

A NOTE ON STABILITY CALCULATIONS AND TIME LAG*

By SEYMOUR SHERMAN (*University of Chicago*)

In recent technical literature¹ covering widely different fields, investigations have appeared of the zeros of particular exponential sums, one example of which is

$$az^2 + bz + \beta ze^{-z} + c. \quad (1)$$

The question is: what are the conditions on a , b , β , and c which are necessary and sufficient that the real parts of all the roots be negative, thus indicating stability. There also have appeared papers in pure mathematics² which discuss similar problems and which supply useful techniques for their solutions. It is the purpose of this note to indicate how one such technique, which shall be referred to as the Cauchy-Sturm method, may be applied to a discussion of the zeros of transcendental expressions such as (1).

Equation (1) arises in the study of control systems with retarded action or time lags.³ Several attempts have been made to study the zeros of this function and the results have not been consistent. Minorsky,⁴ in one of his papers, expands the function in a power series

$$\begin{aligned} f(z) &= az^2 + bz + \beta ze^{-z} + c \\ &= (c + (b + \beta)z + (a - \beta)z^2 + \frac{\beta z^3}{2!} - \frac{\beta z^4}{3!} + \frac{\beta z^5}{4!} - \frac{\beta z^6}{5!} + \dots) \end{aligned}$$

He then attempts to approximate the zeros of $f(z)$ by taking zeros of partial sums. For nonzero c and β we can choose a partial sum of degree n ($n \geq 3$) such that c and $(-1)^{n-1}\beta$ have opposite signs and so the partial sum

$$c + (b + \beta)z + (a - \beta)z^2 + \frac{\beta z^3}{2!} + \dots + (-1)^{n-1} \frac{\beta z^n}{(n-1)!}$$

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¹ H. Bateman, *The control of an elastic fluid*, Bull. Amer. Math. Soc. **51**, 601-646 (1945), especially pp. 618-626. Further references given there.

A. Callender, D. R. Hartree, A. Porter, *Time-lag in a control system*, Trans. Roy. Soc. London (A) **235**, 415-444 (1936).

A. Callender, and A. B. Stevenson, Proc. Soc. Chem. Industry (Chem. Engg. Group) **18**, 108 (1936).

D. R. Hartree, A. Porter, A. Callender, A. B. Stevenson, *Time-lag in a control system*, Proc. Roy. Soc. London (A) **161**, 460-476 (1937).

N. Minorsky, (i) *Control problems*, J. Franklin Inst. **232**, 519-551 (1941), especially pp. 524-529.

N. Minorsky, (ii) *Self-excited oscillations in dynamical systems possessing retarded actions*, J. Appl. Mech. **64**, A65-A71 (1942).

F. Reinhardt, *Parallelbetrieb von Synchrongeneratoren mit Kraftmaschinenregelung konstanter Verzögerungszeit*, Wissenschaftl. Veröffentl. Siemens Werke **18**, 24-44 (1939).

² R. E. Langer, *On the zeros of exponential sums and their integrals*, Bull. Amer. Math. Soc. **37**, 213-239 (1931). Further bibliography can be found there.

³ For example the roots of (1) determine the stability of the following system with retarded viscous term:

$$az''(t) + bz'(t) + \beta z'(t-1) + c = 0,$$

where $z(0)$ is given and $z'(t) = 0$, $-1 < t \leq 0$.

⁴ N. Minorsky, (i) *supra*.

would have at least one positive zero. This would seem to suggest that dynamical systems with retarded viscous terms are necessarily unstable. However, from a theorem of function theory,⁵ we know

For every power series, every point of the circle of convergence is a limit point of zeros of partial sums.

Although function (1) is entire and so has no circle of convergence, the theorem stated above shakes our faith in the value of approximating the zeros of function by taking the zeros of partial sums of its Taylor series.

