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THE REFLECTION OF AN ELECTROMAGNETIC PLANE WAVE BY AN INFINITE SET OF PLATES, I*

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1. Introduction. It has been shown by J. Schwinger that a special class of boundary value problems in electrodynamics can be formulated mathematically as Wiener-Hopf[4] integral equations. These problems may be described as follows. A plane wave is incident upon a number of semi-infinite parallel metallic structures of zero thickness and perfect conductivity. By parallel structures we mean parallel planes or cylinders with parallel axes. It is then possible to express the electric or magnetic field at all points in space in terms of the surface current density on the metal with the aid of an appropriate Green's function. The vanishing of the components of the electric field which are tangential to the semi-infinite cylindrical metallic surfaces, leads to a system of inhomogeneous integral equations for the various surface current densities. This system of integral equations assumes the general form

$$g_i(x) = \sum_{j=1}^{n} \int_0^\infty K_{ij}(x-y) f_j(y) dy, \qquad x > 0, \quad i = 1, \cdots, n,$$

where the $f_j(y)$ are unknown functions, while the $K_{ij}(x)$ and $g_i(x)$ are known. The particular problem which we shall discuss below possesses certain periodicities, and for this case we find it possible to reduce the system to a single integral equation of the form

$$g(x) = \int_0^\infty K(x-y) f(y) dy, \qquad x > 0, \tag{1.1}$$

that is, an inhomogeneous Wiener-Hopf integral equation. Here $f(y)$ is unknown, while $K(x)$ and $g(x)$ are known functions.

The advantage of formulating this particular class of boundary value problems as Wiener-Hopf integral equations is that such equations are susceptible to a rigorous

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[3] This paper is based on work done for the Office of Scientific Research and Development under contract OEMsr-262 with the Massachusetts Institute of Technology.

[4] R. E. A. C. Paley and N. Wiener, *The Fourier transform in the complex domain*, Am. Math. Soc. Colloquium Publication, 1934, Ch. IV.

E. C. Titchmarsh, *Theory of the Fourier integral*, Oxford University Press, Ch. XI, 1937.

J. S. Schwinger, *The theory of guided waves*, Radiation Laboratory Publication. To be published.

solution. We may thus find the functional form of the various surface current densities as well as the electric field. However, in such problems as we have described above, the physically interesting quantities may be calculated from the far field and these quantities in turn are closely related to the Fourier transform of the surface current densities. Since Eq. (1.1) is solved by transform techniques, these quantities can be obtained immediately.

The problem which we treat here is the following. A plane monochromatic electromagnetic wave whose direction of propagation lies in the plane of the paper, is incident upon an infinite set of staggered, equally spaced, semi-infinite metallic plates of zero thickness and perfect conductivity. These plates extend indefinitely in a direction perpendicular to the plane of the paper. (See Fig. 1 for a side view.) The angle of stagger with respect to a fixed direction (that of the cross section of the plates in Fig. 1) is α, while the direction of propagation with respect to this fixed line is θ, where $\alpha-\pi<\theta<\alpha$ and $0<\alpha\leqq\pi/2$. This structure has some properties which are analo-

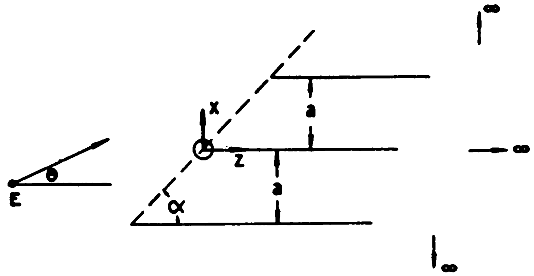

FIG. 1.

gous to those of metal mirrors and gratings. Thus when it is excited by a plane wave with arbitrary direction of propagation, there will be reflected plane waves in certain directions depending on the relative dimensions, the wave length and the direction of incidence.

2. Formulation of the problem. We assume that the electric field of the incident wave has only one component, namely, the component which is perpendicular to the plane of the paper. Since the incident electric field is independent of y and the boundary conditions on the plates must be fulfilled independently of y, no other components of the electric field will be excited. Thus all components of the magnetic field can be derived from this single component of the electric field $E_y(x, z)=\phi(x, z)$. For this case both of the components of the magnetic field lie in the plane of the paper and we shall refer to this problem as an "H plane" problem.

If we now write the Maxwell equations[5] in the form

$$\nabla\times\mathbf{E} = ik\mathbf{H}$$

and

$$\nabla\times\mathbf{H} = -ik\mathbf{E},$$

where $k=2\pi/\lambda$, and λ is the free space wave-length, we see immediately that the only components of the magnetic field are

[5] The time dependence of all field quantities is taken to be e^{-ikct} and may therefore be suppressed. c is the velocity of light. In the engineering literature, the time dependence is written as $\exp(ikct)$. In order to convert our final results to standard engineering form, one merely replaces i by $-j$.

$$ikH_x = -\frac{\partial\phi}{\partial z}$$

and

$$ikH_z = \frac{\partial\phi}{\partial x}\cdot$$

Upon eliminating H_x and H_z from the above equations we obtain the two dimensional wave equation,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} + k^2\phi = 0$$

which is to be solved subject to the boundary condition, $\phi = 0$ on the metal plates since ϕ is the tangential component of the electric field. There are also conditions at infinity on the function $\phi(x, z)$ which we shall discuss later when we have need of them.

We now formulate the equation which expresses the electric field in terms of the surface current density on the metal plates. To this end, we start by modifying the structure in Fig. 1, so that there are now only a finite number of parallel plates, each of which is taken to be finite in length. The length of each plate is such that the amplitudes of the attenuated modes are negligibly small relative to the amplitude of the propagated mode in the parallel plate region before the end of the structure is

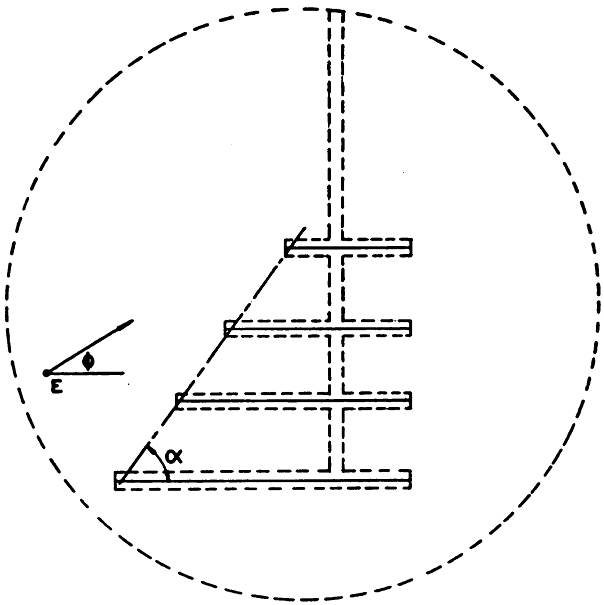

FIG. 2.

