$$p_0(x) = 1/(6N)^{1/2}, \qquad p_1(x) = 2^{1/2}x/(6N)^{1/2}, \qquad p_2(x) = (3/2)2^{1/2}(2x^2 - 1)/(6N)^{1/2},$$
$$p_3(x) = 2(2/65)^{1/2}(18x^3 - 11x)/(6N)^{1/2},$$
$$p_4(x) = (1/4)(2/31)^{1/2}(288x^4 - 306x^2 + 65)/(6N)^{1/2}.$$

The roots of $p_4(x)$ are

$$x_1 = -.8769, \qquad x_2 = -.5418, \qquad x_3 = .5418, \qquad x_4 = .8769.$$

Thus, from (9c) we see that

$$\lambda_1 = \lambda_4 = (.2172)(6N), \qquad \lambda_2 = \lambda_3 = (.2828)(6N).$$

Thus, choosing the function $f(\nu)$ to be

$$f(\nu) = f(x\nu_L) = k(x\theta)^2/\text{sink}^2\,\theta x,$$

where $\theta = h\nu_L/2kT$ we obtain from (1), (11), and (18

$$C_v \simeq 3Nk\{(.4344)(.8769\theta)^2/\sinh^2(.8769\theta) + (.5656)(.5418\theta)^2/\sinh^2(.5418\theta)\}.$$

Qualitatively our method is equivalent to replacing the entire frequency spectrum by a small number, say n, of specially chosen sharp frequencies. These frequencies and their weight factors are chosen so that the values obtained for all averages over polynomials of degree $(2n-1)$, or less, are exact. In conclusion, it might be mentioned that in general analogous methods can be used in evaluating averages over characteristic values of linear operators.

THE SUBSONIC FLOW ABOUT A BODY OF REVOLUTION*

By E. V. LAITONE (*Cornell Aeronautical Laboratory*)

In cylindrical coordinates the Laplace differential equation, which defines the irrotational incompressible fluid flow, becomes

$$\phi_{xx} + \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = 0, \tag{1}$$

where the last term vanishes when the flow has axial symmetry about the x axis.

In this case a solution of Eq. (1) based on a source distribution $f(x)$ per unit length along the x axis from $x=0$ to $x=L$ is

$$\begin{aligned} \phi &= u_\infty x - \frac{1}{4\pi}\int_0^L \frac{f(\xi)d\xi}{[(x-\xi)^2 + r^2]^{1/2}}, \\ u = \phi_x &= u_\infty + \frac{1}{4\pi}\int_0^L \frac{f(\xi)(x-\xi)d\xi}{[(x-\xi)^2+r^2]^{3/2}}, \\ v = \phi_r &= \frac{r}{4\pi}\int_0^L \frac{f(\xi)d\xi}{[(x-\xi)^2+r^2]^{3/2}}, \end{aligned} \tag{2}$$

where $v/u = (dr/dx)_0$ satisfies the fixed boundary conditions given by the body shape.

* Received Dec. 9, 1946.

The linearized differential equation for the velocity potential of a compressible fluid flow with axial symmetry is given by

$$\beta^2\phi_{xx} + \phi_{rr} + \frac{1}{r}\phi_r = 0, \tag{3}$$

where for subsonic flow

$$\beta = \sqrt{1 - M^2} > 0.$$

The first method of Goldstein and Young[1] may be used to convert Eq. (2) into a linear perturbation solution of Eq. (3) in the following manner

$$\begin{aligned}
\Delta\phi &= -\frac{1}{4\pi\beta}\int_0^L \frac{f(\xi)d\xi}{[(x-\xi)^2 + (\beta r)^2]^{1/2}},\\
\Delta u &= \frac{1}{4\pi\beta}\int_0^L \frac{f(\xi)(x-\xi)d\xi}{[(x-\xi)^2 + (\beta r)^2]^{3/2}},\\
\Delta v &= \frac{\beta r}{4\pi}\int_0^L \frac{f(\xi)d\xi}{[(x-\xi)^2 + (\beta r)^2]^{3/2}},\\
\left(\frac{\Delta v}{u_\infty}\right)_0 &= \left(\frac{dr}{dx}\right)_0\left[1 + \left(\frac{\Delta u}{u_\infty}\right)_0\right].
\end{aligned} \tag{4}$$

Equation (4) can then provide a solution for a fixed given body shape for all Mach numbers less than unity ($0<\beta\leqq 1$) as shown in ref. 1.