Reinhardt,⁶ in his discussion of the same equation, first considers a particular case of equation (1), namely

$$z^2 + .5z + .5ze^{-z} + 1 = 0, \quad (2)$$

and among the infinite number of zeros of $f(z)$ chooses one which has the largest real part and small imaginary part (most unstable and corresponding to low frequency). Arguing that this root, corresponding as it does to a low resonant frequency, is physically the most significant, Reinhardt studied the zeros of (1) which for other choices of parameters could be expanded in a series about the "original" zero. Thus he studied one of the infinite number of zeros of (1) intensively and later made approximations to the others. He discovered that for some choices of a , b , β , and c this root had a positive real part and for other choices of these parameters this root had a negative real part. Thus, the results of Reinhardt and Minorsky are inconsistent. Since both of their arguments are approximate a further study is indicated. Minorsky has published another analysis⁷ of this subject which allows for the possibility of stability.

A method frequently useful for counting the zeros of an analytic function $f(z)$ in a simply connected domain D bounded by curve C is Cauchy's⁸ index theorem:

If $w=f(z)$ is an analytic function of z in a simply connected domain D bounded by a closed curve C , $f(z) \neq 0$, $z \in C$, and z traverses C in a counterclockwise direction, then $f(z)$ will traverse a closed curve in the w -plane and the number of zeros of $f(z)$ in D is equal to the number of times the w -contour encircles the origin.

This theorem is at the heart of Nyquist's⁹ criterion for the stability of amplifiers and Routh's¹⁰ stability criterion. An attempt will be made to apply Cauchy's Theorem to the zeros of (1). We first note that as z traverses C in a counterclockwise sense w may cross the real axis. Let γ be the number of times w crossed the real axis in a counterclockwise direction relative to the origin (i.e., from quadrant IV to quadrant I or from quadrant II to quadrant III) and let α be the number of times w crossed the real axis in a clockwise direction relative to the origin (i.e., from quadrant I to quadrant IV or from quadrant III to quadrant II). The number of zeros of $f(z)$ in D is then equal to $1/2(\gamma - \alpha)$.

Conditions are sought on a , b , β , and c (all nonzero) in order that all of the roots of

$$w(z) = az^2 + bz + \beta ze^{-z} + c = 0$$

⁵ E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, 1939, p. 238.

⁶ See (1) *supra*.

⁷ N. Minorsky, (ii) *supra*.

⁸ H. W. Bode, *Network analysis and feedback amplifier design*, D. Van Nostrand Co., 1945, chap. 8. See Titchmarsh, *supra* pp. 115-116.

⁹ See Bode *supra*.

¹⁰ See E. J. Routh, *Dynamics of a system of rigid bodies*, Macmillan and Co., 1892, Part II, pp. 191-202.

have negative real parts. We assume that $w(z)$ has no zeros on the imaginary axis.

Our domain D is the semicircle

$$D: \mathcal{R}(z) > 0, \quad |z| < R$$

in the z -plane.

For $R(>0)$ sufficiently large, if $|z| \geq R$ and¹¹ $\mathcal{R}(z) \geq 0$, then

$$|az^2| > |\beta ze^{-z} + bz + c|.$$

Since $az^2 \neq 0$ for $|z| \geq R$, $\mathcal{R}(z) \geq 0$, we have, by arguing from Rouché's theorem,¹² that $w(z) \neq 0$, for z in this region. If we choose the R given above, all the zeros of $w(z)$ lying in the right half-plane will lie in D and so we apply Cauchy's Index Theorem to this region. The boundary curve C may be broken into two parts

$$A; \mathcal{R}(z) = 0 \quad |z| \leq R$$

and

$$B; \mathcal{R}(z) > 0 \quad |z| = R.$$

We consider A for large R . Let $z = iy$ and

$$w(iy) = -ay^2 + c + \beta y \sin y + i(by + \beta y \cos y).$$

Note that $\mathcal{R}(w)$ is an even function of y and $\Im(w)$ is an odd function of y .