reached. (See Fig. 2 for a side view.) If we employ the free space Green's function, we may express $\phi(x, z)$ in terms of $\partial\phi/\partial n$, the normal derivative on the metallic plates. We have from Green's theorem

$$\phi(x, z) = \int_C \left(G\frac{\partial\phi}{\partial n'} - \phi\frac{\partial G}{\partial n'}\right)ds',$$

where the contour C is the one indicated by the dotted line in Fig. 2, ds' is the element of arc length along it and $G(x, z, x', z')$ is the free space Green's function. The outer boundary of the contour C is taken to be a circle of large radius. This is merely for convenience and the outer boundary might have been any other closed curve. $G(x, z, x', z')$ satisfies the homogeneous wave equation

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial z^2} + k^2 G = 0$$

save for the point $x = x'$, $z = z'$. At this point

$$\int_{-\infty}^{\infty} \frac{\partial G}{\partial x}\bigg|_{x=x'-0}^{x=x'+0} dz' = -1$$

and

$$\int_{-\infty}^{\infty} \frac{\partial G}{\partial z}\bigg|_{z=z'-0}^{z=z'+0} dx' = -1.$$

This may be expressed symbolically by saying that $G(x, z, x', z')$ satisfies the inhomogeneous wave equation

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial z^2} + k^2 G = -\delta(x - x')\delta(z - z'),$$

where $\delta(x - x')$ is the Dirac delta function and is zero everywhere save at $x = x'$, where it becomes infinite in such a fashion as to make the integral

$$\int_{-\infty}^{\infty} \delta(x - x')dx' = 1.$$

On the plates $\phi(x, z) = 0$, while $\partial\phi/\partial n'$ is the tangential component of the magnetic field on the plates. Since the tangential component of the magnetic field suffers a discontinuity which is proportional to the surface current density when we go from one side of a given plate to the other side of it, the only contribution we get from the integration along the metallic plates is

$$\sum_{m=p}^{q} \int G(x, z, ma, z')I_m(z')dz',$$

and the limits of integration are those which cover the full length of each plate. The sum is carried out over the finite number of plates as shown in Fig. 2. $I_m(z)$ is proportional to the surface current density on the mth metal plate. There is complete cancellation of the integrals taken along the paths which lead from one plate to the next or which lead from the end plates to the large circle enclosing all of the plates.

We now calculate the contribution from the large circle. In the first place, the free space Green's function which represents an outgoing wave for $\sqrt{x^2+z^2} \gg \sqrt{x'^2+z'^2}$ is $G(x, z, x', z') = (i/4)H_0^{(1)}[k\sqrt{(x-x')^2+(z-z')^2}]$ where $H_0^{(1)}$ is the Hankel function of the first kind. The contribution from the large circle is

$$\int_0^{2\pi} \left[G(r, r', \beta, \beta') \frac{\partial\phi(r', \beta')}{\partial r'} - \phi(r', \beta') \frac{\partial G(r, r', \beta, \beta')}{\partial r'}\right] r' d\beta', \qquad (2.1)$$

where

$$G(r, r', \beta, \beta') = \frac{i}{4} H_0^{(1)}[k\sqrt{r^2 + r'^2 - 2rr' \cos(\beta - \beta')}]$$

and $x = r \sin \beta$, $z = r \cos \beta$. If we now expand $G(r, r', \beta, \beta')$ in terms of cylindrical waves we have

$$G(r, r', \beta, \beta') = \frac{i}{4} \sum_{m=-\infty}^{\infty} H_m^{(1)}(kr')J_m(kr)e^{im(\beta-\beta')}, \qquad r < r'.$$

Furthermore, for any point outside of the region of the plates

$$\phi(x, z) = e^{ik(z \cos \theta + x \sin \theta)} + \sum_{m=-\infty}^{\infty} \alpha_n H_n^{(1)}(kr)e^{in\beta}, \tag{2.2}$$

where the first term represents the incident plane wave whose direction of propagation is θ, while the second term represents the scattered wave. We shall not be interested in the explicit form of the α_n's and indeed, we shall show that they do not enter explicitly into the formulation of the integral equation. The expression for the plane wave, $e^{ik[z \cos \theta + x \sin \theta]}$ may be expanded in terms of cylindrical waves by noting that

$$e^{ik(z \cos \theta + x \sin \theta)} = e^{ikr \cos(\theta - \beta)} = \sum_{m=-\infty}^{\infty} e^{+im\pi/2} J_m(kr)e^{im(\theta-\beta)}.$$

If we now evaluate the integrals in (2.1) we get immediately

$$e^{ik(z \cos \theta + x \sin \theta)} = \phi_{\text{inc}}(x, z)$$

i.e., the incident field.

For our final equation we then have

$$\phi(x, z) = \phi_{\text{inc}}(x, z) + \sum_{m=p}^{q} \int I_m(z')G(x, z, ma, z')dz'.$$

If we now let q become positively infinite, p negatively infinite, and let each plate extend indefinitely to the right, we can then express $\phi(x, z)$, the y component of the electric field, in terms of the incident field and the surface current density on the plates, that is,

$$\phi(x, z) = \phi_{\text{inc}}(x, z) + \frac{i}{4} \sum_{m=-\infty}^{\infty} \int_{mb}^{\infty} I_m(z')H_0^{(1)}[k\sqrt{(z - z')^2 + (x - ma)^2}]dz', \tag{2.3}$$

where $a = b \tan \alpha$. We now impose the electromagnetic boundary condition, namely that $\phi(x, z)$ vanishes on the metallic plates, and we get a system of simultaneous integral equations of the Wiener-Hopf type for $I_m(z)$. That is, for $x = na$

$$0 = \phi_{\text{inc}}(na, z) + \frac{i}{4} \sum_{m=-\infty}^{\infty} \int_{mb}^{\infty} I_m(z')H_0^{(1)}[k\sqrt{(z - z')^2 + (n - m)^2a^2}]dz' \tag{2.4}$$

for all n with $z > nb$, $n = 0, \pm 1, \pm 2, \cdots$.[6] Due to the periodic nature of the structure, the infinite set of simultaneous integral equations can be cast into the form (1.1).

[6] It is possible to obtain the integral equation (2.3) directly from the infinite structure indicated in Fig. 1. We have intentionally avoided this because it requires a more detailed knowledge of the field at infinity.