If the substitution $\xi = x + \beta r z$ is introduced into Eq. (4) and the Taylor Expansion is written as

$$f(\xi) = f(x + \beta r z) = \sum_{n=0}^{\infty} \frac{(\beta r z)^n}{n!} f^{(n)}(x),$$

then Eq. (4) becomes

$$\begin{aligned}
\Delta u &= -\frac{1}{4\pi\beta}\sum_{n=0}^{\infty}\frac{(\beta r)^{n-1}f^{(n)}(x)}{n!}\int_{-x/\beta r}^{(L-x)/\beta r}\frac{z^{n+1}dz}{(1+z^2)^{3/2}},\\
\Delta v &= \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(\beta r)^{n-1}f^{(n)}(x)}{n!}\int_{-x/\beta r}^{(L-x)/\beta r}\frac{z^{n}dz}{(1+z^2)^{3/2}},
\end{aligned} \tag{5}$$

or, to the first order terms in βr,

$$\begin{aligned}
\Delta u = -\frac{1}{4\pi\beta}\Bigg\{&\frac{f(x)}{x}\cdot\left(\frac{1-2x/L}{1-x/L}\right) + f'(x)\left[\log(4x/L)(1-x/L) - 2\log\beta r/L - 2\right]\\
&+ f''(x)\cdot(L/2)(1-2x/L) + f'''(x)\cdot(L^2/12)\left[(1-x/L)^2 + (x/L)^2\right] + \cdots\\
&+ f^{(n)}(x)\cdot\left[\frac{(\beta r)^{n-1}}{n!}\int_{-x/\beta r}^{(L-x)/\beta r}\frac{z^{n+1}dz}{(1+z^2)^{3/2}}\right] + O(\beta r)\Bigg\},
\end{aligned} \tag{6}$$

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[1] S. Goldstein and A. D. Young, *The linear perturbation theory of compressible flow, with applications to wind tunnel interference*, Brit., A.R.C., R. & M. **1909** (1943).

$$\Delta v = \frac{1}{4\pi}\left\{2f(x)/\beta r - f''(x)\left[\beta r \log(\beta r/L)\right] + O(\beta r)\right\}, \tag{7}$$

for $0<x<L$.

On the body surface to a first order approximation

$$(\Delta v/u_\infty)_0 = (dr/dx)_0 = f(x)/2\pi\beta r_0 u_\infty, \tag{8}$$

or

$$f(x) = 2\pi\beta u_\infty r_0 (dr/dx)_0 = \beta u_\infty S', \tag{9}$$

where

$$S = \pi r_0^2.$$

Then

$$2\left(\frac{\Delta u}{u_\infty}\right) = -\frac{1}{\pi}\left\{\frac{S'}{2x}\cdot\left(\frac{1-2x/L}{1-x/L}\right) - S''\cdot\left(1+\log\frac{\beta r}{L} - \log 2\sqrt{\frac{x}{L}\left(1-\frac{x}{L}\right)}\right)\right.$$
$$\left. + \frac{S^{(3)}}{4}L\left(1-\frac{2x}{L}\right) + \frac{S^{(4)}}{24}L^2\left(1-\frac{2x}{L}+\frac{2x^2}{L^2}\right) + \cdots\right\}, \tag{10}$$

for any x other than 0 or L.

Eqs. (8) and (10) provide the pressure distribution on (or along the streamlines $r\to 0$) a slender symmetrical body of revolution in either incompressible or subsonic potential flow. They also provide a first order Mach number correction for the subsonic potential flow on any body of revolution. For example, the surface pressure coefficient (C_p) at $x=L/2$ for a symmetrical body of revolution at any Mach number (M) less than unity would be given by

$$\left(\frac{C_p}{C p_{M=0}}\right)_{\max} = 1 + \frac{\log\sqrt{1-M^2}}{1-(L^2S^{(4)}/48S'' + \cdots) + \log(r/L)}. \tag{11}$$

Eq. (11) agrees with the expression obtained by Lees[2] for a slender prolate spheroid ($S^{(n)}=0$ for $n>2$).