In practical application frequently $a > 0$, $c > 0$. Let us consider the special case $b \geq |\beta| > 0$. If $y = 0$, then $w(0) = c > 0$. If $0 < y \leq R$, then $\Im(w) = y(b + \beta \cos y) \geq 0$ and w is in either quadrant I or quadrant II. For large R , $w(iR)$ is in the second quadrant. If $-R \leq y < 0$, then $\Im(w) = y(b + \beta \cos y) \leq 0$ and w is in either the third or fourth quadrants. Thus, as z traverses A from $+iR$ to $-iR$, w crosses the real axis once in a clockwise direction relative to the origin (from quadrant I to IV). On the other hand, for large R , $w(z)$ is dominated by az^2 and as z traverses B from $-iR$ to $+iR$, the net number of times that w crosses the real axis is just once in a counterclockwise direction relative to the origin (from quadrant IV to I). Since $1/2(\gamma - \alpha) = 0$ for C , $f(z)$ has no zeros in D , therefore all the zeros of $f(z)$ have negative real parts. It should be noted that if we remove the restriction of $c > 0$, then¹³ $\gamma - \alpha = 1 - \text{sgn}[c(b + \beta)]$.

We may rephrase the results as follows: *In a one degree of freedom mechanical system with positive mass a , positive spring constant c and positive damping coefficient b , and with retarded (unit time lag) coefficient β , if the damping coefficient is greater than or equal to the absolute value of the retarded damping coefficient, then the system is stable.*¹⁴

Suppose we relax the restrictions of the preceding two paragraphs but still require $a > 0$ and consider (1) on curve C . In order to compute $1/2(\gamma - \alpha)$ for the line

¹¹ If $z = x + iy$, x, y real, then

$$\mathcal{R}(z) = x \quad \text{and} \quad \Im(z) = y.$$

¹² See Titchmarsh, p. 116.

¹³ $\text{sgn}(x)$ is a real valued function of a real variable defined as follows:

$$\text{sgn}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

¹⁴ This is consistent with Minorsky (ii) p. A69. Note that if $b \leq -|\beta| < 0$ (the damping coefficient less than or equal to the negative of the absolute value of retarded damping coefficient), then the system is unstable with two zeros of (1) in the right half-plane.

segment A and arc B we wish first to note where w crosses the real axis. Because $\Re(w(iy))$ and $\Im(w(iy))$ are even and odd functions of y respectively, we need only consider those positive values of y for which $\Im(w(iy)) = y(b + \beta \cos y) = 0$.

If we temporarily ignore the case previously considered where $y(b + \beta \cos y)$ had no positive zeros, but assume merely that $\beta \neq 0$, then the positive roots of $y(b + \beta \cos y)$ will be of the form

$$\arccos\left(\frac{-b}{\beta}\right) + 2m\pi, \quad m = 0, 1, 2, \dots,$$

where we permit \arccos to take two positive values between 0 and 2π (including the latter). We number the positive zeros of $y(b + \beta \cos y)$ in order of increasing size y_1, y_2, \dots . Let y_M be such a positive zero of $b + \beta \cos y$ that

$$(1) \quad \beta \sin y_M < 0$$

and

$$(2) \quad -ay^2 + c + \beta y \sin y < 0 \text{ for }^{15} y \geq y_M.$$

We might, if we so wished, have chosen y_M so large that $y_M < R < y_{M+1}$. For the purposes of our subsequent calculation this would be inconvenient, but since it is convenient for the purpose of the argument, we shall make this assumption during the proof. We now calculate $\gamma - \alpha$ for curve C . In other words we have to consider the number of times and the direction relative to the origin in which w crosses the real axis as z traverses C in a counterclockwise direction relative to D . The only values of z along A for which w hits the real axis are: $0, \pm y_j; j = 0, 1, 2, \dots, M$. Let us consider y_j , which is by definition positive. The contribution to $\gamma - \alpha$ due to this crossing is

$$\begin{aligned} & -\operatorname{sgn}(-ay_j^2 + c + \beta y_j \sin y_j) \operatorname{sgn}\left[\frac{d}{dy}\{y(b + \beta \cos y)\}\right]_{y=y_j} \\ &= -\operatorname{sgn}[-ay_j^2 + c + \beta y_j \sin y_j] \operatorname{sgn}\left[\frac{d}{dy}(b + \beta \cos y)\right]_{y=y_j} \\ &= +\operatorname{sgn}[(-ay_j^2 + c + \beta y_j \sin y_j)(\beta \sin y_j)]. \end{aligned}$$