We close our discussion of the formulation of the integral equation (2.3) with some remarks about the range of values of a/λ which is allowed. In the parallel plate regions for z large and positive, $\phi(x, z)$ is asymptotic to $\sin(\pi x/a)e^{i\kappa z}$ where $\kappa=\sqrt{k^2-(\pi/a)^2}$. If now $k<\pi/a$, i.e., $\lambda>2a$, κ will be pure imaginary and hence for z sufficiently large and positive, $\phi(x, z)$ will vanish exponentially. In this case, the parallel plate regions cannot sustain a propagating mode. If $k>\pi/a$, i.e., $2a<\lambda$, then κ is real and the parallel plate region can sustain at least one mode consistent with the polarization which we have employed. In order that a second mode not propagate in this parallel plate region, we must further assume that $a<\lambda$. We also assume that there is a single reflected wave. Such a restriction puts further limitations on a/λ as well as on θ. These restrictions will appear when we have obtained the solution of the problem.

3. Fourier transform solution of the integral equation. Before we turn to the Fourier transform solution of the integral equation (2.4) we shall first convert it into one of the Wiener-Hopf type. We note that the surface current density of the mth plate has the same magnitude as that of the zeroth plate provided we measure the distance along the mth plate from its edge. Hence, the surface current density on the mth plate differs from that of the zeroth plate only by a phase factor. This phase factor arises because the amplitude of the incident wave differs from plate edge to plate edge by the factor

$$e^{ik(b\cos\theta+a\sin\theta)}.$$

Thus

$$I_m(z-mb) = I_0(z)e^{ikm(b\cos\theta+a\sin\theta)},$$

where $I_0(z)$ is the surface current density on the zeroth plate. Equation (2.4) may then be rewritten as

$$0 = \phi_{\text{inc}}(z, na) + \frac{i}{4}\sum_{m=-\infty}^{\infty}\int_0^\infty I_0(z)e^{ik\rho m}H_0^{(1)}[k\sqrt{(z-z'-mb)^2+(n-m)^2a^2}]dz', \quad (3.1)$$

where $\rho=b\cos\theta+a\sin\theta$. If we replace z by $z+nb$, Eq. (3.1) will read

$$0 = e^{ik[(z+nb)\cos\theta+na\sin\theta]}$$
$$+\frac{i}{4}\sum_{m=-\infty}^{\infty}\int_0^\infty I_0(z')e^{ik\rho m}H_0^{(1)}[k\sqrt{\{(n-m)b+(z-z')\}^2+(n-m)^2a^2}]dz', z>0.$$

Finally, when we divide the last equation by $e^{ik\rho n}$ and put $m-n=q$, we get

$$0 = e^{ikz\cos\theta} + \frac{i}{4}\sum_{q=-\infty}^{\infty}\int_0^\infty I_0(z')e^{ik\rho q}H_0^{(1)}[k\sqrt{q^2a^2+(qb+z-z')^2}]dz' \quad (3.2)$$

and this equation is of the Wiener-Hopf type.

In order to put this equation into a form which amenable to solution by Fourier transform methods, we extend it for negative z to be

$$\phi_1(z) = \frac{i}{4}\sum_{q=-\infty}^{\infty}\int_0^\infty I_0(z')e^{ik\rho q}H_0^{(1)}[k\sqrt{q^2a^2+(qb+z-z')^2}]dz', \quad z<0, \quad (3.3)$$

where $\phi_1(z)$ is an unknown function which is, save for a phase factor, the tangential component of the scattered electric field at $x=na$. In view of the periodic nature of the structure, the dependence of the integral equation on n is not explicit. We may now replace Eqs. (3.2) and (3.3) by the equation

$$\phi_1(z) = \phi_0(z) + \frac{i}{4} \sum_{q=-\infty}^{\infty} \int_{-\infty}^{\infty} I_0(z') e^{ik\rho q} H_0^{(1)}[k\sqrt{q^2a^2 + (qb + z - z')^2}]dz', \qquad (3.4)$$

where now

$$\phi_1(z) \equiv 0 \quad \text{for} \quad z > 0,$$
$$I_0(z) \equiv 0 \quad \text{for} \quad z < 0,$$
$$\phi_0(z) \equiv \begin{cases} 0 & \text{for} \quad z < 0, \\ e^{ikz} \cos\theta & \text{for} \quad z > 0. \end{cases}$$

For analytical convenience, it is now assumed that k has a small positive imaginary part. This is tantamount to assuming that the medium is slightly absorbing.

Before we can apply the Fourier transform in the complex plane to the solution of Eq. (3.4) it is necessary to study the growth order of the functions $\phi_1(z)$, $I_0(z)$ and $\phi_0(z)$. It is clear from a direct study of the integral Eqs. (3.2) and (3.3) that these functions are integrable for all finite z. The half planes of regularity of the Fourier transforms of $\phi_0(z)$, $\phi_1(z)$ and $I_0(z)$ are, of course, determined from their growth orders at infinity and we now proceed to determine these orders. Since we know $\phi_0(z)$ explicitly, it is clear that its Fourier transform is

$$\int_0^{\infty} e^{-iwz}\phi_0(z')dz' = \frac{1}{i[w - k\cos\theta]}$$

and is regular in a lower half of the w plane defined by the inequality $\Im m w < \Im m(k\cos\theta)$. Save for a translation on the z variable and a phase factor which is independent of z, $I_0(z)$ is, in certain units, the surface current density on any metallic plate. For z sufficiently large and positive, $I_0(z)$ is asymptotic to the surface current density in any of the parallel plate regions, that is, it is asymptotic to $e^{i\kappa z}$. Since $I_0(z)$ is integrable at the origin, the Fourier transform of $I_0(z)$, that is

$$\int_0^{\infty} I_0(z')e^{-iwz'}dz',$$

is regular in some half plane defined by

$$\Im m w < \Im m(\kappa) \sim \frac{\Re e k \Im m k}{|\kappa|} > \Im m k,$$

since $\Re e(k)/|\kappa| > 1$.

We now investigate the asymptotic form of $\phi_1(z)$ for z large and negative. Before doing this, however, it is convenient to give another representation of the kernel of the integral equation (3.4). The kernel

$$\frac{i}{4} \sum_{q=-\infty}^{\infty} e^{ik\rho q} H_0^{(1)}[k\sqrt{q^2a^2 + (qb + z)^2}]$$

has the Fourier integral representation

$$\frac{i}{4\pi}\int_C e^{iwz}\sum_{q=-\infty}^{\infty}\frac{e^{ik\rho q+i|q|a\sqrt{k^2-w^2}-iwqb}}{\sqrt{k^2-w^2}}\,dw, \tag{3.5}$$

where C is a contour which lies in the strip of regularity of the sum in (3.5). It is closed in the upper or lower half planes by a large semi-circle which passes between the poles of this sum depending upon whether $z>0$ or $z<0$. The strip of regularity is, of course, determined by the region in which the infinite series in (3.5) converges. A direct study of this series will reveal that the ordinates of convergence are given by the inequality, $\mathfrak{Im}k \cos(2\alpha-\theta)<\mathfrak{Im}w<\mathfrak{Im}k\cos\theta$. This now clarifies the reason why we imposed a small but positive imaginary part on k. Had we not done this, the series would only converge on the real axis of the w plane and as we shall see in the actual solution of the Wiener-Hopf equation, this situation would have presented us with some analytical difficulties.

