(Gl/a)\sin^2(Cl/a)}\Big\}. \quad (21)
\end{aligned}
$$

Some applications of the solution of the problem of the infinitely long cylinder. The problems of a couple acting on an infinitely long cylinder in the direction of either the generatrix or the circumference can be analyzed by using the solution given by Eq. (16) for a single load. The action of the couple is equivalent to that of the two forces P shown in Fig. 8, where $\lim_{\Delta x\to 0} P\Delta x = T_c$.

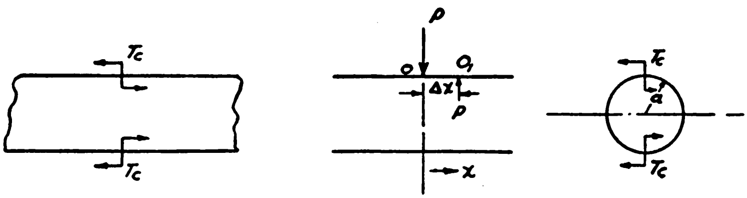

FIG. 8. Two couples acting on an infinitely long cylinder.

It is easy to see that the deflection for the case when the force P is applied at the point O_1, at a distance Δx from the origin, can be obtained from the deflection w, given in Eq. (16), by writing $x-\Delta x$ instead of x and also $-P$ instead of P. This and the original w are then added. The radial deflection due to the two equal and opposite forces applied at O and O_1 is now obtained in the form

$$- w_T = w(x, s) - w(x - \Delta x, s).$$

When Δx is very small, this approaches the value

$$w_T = \frac{dw(x, s)}{dx}\Delta x.$$

As T_c is the moment of the applied torque and is equal to $P\Delta x$, the radial deflection due to this torque is

$$w_{T_1} = \frac{T_c}{P}\frac{dw}{dx}, \tag{22}$$

where w is the radial deflection due to the concentrated load P.

For the radial deflection due to the couple acting along the circumferential direction one finds similarly (Fig. 8) that

$$w_{T_2} = \frac{T_c}{P}\frac{dw}{ds}. \tag{23}$$

Substituting w from Eq. (16) in Eqs. (22) and (23) one obtains for the couple acting along the circumferential direction,

$$\begin{aligned}\frac{w_{T_1}/h}{T_c/Eh^2} = \frac{6(1-\nu^2)}{\pi}\left(\frac{a}{h}\right)^2 \sum_{n=2,4\cdots}^{\infty} \frac{\cos ns/a}{R_2 n^2}\{&e^{-Bx/a}\cos Ax/a[A(\phi G - \eta C) - B(\phi C + \eta G)] \\ &- e^{-Bx/a}\sin(Ax/a)[(\phi C + \eta G)A + B(\phi G - \eta C)] \\ &+ e^{-Gx/a}\cos(Cx/a)[C(\eta A + \phi B) - G(\phi A - \eta B)] \\ &- e^{-Gx/a}\sin(Cx/a)[C(\phi A - \eta B) + G(\eta A + \phi B)]\}\end{aligned} \tag{24}$$

while for the couple acting along the generatrix direction,

$$\begin{aligned}\frac{w_{T_2}/h}{T_c/Eh^2} = -\frac{6(1-\nu^2)}{\pi}\left(\frac{a}{h}\right)^2 \sum_{n=2,4\cdots}^{\infty} \frac{\sin ns/a}{R_2 n}\{&[(\phi C + \eta G)\cos(Ax/a) \\ &+ (\phi G - \eta C)\sin(Ax/a)]e^{-Bx/a} + [(\phi A - \eta B)\cos(Cx/a) \\ &+ (\eta A + \phi B)\sin(Cx/a)]e^{-Gx/a}\}.\end{aligned} \tag{25}$$

In the case when $x/a = 0$,

$$\frac{w_{T_1}/h}{T_c/Eh^2} = 0, \quad \text{at any } s/a.$$

Hence the condition that the slope of the deflection curve dw/dx must vanish under the concentrated load $(x/a=0)$ is satisfied.

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