The flow about any body of revolution is given to the first order by Eqs. (4) and (9) as

$$2\left(\frac{\Delta u}{u_\infty}\right) = \frac{1}{2\pi}\int_0^L \frac{S'(\xi)(x-\xi)d\xi}{[(x-\xi)^2+(\beta r)^2]^{3/2}},$$
$$\left(\frac{\Delta v}{u_\infty}\right) = \frac{\beta^2 r}{4\pi}\int_0^L \frac{S'(\xi)d\xi}{[(x-\xi)^2+(\beta r)^2]^{3/2}}. \tag{12}$$

Eq. (12) shows that at large transverse distances from a body of revolution

$$\frac{\Delta u}{u_\infty} = \frac{1}{4\pi\beta^3 r^3}\int_0^L S'(\xi)\cdot(x-\xi)d\xi,$$

[2] Lester Lees, *A discussion of the application of the Prandtl-Glauert method to subsonic compressible flow over a slender body of revolution*, NACA, TN **1127** (1946).

so that the subsonic flow wind tunnel wall correction for a body of revolution would vary as $1/(1-M^2)^{3/2}$.

Eq. (7) may be written

$$f(x) = 2\pi\beta r r'(u_\infty + \Delta u) + \frac{(\beta r)^2}{2} f''(x) \log(\beta r/L) + O(\beta^2 r^2 f).$$

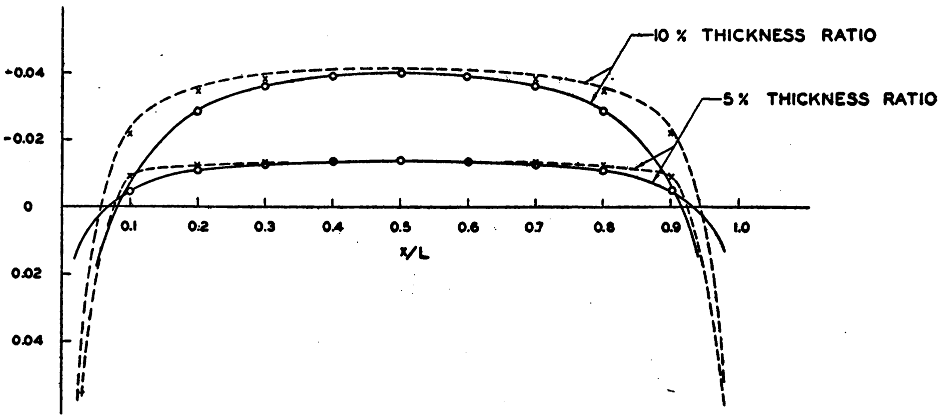

FIG. 1. Prolate spheroid.
—— Equation 12
○ Equation 10
– – – Exact Potential Flow Solution
× Equation 14, [Equation 10—$(dr/dx)^2$]