We may now write the sum in the integral (3.5) in closed form as

$$\frac{i}{4\pi}\int_C \frac{e^{iwz}\sin a\sqrt{k^2-w^2}\,dw}{\sqrt{k^2-w^2}[\cos a\sqrt{k^2-w^2}-\cos(k\rho-wb)]}.$$

For $z<0$, we close the path C in the lower half of the w plane. The poles in the lower half plane are $w=k\cos(2\alpha-\theta)$ and two infinite sequences of poles both of which have negative imaginary parts. We shall have more to say about this double set of poles presently. Suffice it to be noted at this point, that the kernel has a second representation which for $z<0$ may now be written as

$$\frac{e^{ikz\cos(2\alpha-\theta)}}{2ak\sin(\alpha-\theta)}+\text{terms which attenuate exponentially for } z \text{ large and negative.}$$

It is clear then, that for z large and negative, $\phi_1(z)$ is asymptotic to

$$\int_0^\infty \frac{e^{ik(z-z')\cos(2\alpha-\theta)}}{2ak\sin(\alpha-\theta)}I_0(z')dz',$$

and thus, the Fourier transform of $\phi_1(z)$, i.e.,

$$\int_{-\infty}^0 e^{-iwz}\phi_1(z)dz,$$

is regular in the upper half of the w plane $\mathfrak{Im}w>\mathfrak{Im}k\cos(2\alpha-\theta)$.

The Fourier transforms involved in this problem then have a common strip of regularity, $\mathfrak{Im}((k\cos(\theta-2\alpha))<\mathfrak{Im}w<\mathfrak{Im}(k\cos\theta)$ and it is thus permissible to apply the Fourier transform to the integral equation (3.4) within this strip.

Let $\Phi_1(w)$ be the Fourier transform of $\phi_1(z)$ and $J(w)$ the Fourier transform of $I_0(z)$. The Fourier transform of the integral equation (3.4) is then

$$\Phi_1(w)=\frac{1}{i(w-k\cos\theta)}+\frac{J(w)\sin a\sqrt{k^2-w^2}}{2\sqrt{k^2-w^2}\,[\cos a\sqrt{k^2-w^2}-\cos(k\rho-wb)]}. \tag{3.6}$$

The Wiener-Hopf theory now tells us that we can split this transform equation into two parts. One part will be regular in an upper half plane, $\Im m w > \Im m k \cos(\theta - 2\alpha)$, the other in a lower half plane $\Im m w < \Im m k \cos\theta$ and both of these half planes have a common region of regularity. It is well to note here that we use the term regularity in a slightly extended sense. We imply by regularity that the function has neither zeros, branch points nor poles in the region of regularity. That is, the function as well as its reciprocal is "regular" in the conventional sense of the term. Suppose we assume that we can write

$$\frac{K_-(w)}{K_+(w)} = \frac{\sin a\sqrt{k^2 - w^2}}{\sqrt{k^2 - w^2}\left[\cos a\sqrt{k^2 - w^2} - \cos(k\rho - wb)\right]},$$

where $K_-(w)$ is regular in the proper lower half plane and $K_+(w)$ is regular in the proper upper half plane and that there is a common strip of regularity for both $K_-(w)$ and $K_+(w)$. Then

$$\Phi_1(w)K_+(w) = \frac{K_+(w)}{i(w - k\cos\theta)} + \frac{J(w)K_-(w)}{2}\cdot \tag{3.7}$$

The left side of Eq. (3.7) is regular in an upper half plane while the second term on the right side is regular in a lower half plane. The term

$$\frac{K_+(w)}{i(w - k\cos\theta)}$$

is only regular in the strip of regularity. This function may be decomposed into two functions in such a manner that one function is regular in the appropriate upper and the other in the appropriate lower half plane, since

$$\frac{K_+(w)}{i(w - k\cos\theta)} = \frac{K_+(w) - K_+(k\cos\theta)}{i(w - k\cos\theta)} + \frac{K_+(k\cos\theta)}{i(w - k\cos\theta)}\cdot$$

The first term on the right no longer has a singularity at $w = k\cos\theta$, but is regular in the upper half plane and the second term is regular in the lower half plane. Thus Eq. (3.7) can be rewritten in the form

$$\Phi_1(w)K_+(w) - \frac{K_+(w) - K_+(k\cos\theta)}{i(w - k\cos\theta)} = \frac{J(w)K_-(w)}{2} + \frac{K_+(k\cos\theta)}{i(w - k\cos\theta)}\cdot \tag{3.8}$$

The right side of the equation is regular in the lower half plane $\Im m w < \Im m k \cos\theta$ while the left side is regular in the upper half plane $\Im m w > \Im m k \cos(\theta - 2\alpha)$. Both sides have a common strip of regularity and hence the left side of (3.8) is the analytical continuation of the right side. Such an equality can only hold if both sides of Eq. (3.8) are equal to an integral function, that is, a function regular everywhere in the complex w plane. We have then

$$\frac{J(w)K_-(w)}{2} + \frac{K_+(k\cos\theta)}{i(w - k\cos\theta)} = \text{integral function} \tag{3.9}$$

and also

$$\Phi_1(w)K_+(w) - \frac{K_+(w) - K_+(k\cos\theta)}{i(w - k\cos\theta)} = \text{integral function.} \tag{3.10}$$

We shall now show that it is possible to decompose the function

$$\frac{\sin a\sqrt{k^2 - w^2}}{\sqrt{k^2 - w^2}\,[\cos a\sqrt{k^2 - w^2} - \cos (k\rho - wb)]}$$

into two functions, one of which is regular in the lower half plane $\Im m w < \Im m k \cos \theta$, while the other is regular in the upper half plane $\Im m w > \Im m k \cos (\theta - 2\alpha)$. The denominator of the fraction may be written as

$$\begin{aligned}
&\cos a\sqrt{k^2 - w^2} - \cos (k\rho - wb) \\
&= 2 \sin \frac{[a\sqrt{k^2 - w^2} + k\rho - wb]}{2} \sin \frac{[k\rho - wb - a\sqrt{k^2 - w^2}]}{2} \\
&= \tfrac{1}{2}[a\sqrt{k^2 - w^2} + k\rho - wb][k\rho - wb - a\sqrt{k^2 - w^2}] \\
&\quad \times \prod_{n=1}^{\infty}\left[1 - \frac{(a\sqrt{k^2 - w^2} + k\rho - wb)^2}{4n^2\pi^2}\right]\prod_{n=1}^{\infty}\left[1 - \frac{(k\rho - wb - a\sqrt{k^2 - w^2})^2}{4n^2\pi^2}\right] \\
&= \tfrac{1}{2}[(k\rho - wb)^2 - a^2(k^2 - w^2)]\prod_{n=1}^{\infty}\left[1 - \frac{a\sqrt{k^2 - w^2} + k\rho - wb}{2n\pi}\right]e^{(a\sqrt{k^2-w^2}+k\rho-wb)/2n\pi} \\
&\quad \times \prod_{n=-\infty}^{-1}\left[1 - \frac{a\sqrt{k^2 - w^2} + k\rho - wb}{2n\pi}\right]e^{(a\sqrt{k^2-w^2}+k\rho-wb)/2n\pi} \\
&\quad \times \prod_{n=1}^{\infty}\left[1 - \frac{k\rho - wb - a\sqrt{k^2 - w^2}}{2n\pi}\right]e^{(k\rho-wb-a\sqrt{k^2-w^2})/2n\pi} \\
&\quad \times \prod_{n=-\infty}^{-1}\left[1 - \frac{k\rho - wb - a\sqrt{k^2 - w^2}}{2n\pi}\right]e^{(k\rho-wb-a\sqrt{k^2-w^2})/2n\pi}.
\end{aligned}$$