Because $\Re(w(iy))$ and $\Im(w(iy))$ are, respectively, even and odd functions of y , the contribution to $\gamma - \alpha$, because of the crossing at $z = -iy_j$ is also

$$\operatorname{sgn}[(-ay_j^2 + c + \beta y_j \sin y_j)(\beta \sin y_j)].$$

The definition of y_M was so arranged that the net contribution to $\gamma - \alpha$ because of the crossing of the real axis corresponding to $z = 0$ and those crossings corresponding to z on B totals to $1 - \operatorname{sgn}[c(b + \beta)]$, as in the case $a > 0, b \geq |\beta| > 0$, previously considered. We now have¹⁶

¹⁵ If $y > (2c/2\sqrt{ac} - \beta) > 0$, then $-ay^2 + c + \beta y \sin y < 0$. Thus in many cases the condition $y > (2c/2\sqrt{ac} - \beta) > 0$ may be used rather than (2). There are other approximations to (2) which might prove convenient in different cases.

¹⁶ If $\Im[w(iy)] \neq 0$ for any real value of y or if $\Im[w(iy)] = 0$ implies that $\sin y = 0$, then substitute 0 for $2\sum_{j=1}^M \operatorname{sgn}\{[-ay_j^2 + c + \beta y_j \sin y_j][\beta \sin y_j]\}$.

$$(\gamma - \alpha) = 1 - \operatorname{sgn} [c(b + \beta)] + 2 \sum_{i=1}^M \operatorname{sgn} \{ [-ay_i^2 + c + \beta y_i \sin y_i][\beta \sin y_i] \}.$$

Thus a one degree of freedom mechanical system with positive mass a , and nonzero spring constant c , damping constant b , retarded (unit time lag) damping constant β , and with $w(iy) \neq 0$, y real, is stable if, and only if,¹⁶

$$1 - \operatorname{sgn} [c(b + \beta)] + 2 \sum_{i=1}^M \operatorname{sgn} \{ [-ay_i^2 + c + \beta y_i \sin y_i][\beta \sin y_i] \} = 0. \quad (*)$$

Only minor modifications are required in order to take care of the case $a < 0$ or the degenerate cases where one or more of the coefficients is zero. The expression (*) may readily be calculated since it involves only four readings from a trigonometric table and subsequent evaluations of the sign of quadratic expressions.

For dynamical systems with a larger number of degrees of freedom or with more lag terms, we get higher powers of z or more exponential terms and the application is complicated but not hopeless. A machine¹⁷ of the isograph type should prove helpful where extended calculations on complicated systems are being considered.

Stability calculations are always easier than control calculations. The usual design procedure would be to discover a range of a , b , β , and c corresponding to stable responses and then to investigate the detailed response for a few choices of a , b , β , and c . Thus we would expect that for any particular choice of a , b , β , and c the control calculation will be more laborious. In the stability calculation of this paper we have been, in effect, investigating the relations between the parameters a , b , β , and c and the asymptotic character of the solutions of

$$az''(t) + bz'(t) + \beta z'(t - 1) + cz(t) = 0,$$

subject to the boundary conditions:

$$z(0) \neq 0, \quad \text{and} \quad z'(t) = 0, \quad -1 < t \leq 0.$$

For many design purposes information more specific than the asymptotic character of the oscillation might be needed. For instance, after having chosen a , b , β , and c so that the transient oscillations are stable (all of the roots of (1) have negative real parts), one may be interested in the detailed response $z(t)$ of the system to a given impressed force $f(t)$. This is the control problem. We therefore seek the solution of