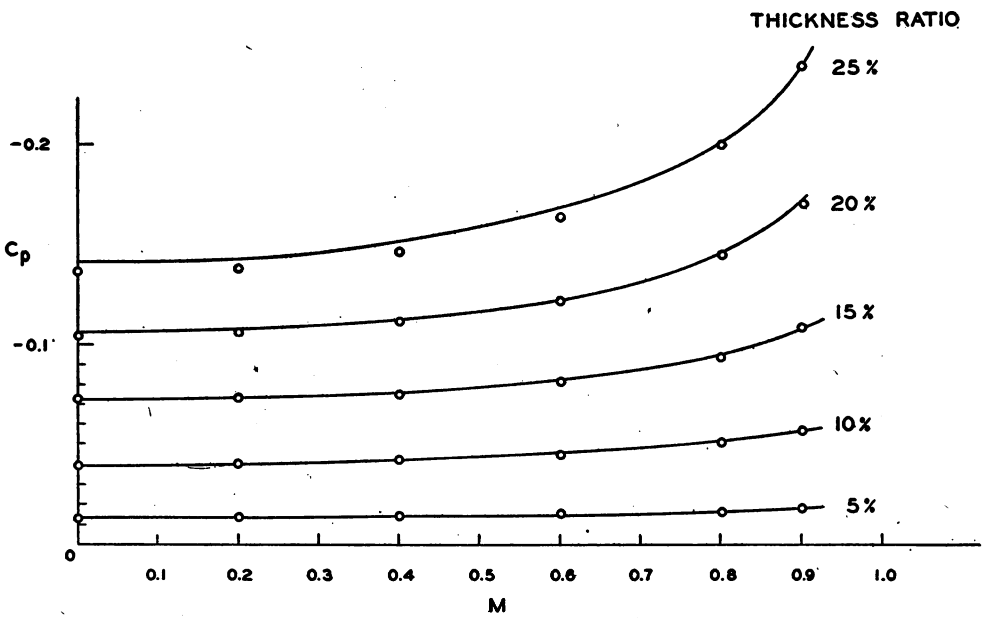

FIG. 2. Prolate spheroid, $x = L/2$.
—— Equation 12
○ Equation 10

Therefore, for $0<x<L$,

$$f(x) = 2\pi u_\infty \beta r r'[1 + (\log \beta r/L)(3\beta^2 r r''/2 + \beta^2 r^2 r^{(3)}/2r' + r r'' + r'^2) + O(r^2)]$$
$$= \beta u_\infty S'(x) + O(r^4 \log r), \quad \text{for} \quad \beta > 0. \tag{13}$$

Eq. (13) shows that at least for incompressible flow ($\beta = 1$) Eq. (12) will provide the surface velocities accurate to the order $r^4 \log r = 0$. A corresponding statement cannot be made for subsonic flow however since the compressibility effects of the flow have been considered only to the first order in Eq. (3).

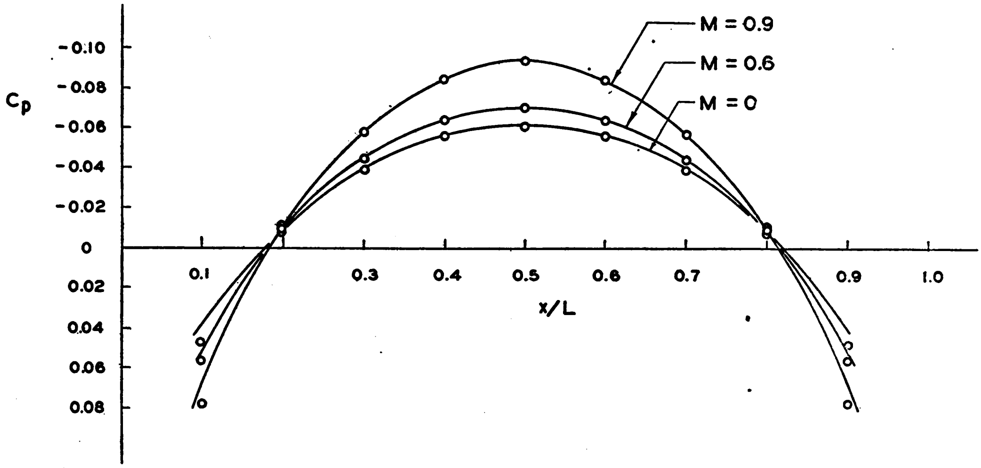

FIG. 3. 10% parabolic arc body of revolution
—— Equation 12—$(dr/dx)^2$
○ Equation 10—$(dr/dx)^2$

The pressure coefficient for compressible flow is given to the second order by

$$C_p = -2(\Delta u/u_\infty) - (\Delta v/u_\infty)^2 - (\Delta u/u_\infty)^2(1 - M^2) + O(\phi_x^3, \phi_r^4),$$

which in view of Eqs. (8) and (10) becomes

$$C_p = -2(\Delta u/u_\infty) - (r')^2 + O(r^4 \log r). \tag{14}$$

Figure 1 shows a comparison of the exact incompressible potential flow pressure distribution on a prolate spheroid (ellipsoid of revolution) with that obtained from Eqs. (10), (12) and (14). Figure 2 shows the effect of Mach number on the maximum C_p of various thickness ratio prolate spheroids as computed from Eqs. (10) and (12). Figure 3 shows the effect of Mach number on the pressure distribution of a parabolic arc body of revolution as obtained from Eq. (14) in conjunction with Eqs. (10) and (12).