The exponential factors in each of these products has been inserted to render the products absolutely convergent. The above expression may now be rewritten to read

$$\tfrac{1}{2}(a^2 + b^2)(w - \sigma_1)(w - \sigma_2) \prod_{n=-\infty}^{\infty}{}'\left[\left\{1 - \frac{k\rho - wb}{2n\pi}\right\}^2 - \frac{a^2(k^2 - w^2)}{4n^2\pi^2}\right]e^{(k\rho-wb)/n\pi} \tag{3.11}$$

where the prime on the products denotes the absence of the term $n = 0$ in the product. The infinite product in the last expression may now be expressed in a manner such that it puts into evidence the portion which is regular in the correct upper half and lower half planes. Indeed we may express (3.11) as

$$\begin{aligned}
&\tfrac{1}{2}(a^2 + b^2)(w - \sigma_1)(w - \sigma_2) \prod_{n=-\infty}^{\infty}{}' [\Delta_n - i\Psi_n]e^{[(k\rho-wb+wai)/2\pi n]+i(\pi/2-\alpha)} \\
&\qquad\qquad \times \prod_{n=-\infty}^{\infty}{}' [\Delta_n + i\Psi_n)e^{[(k\rho-wb-wai)/2n\pi]-i(\pi/2-\alpha)},
\end{aligned}$$

where now

$$\sigma_1 = k \cos \theta, \qquad \sigma_2 = k \cos (2\alpha - \theta),$$

and

$$\Delta_n = \sqrt{\sin^2 \alpha\left(1 - \frac{k\rho}{2\pi n}\right)^2 - \left(\frac{ak}{2\pi n}\right)^2}, \qquad \Psi_n = \cos \alpha\left(1 - \frac{k\rho}{2\pi n}\right) + \frac{wa \csc \alpha}{2\pi n},$$

where again, the exponential factors following the infinite products have been chosen to insure the absolute convergence of the product. One should note at this point that the choice of these exponential factors is not unique and indeed need only be asymptotic to the factors which we have chosen. However, we shall see that a second integral function $\chi(w)$, introduced into the decomposition of $K(w)$, is determined in terms of the factors which we have chosen. We have finally that the factor

$$(w-\sigma_1)\prod_{n=-\infty}^{-1}[\Delta_n - i\Psi_n]e^{[(k\rho-wb+wai)/2\pi n]+i(\pi/2-\alpha)}\prod_{n=1}^{\infty}[\Delta_n + i\Psi_n]e^{[(k\rho-wb-wai)/2\pi n]-i(\pi/2-\alpha)}$$

has no zeros in the lower half plane $\mathfrak{Im}\,w<\mathfrak{Im}\,k\cos\theta$, while the factor

$$(w-\sigma_2)\prod_{n-1}^{\infty}[\Delta_n - i\Psi_n]e^{[(k\rho-wb-wai)/2\pi n]+i(\pi/2-\alpha)}\prod_{n=-\infty}^{-1}[\Delta_n + i\Psi_n]e^{[(k\rho-wb-wai)/2\pi n]-i(\pi/2-\alpha)}$$

has no zeros in the upper half plane $\mathfrak{Im}\,w>\mathfrak{Im}[k\cos(\theta-2\alpha)]$. The factorization of

$$\frac{\sin a\sqrt{k^2-w^2}}{\sqrt{k^2-w^2}}$$

is more direct, for

$$\frac{\sin a\sqrt{k^2-w^2}}{\sqrt{k^2-w^2}} = a\prod_{n=1}^{\infty}\left[1-\frac{a^2(k^2-w^2)}{n^2\pi^2}\right]$$

$$= \frac{a^3}{\pi}(w-\kappa)e^{-iaw/\pi}(w+\kappa)e^{iaw/\pi}\prod_{n=2}^{\infty}\left[\sqrt{1-\left(\frac{ak}{\pi n}\right)^2}+\frac{iaw}{\pi n}\right]e^{-iaw/\pi n}$$

$$\times\prod_{n=2}^{\infty}\left[\sqrt{1-\left(\frac{ak}{\pi n}\right)^2}-\frac{iaw}{\pi n}\right]e^{iaw/\pi n}.$$

The factor

$$\frac{a}{\pi}(w-\kappa)e^{-iaw/\pi}\prod_{n=2}^{\infty}\left[\sqrt{1-\left(\frac{ak}{\pi n}\right)^2}+\frac{iaw}{\pi n}\right]e^{-iaw/\pi n}$$

has no zeros in the lower half plane $\mathfrak{Im}\,w<\mathfrak{Im}\,\kappa$, while the factor

$$\frac{a}{\pi}(w+\kappa)e^{iaw/\pi}\prod_{n=2}^{\infty}\left[\sqrt{1-\left(\frac{ak}{\pi n}\right)^2}-\frac{iaw}{\pi n}\right]e^{iaw/\pi n}$$

has no zeros in the upper half plane $\mathfrak{Im}\,w>\mathfrak{Im}(-\kappa)$. We thus find that

$$K_-(w) = \frac{\prod_{n=2}^{\infty}\left[\sqrt{1-\left(\frac{ak}{\pi n}\right)^2}+\frac{iaw}{\pi n}\right]e^{-iaw/\pi n}\frac{a}{\pi}(w-\kappa)e^{-iaw/\pi}e^{\chi(w)}}{(w-\sigma_1)\prod_{n=-\infty}^{-1}[\Delta_n - i\Psi_n)e^{[(k\rho-wb+wai)/2\pi n]+i(\pi/2-\alpha)}\prod_{n=1}^{\infty}[\Delta_n + i\Psi_n]e^{[(k\rho-wb-wai)/2\pi n]-i(\pi/2-\alpha)}}$$