$$az''(t) + bz'(t) + \beta z'(t - 1) + c = f(t),$$

subject to the boundary conditions:

$$z(0) \text{ given, } z'(t) = 0, \quad -1 < t \leq 0.$$

We consider first the response of the system in the first second. During that time it will act like a classical one degree of freedom system with constant mass, viscosity, and elastic coefficients a , b , and c , and variable impressed force $f(t)$. The response is given by the solution of

$$az''(t) + bz'(t) + cz(t) = f(t),$$

¹⁷ See T. C. Fry, *Some numerical methods for locating roots of polynomials*, Quart. Appl. Math. **3**, 89-105 (1945), especially pp. 100-103.

where $z_1(0)$, given and $z_1'(0) = 0$. The solution of this equation can be found in any standard text¹⁸ on differential equations, but, depending on the nature of $f(t)$, might best be done by numerical integration. In any event, we have $z(t) = z_1(t)$, $0 \leq t \leq 1$. Now during the second second we can again consider our system as a one degree of freedom system with the same mass, viscosity, and elastic coefficients, but with an impressed force which depends on $f(t)$ and velocity at time $t-1$. Consider the equation

$$az_2''(t) + bz_2'(t) + cz(t) = f(t+1) - \beta z_1'(t),$$

for $0 \leq t \leq 1$, where $z_2(0) = z_1(1)$, $z_2'(0) = z_1'(1)$, and $z_1'(t)$ are derived from the solution of the previous equation. Again by standard methods we solve for $z_2(t)$, $0 \leq t \leq 1$. This is related to the actual response $z(t)$ during the second second by

$$z(t) = z_2(t-1), \quad 1 \leq t \leq 2.$$

We can continue this process¹⁹ for larger values of t if so desired.

THE CENTER OF SHEAR AND THE CENTER OF TWIST*

By A. WEINSTEIN (*Carnegie Institute of Technology*)

In two recent papers W. R. Osgood¹ and J. N. Goodier² reconsider the much discussed question of the center of shear and center of twist, the former author pointing out the disagreement in the literature as to the location of the center of shear. However no mention is made of the important paper by P. Cicala³ which, together with a paper of Trefftz,⁴ will form the basis of the following remarks.

R. V. Southwell⁵ has clearly pointed out that the two centers, which are intuitively well known to engineers, constitute two different concepts and are not just synonyms for the same point. The center of twist is the point at rest in every section of a uniform beam subject to a twist by a terminal couple and rigidly clamped at the other end. The center of shear (called also flexural center) is the point at which an applied shearing force would produce a flexure without torsion. However, neither of these points can be explicitly computed, since the displacements of a rigidly clamped beam under torsion are not known and, on the other hand, the concept of flexure without torsion is still to be exactly defined. Nevertheless Southwell, using Maxwell's reciprocal relations in a summary way, makes plausible the coincidence of both centers.

As Goodier points out, Saint Venant's theory of torsion and flexure of beams does

¹⁸ E. L. Ince, *Ordinary differential equations*, Dover Publications, 1944, chap. 6.

¹⁹ Analogous stepwise integration can be found in R. M. Head, *Lag determination of altimeter systems*, J. Aeronaut. Sci. **12**, 85-93 (1945) and C. C. Kennedy, *Measuring the Coulomb and viscous components of friction*, Instruments **15**, 404-410 (1942).

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¹ W. R. Osgood, J. Appl. Mech. **10**, A-62-A-64 (1943).

² J. N. Goodier, J. Aeron. Sci. **11**, 272-280 (1944).

³ P. Cicala, Atti R. Acc. Sci. Torino **70**, 356-371 (1935).

⁴ E. Trefftz, Z. angew. Math. Mech. **15**, 220-225 (1935).

⁵ R. V. Southwell, *An introduction to the theory of elasticity*, 2nd ed., 1941, p. 29.