is free of zeros and poles in the lower half plane $\mathfrak{Im}\,w<\mathfrak{Im}\,k\cos\theta$. The factor $e^{\chi(w)}$ will be determined so as to make $K_-(w)$ have algebraic growth as $|w|\to\infty$ for $\mathfrak{Im}\,w<0$. With $\chi(w)$ so chosen, the integral function sought can only be of algebraic growth for $|w|\to\infty$. $K_-(w)$ is regular in the lower half plane $\mathfrak{Im}\,w<\mathfrak{Im}\,k\cos\theta$. Finally,

$$K_+(w) = \frac{(a^2+b^2)(w-\sigma_2)e^{\chi(w)}\prod_{n=1}^{8}[\Delta_n - i\Psi_n]e^{[(k\rho-wb+wai)/2\pi n]+i(\pi/2-\alpha)}\prod_{n=-\infty}^{-1}[\Delta_n + i\Psi_n]e^{[(k\rho-wb+wai)/2\pi n]-i(\pi/2-\alpha)}}{2a\prod_{n=2}^{\infty}\left[\sqrt{1-\left(\frac{ak}{\pi n}\right)^2}-\frac{iaw}{\pi n}\right]e^{iaw/\pi n}\frac{a}{\pi}(w+\kappa)e^{iaw/\pi}}$$

has no zeros or poles in the upper half plane $\mathfrak{Im}\, w > \mathfrak{Im}\, k \cos(\theta - 2\alpha)$.

We shall now discuss the asymptotic form of $K_-(w)$ as $|w| \to \infty$, $\mathfrak{Im}\, w < 0$. This procedure will enable us to determine the unknown integral function $\chi(w)$. It has been shown by Schwinger[7] that functions of the form of $K_-(w)$ are independent of ka for $|w| \to \infty$, $\mathfrak{Im}\, w < 0$, and $\pi < ak < 2\pi$. Thus

$$K_-(w) = \frac{e^{\chi(w)} \prod_{n=2}^{\infty} \left[1 + \frac{iaw}{\pi n}\right] e^{-iaw/\pi n} \frac{a}{\pi} e^{-iaw/\pi}}{\prod_{n=-\infty}^{-1} \left[\sin\alpha - i\left(\cos\alpha + \frac{wa \csc\alpha}{2\pi n}\right)\right] e^{[(wai - wb)/2\pi n] + i(\pi/2 - \alpha)}} \times \frac{1}{\prod_{n=1}^{\infty} \left[\sin\alpha + i\left(\cos\alpha + \frac{wa\csc\alpha}{2\pi n}\right)\right] e^{-[(wai + wb[/2\pi n] - i(\pi/2 - \alpha)}}. \tag{3.12}$$

The products in (3.12) are now in the form of gamma functions and

$$K_-(w) \sim \frac{e^{\chi(w)} \left(\frac{wa\csc\alpha}{2\pi}\right)^2 e^{iwa\gamma/\pi}\, \Gamma\left(\frac{-wa\csc\alpha e^{-i\alpha}}{2\pi}\right) \Gamma\left(\frac{wa\csc\alpha e^{i\alpha}}{2\pi}\right)}{\frac{iaw}{\pi}\left(1 + \frac{iaw}{\pi}\right) e^{iaw\gamma/\pi}\, \Gamma\left(\frac{iaw}{\pi}\right)},$$

where γ is the Euler-Mascheroni constant. Using the Stirling expansion theorem for $|w| \to \infty$, $\mathfrak{Im}\, w < 0$ we get

$$K_-(w) \sim \frac{e^{\chi(w)} a \csc^2\alpha \left(-\frac{aw\csc\alpha}{2\pi} e^{-i\alpha}\right)^{-[(wa\csc\alpha)/2\pi]e^{-i\alpha} - 1/2} \left(\frac{aw\csc\alpha}{2\pi} e^{i\alpha}\right) e^{[(wa\csc\alpha)/2\pi]e^{i\alpha} - 1/2}}{4\pi^2 i \left(\frac{iaw}{\pi}\right)^{(iaw/\pi) - 1/2}}$$

$$\sim \frac{C e^{\chi(w) + iaw/\pi[(\alpha - \pi/2)\cot\alpha + \ln(\csc\alpha)/2]}}{w^{1/2}},$$

where C is a constant. Thus if we choose

$$\chi(w) = \frac{-iaw}{\pi}\left[\left(\alpha - \frac{\pi}{2}\right)\cot\alpha - \ln 2\sin\alpha\right],$$

$K_-(w)$ will have algebraic growth for $|w|$ large, $\mathfrak{Im}\, w < 0$.

Now $J(w)$, which is proportional to the Fourier transform of the surface current density on the various plates, approaches zero for $|w|$ large, $\mathfrak{Im}\, w < 0$. This assumes, of course, that $I_0(z)$ can at most be of exponential growth for z large and positive and is integrable for z finite. Thus $K_-(w)J(w)$ approaches zero for $|w|$ large and $\mathfrak{Im}\, w < 0$. If we now return to Eq. (3.12) we see that as $|w|$ becomes large, $\mathfrak{Im}\, w < 0$, the integral function in (3.9) is asymptotic to zero. We may now apply the same argument to Eq. (3.10) and find that the integral function is again asymptotic to zero. But by a theorem of Liouville, and analytic function which is bounded in the entire complex plane is constant and in this case the constant must be zero. We thus have

$$J(w) = \frac{2iK_+(k\cos\theta)}{K_-(w)(w - k\cos\theta)}.$$

If we were interested in the explicit form of the surface current density, we could obtain it from $J(w)$ by evaluating the Fourier inversion integral

[7] J. S. Schwinger, loc. cit.

$$\frac{i}{\pi}\int_C \frac{K_+(k\cos\theta)e^{iwz}dw}{K_-(w)(w - k\cos\theta)},$$

where C is a contour which may be taken as a straight line within the strip of regularity of the Fourier transforms of $I(z)$, $\phi_1(z)$, $\phi_0(z)$ and $K(z)$. The contour is closed above by a semi-circle, which by familiar arguments in contour integration may be shown to make no contribution to the value of the integral. In the next section we shall show that it is possible to find the reflection and transmission coefficients without evaluating this integral in detail.

4. Investigation of the far fields. In order to find the reflection and transmission coefficients, we now investigate the asymptotic form of $\phi(x, z)$ for $|z|$ large. To this end we note that Eq. (2.3) can be written in Fourier integral representation as

$$\phi(x, z) = \phi_{inc}(x, z) + \frac{i}{4\pi}\int_C \sum_{m=-\infty}^{\infty} \frac{e^{iwz+ikm\rho-iwmb+|x-ma|\sqrt{k^2-w^2}}J(w)dw}{\sqrt{k^2-w^2}},$$

where C is the contour which we described at the end of Section 3. This in turn, may simplified to

$$\phi(x, z) = \phi_{inc}(x, z)$$
$$-\frac{i}{4\pi}\int_C J(w)e^{iwz+i(k\rho-wb)}\frac{[\sin\sqrt{k^2-w^2}(x-an-a) + e^{i(k\rho-wb)}\sin\sqrt{k^2-w^2}(an-x)]dw}{\sqrt{k^2-w^2}[\cos a\sqrt{k^2-w^2} - \cos(k\rho - wb)]}, \quad (4.1)$$

where n is the greatest integer contained in x/a. From (4.1) one can get the asymptotic form of $\phi(x, z)$ as z becomes large and positive. Since $J(w)$ is regular in the lower half of the w plane $\Im m w < \Im m k\cos\theta$, we can close the contour C by a large semicircle which passes between the poles in the upper half plane. For $na < x < (n+1)a$ it can be seen that due to the form of the integrand, there is no contribution from this circular arc as its radius becomes infinite. In the upper half plane $\Im m w > \Im m k\cos(2\alpha - \theta)$, there are two poles which correspond to propagating modes, namely $w = k\cos\theta$ and $w = \kappa$. All other modes are attenuated modes in the sense that they have large positive imaginary parts compared to the imaginary parts of $k\cos\theta$ and κ. If we now express $J(w)$ as a function of w and use the above described contour in the evaluation of the integral in (4.1) we have then to consider the asymptotic form of

$$\frac{i}{2\pi}\int_C e^{i[k\rho-wb]n+iwz}\frac{[\sin(x-an-a)\sqrt{k^2-w^2} + e^{i(k\rho-wb)}\sin\sqrt{k^2-w^2}(an-x)]K_+(k\cos\theta)dw}{(w-k\cos\theta)K_+(w)\sin a\sqrt{k^2-w^2}}.$$

This in turn is equal to $[e^{ik(x\sin\theta+z\cos\theta)} - Te^{i\kappa z}\sin\pi x/a$ +terms which approach zero for $z \gg 0]$. For z large and positive, this is asymptotic to

$$\phi_{inc}(x, z) - Te^{i\kappa z}\sin\frac{\pi x}{a}\cdot$$

Hence, save for a numerical factor, the functional form of $\phi(x, z)$ as z becomes infinite is $e^{i\kappa z}\sin\pi x/a$, that is, it represents a travelling wave in the parallel plate region with propagation constant κ, as it should. The amplitude of this wave is

$$T = |T|e^{i\Theta} = \frac{\pi e^{in(k\rho-\kappa b)}(-)^n[1 + e^{i(k\rho-\kappa b)}]K_+(k\cos\theta)}{(\kappa - k\cos\theta)a^2\kappa K_+(\kappa)}$$

and depends of course on the particular parallel plate region for which it has been computed. Since T is the amplitude of the wave transmitted in the parallel plate region it is the transmission coefficient because the amplitude of the incident wave has been taken to be unity. If we now assume that k is real, the magnitude of T is

$$|T| = \frac{2^{3/2}k \sin(\alpha - \theta)}{\sqrt{(k\cos\theta + \kappa)(\kappa - k\cos(2\alpha - \theta))}},$$

a quantity independent of the particular parallel plate region considered. Its phase angle depends, of course, on the particular parallel plate region. We shall not give the phase angle explicitly since we shall not use it in our later discussions.

For z large and negative we close the contour in the lower half of the w plane. There is again no contribution from the circular arc which is drawn between the poles in the lower half plane and so we need only evaluate the residues from the poles in the lower half plane. The dominant contribution now arises from the pole $w = k\cos(\theta - 2\alpha)$ and in this case the dominant term is

$$\frac{K_+(k\cos\theta)e^{ik[x\sin(2\alpha-\theta)+z\cos(2\alpha-\theta)]}}{k[\cos(2\alpha-\theta) - \cos\theta]K_+'[k\cos(2\alpha-\theta)]},$$

all other terms in the integrand approaching zero for z large and negative. Here $K_+'[k\cos(2\alpha-\theta)]$ means, as usual, the derivative of $K_+(w)$ with respect to w evaluated at $w = k\cos(2\alpha-\theta)$. The amplitude of the reflected plane wave is the reflection coefficient R if the amplitude of the incident wave is taken as unity, so that we now have

$$R = \frac{K_+(k\cos\theta)}{k[\cos(2\alpha-\theta) - \cos\theta]K_+'[k\cos(2\alpha-\theta)]}.$$

Assuming, once again that k is real, the reflection coefficient may then be rewritten in complex polar form as follows:

$$R = -e^{i(\Theta_1-\Theta_2)}\sqrt{\frac{(k\cos\theta - \kappa)(k\cos(2\alpha-\theta) + \kappa)}{(k\cos\theta + \kappa)(k\cos(2\alpha-\theta) - \kappa)}},$$

where now

$$\Theta_1 = -\sum_{n=1}^{\infty}\left\{\arc\sin\frac{\cos\alpha + \frac{ka}{2\pi n}\sin(\alpha-\theta)}{\sqrt{1 - \frac{ka}{\pi n}\sin\theta}} - \frac{ka\cos\theta}{2\pi n} - \left(\frac{\pi}{2} - \alpha\right)\right\}$$

$$+\sum_{n=-\infty}^{-1}\left\{\arc\sin\frac{\cos\alpha + \frac{ka}{2\pi n}\sin(\alpha-\theta)}{\sqrt{1 - \frac{ka}{\pi n}\sin\theta}} - \frac{ka\cos\theta}{2\pi n} - \left(\frac{\pi}{2} - \alpha\right)\right\}$$

$$+\sum_{n=2}^{\infty}\left\{\text{arc sin}\frac{ka\cos\theta}{\pi n\sqrt{1-\left(\frac{ka}{\pi n}\right)^2\sin^2\theta}}-\frac{ka}{\pi n}\cos\theta\right\}$$

$$-\frac{ak}{\pi}\cos\theta\left\{1+\left(\alpha-\frac{\pi}{2}\right)\cos\alpha-\ln 2\sin\alpha\right\},$$

and

$$\Theta_2=-\sum_{n=1}^{\infty}\left\{\text{arc sin}\frac{\cos\alpha+\frac{ka}{2\pi n}\sin(\theta-\alpha)}{\sqrt{1-\frac{ka}{\pi n}\sin(2\alpha-\theta)}}-\frac{ka}{2n\pi}\cos(\theta-2\alpha)-\left(\frac{\pi}{2}-\alpha\right)\right\}$$

$$+\sum_{n=-\infty}^{-1}\left\{\text{arc sin}\frac{\cos\alpha+\frac{ka}{2\pi n}\sin(\theta-\alpha)}{\sqrt{1-\frac{ka}{\pi n}\sin(2\alpha-\theta)}}-\frac{ka}{2n\pi}\cos(\theta-2\alpha)-\left(\frac{\pi}{2}-\alpha\right)\right\}$$

$$+\sum_{n=2}^{\infty}\left\{\text{arc sin}\frac{\frac{ka}{\pi n}\cos(2\alpha-\theta)}{\sqrt{1-\left(\frac{ka}{\pi n}\right)^2\sin^2(2\alpha-\theta)}}-\frac{ak}{n\pi}(\cos(2\alpha-\theta)\right\}$$

$$-\frac{ak}{\pi}\cos(2\alpha-\theta)\left\{1+\left(\alpha-\frac{\pi}{2}\right)\cos\alpha-\ln 2\sin\alpha\right\}.$$

It is evident that there will be restrictions on a/λ and θ if we are to have a single reflected plane wave. These restrictions become evident when we study the arc sin sums and observe that conceivably the first term in the sums beginning with index unity can exceed unity. We have tacitly assumed that they do not, for otherwise they would appear in the amplitude factor as real terms. Thus we must see what is implied by the condition that all factors in the infinite products be complex, or equivalently $\Delta_n^2>0$. If we demand that $\Delta_1^2>0$ it is clear that all other $\Delta_n s$, $n=1, 2, \cdots$ will also be >0. The condition $\Delta_1>0$ is equivalent to

(i) $$\frac{ak}{\pi}=\frac{2a}{\lambda}<\frac{\sin\alpha}{\cos^2\frac{1}{2}(\theta-\alpha)}$$

and

(ii) $$\frac{ak}{\pi}>-\frac{\sin\alpha}{\text{sn}^2\frac{1}{2}(\theta-\alpha)}.$$

Condition (ii) is always satisfied since a/λ is always positive. Condition (i) can be more restrictive than the condition $1/2<a/\lambda<1$. For example, if $\theta=\pi/12$, $\alpha=5\pi/12$, then condition (i) implies

$$a/\lambda<.65.$$

5. No propagation in the parallel plate regions. In the Fourier transform solution of the integral equation (3.4) we have assumed that there was only one propagating mode in the parallel plate region, i.e.,

$$I_m(z) \sim e^{i\kappa z} \sin \frac{\pi x}{a}, \qquad \kappa > 0$$

for z large and positive and z in the parallel plate region. Suppose now we dispense with this assumption and ask what form the reflection coefficient takes if we now assume that $\kappa^2 < 0$, i.e., $0 < a/\lambda < 1/2$. In this case κ is, of course, imaginary and

$$I_m(z) \sim e^{-\sqrt{(\pi^2/a^2)-k^2}\, z} \sin \frac{\pi x}{a}$$

for z large and positive and z in the parallel plate region. The result we desire can be obtained most easily by studying the result which we have obtained in Section 3.

We note that if κ is purely imaginary and k is real, the amplitude of the reflection coefficient becomes complex of magnitude unity. Indeed for $\kappa^2 < 0$

$$\sqrt{\frac{(k \cos \theta - \kappa)(k \cos (2\alpha - \theta) + \kappa)}{(k \cos \theta + \kappa)(k \cos (2\alpha - \theta) - \kappa)}}$$

$$= \exp\left\{ i \left[\arc\sin \frac{ka \cos \theta}{\pi\sqrt{1 - \left(\frac{ka}{\pi}\right)^2 \sin^2 \theta}} - \arc\sin \frac{ka \cos (2\alpha - \theta)}{\pi\sqrt{1 - \left(\frac{ka}{\pi}\right)^2 \sin^2 (2\alpha - \theta)}} \right] \right\}.$$

Thus for this situation, the amplitude of the reflection coefficient is -1. The phase angle Θ_1' is given by

$$\Theta_1' = \Theta_1 + \arc\sin \frac{ka \cos \theta}{\pi\sqrt{1 - \left(\frac{ka}{\pi}\right)^2 \sin^2 \theta}},$$

while Θ_2' is now given by

$$\Theta_2' = \Theta_2 + \arc\sin \frac{ka}{\pi} \frac{\cos (2\alpha - \theta)}{\sqrt{1 - \left(\frac{ak}{\pi}\right)^2 \sin^2 (2\alpha - \theta)}}.$$

Hence the reflection coefficient for $0 < a/\lambda < 1/2$ is now $-e^{i(\Theta_1' - \Theta_2')}$. For a single reflected wave, the inequality (i) in Section 4 must still be satisfied, although now it is not as severe.

6. Discussion of results. It should be pointed out that some of the results obtained from our calculations can be interpreted in a simple physical manner. For convenience, in this discussion, instead of the angle θ we use the angle i, which the incident wave makes with the normal to the trace of the edges of the plates. It is readily verified that

$$i = \theta - \alpha + \frac{\pi}{2}$$

and also that the angle r which the reflected wave makes with the normal is also equal to i. The condition that there be only one reflected wave

$$\frac{2a}{\lambda} < \frac{\sin \alpha}{\cos^2 \frac{1}{2}(\theta - \alpha)}$$

is seen to be a result of simple grating theory. If the waves scattered by a uniform grating are not to interfere constructively in the region from which the waves are incident (except for the specular case $r = i$) the condition $a'(1 + \sin i) < \lambda$ must be satisfied where a' is the distance between neighboring scatterers. In our case $a' = a \csc \alpha$. Expressing θ in terms of i, the relation

$$\frac{2a}{\lambda} < \frac{\sin \alpha}{\cos^2 \frac{1}{2}(\theta - \alpha)}$$

is seen to be equivalent to $a \csc \alpha(1 + \sin i) < \lambda$. If this condition is satisfied and the condition for no propagation, $\lambda > 2a$, is also satisfied, the plates act as a perfect plane mirror. However, while the magnitude of the reflected wave is unity, its phase is not π but $\Theta_1' - \Theta_2'$. It is easily shown that it will be π on any plane parallel to that of the trace at a distance d given by

$$(4\pi d/\lambda) \cos i + 2m\pi = \Theta_2' - \Theta_1' \qquad m = 0, \pm 1, \pm 2, \cdots .$$

Therefore, as far as all distant fields are concerned, the plates behave in this case like a perfect plane mirror whose surface coincides with any of the planes given by the above equation.

When transmission is possible in the parallel plate region the wavelength in this region differs from that in free space. One would, therefore, expect to find some analogy with the phenomena associated with a plane interface between two dielectric media. This can be shown for the case $\alpha = \pi/2$. In this case the magnitude of the reflection coefficient is

$$|R| = \frac{k \cos i - \kappa}{k \cos i + \kappa} \cdot$$

This expression is identical with that obtained for the reflection at a dielectric interface of a wave with the electric vector parallel to the interface. The phases are different in the two cases and one can again find a set of planes at a distance d from the trace given by

$$(4\pi d/\lambda) \cos i + 2m\pi = \Theta_2 - \Theta_1, \qquad m = 0, \pm 1, \cdots$$

such that the distant fields are identical in the two cases if we regard any one of the planes as the interface.

The expression for the magnitude of the reflection coefficient should be of use in estimating the reflection of waves incident on a metal lens provided that the radius of curvature of the lens (i.e., the angle α) does not vary too rapidly.