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THE INFLUENCE OF THE WIDTH OF THE GAP UPON THE THEORY OF ANTENNAS*

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1. Introduction. The modern theories of antennas differ from the older sinusoidal theory in one essential point: they take into consideration the existence of the gap as the region from which the energy radiates, whereas the sinusoidal theory[1] spreads the sources over the whole length of the antenna.

The gap, in all these modern theories, is characterized by a *given* difference of potential $(-V)$ and by this difference only. The dimensions of the gap and the electric field in the gap do not enter the picture. If we denote by d the dimensions of the gap, and by E_z the tangential component of the electric field along the short line representing the length of the gap, then the *physical* quantity which enters the theory is:

$$-V = \int_{-d/2}^{d/2} E_z dz, \tag{1.1}$$

and the knowledge of E_z and d is not assumed. Neither E_z nor d appears in the theories of antennas, because the gap is represented by what I shall call a *δ-gap*. Its model is obtained by the following limiting process: let us assume that the dimensions of the gap (d) tend to zero. Let us assume further that at the same time E_z increases, but in such a way that the integral (1.1) remains constant. In other words, E_z becomes proportional to a Dirac function:

$$E_z = -V\delta(z), \tag{1.2}$$

where $\delta(z)$ is defined by

$$\delta(z) = 0 \quad \text{for} \quad z \neq 0 \tag{1.3}$$

$$\int_{-\infty}^{\infty} \delta(z) dz = 1. \tag{1.4}$$

This δ-gap model is the basis of nearly all recent investigations on the theory of antennas. In the final solution neither d or E_z appears. They were wiped out by the limiting process $d \to 0$.

Most of the investigations based on this δ-gap model employ an approximation procedure which is so complicated that it becomes practically impossible to carry it

* Received Aug. 9, 1946. This paper was written in 1942 and issued as a classified report by the National Research Council of Canada.

[1] J. Labus, Hochfrequenz Technik und Elektroakustik **41**, 17–23 (1933).

beyond the first few steps. Even the first approximation gives a good agreement with experiment. But the question of convergence is hardly investigated. There is a danger that pushing the approximation further would spoil rather than improve such an agreement.

From the theoretical point of view this is an essential point to which little attention has been paid.

All the existing theories of antennas can, so far as the author knows, be classified as follows:

1. The sinusoidal theory which assumes a sinusoidal distribution of the current and ignores the gap and the boundary conditions on the surface of the antenna.[1]

2. The theories based upon the model of a δ-gap, leading to an integral equation and its approximate[2] solution. In connection with these theories, that of King[3] may be mentioned, which though taking into account the boundary conditions on the surface, ignores the role and the structure of the gap.

3. The theory of Stratton and Chu[4] based upon the model of a δ-gap and upon a rigorous solution of Maxwell's equations. This theory is, however, restricted to an infinite cylinder, sphere and spheroid. The last case is the only one of any practical value for the construction of antennas. The advantage of the method of Stratton and Chu is that it avoids approximate solutions and therefore most easily allows investigation of the convergence of expressions which appear in it.

4. The theory of Schelkunoff[5] which represents the antenna as consisting of two thin cones, and the gap as a singularity of the field at the point where the two tips meet. This is the only theory which takes into account the gap and does not represent it by a δ-model. The drawback of this theory is its great complexity and also the fact that it ignores the boundary conditions on the spherical ends of the cones.

The present paper, which tries to clear up the problem of convergence and that of the structure of the gap, is based exclusively on Stratton and Chu's papers, especially those referring to the sphere and spheroid. I shall quote them as SC II and SC III respectively. The solutions given in SC II and III are not based upon approximation methods and they satisfy the boundary conditions rigorously. The restriction of the shape to a sphere or spheroid matters little in an investigation such as this which does not pretend to be of immediate practical value. It is obvious that the general results obtained here for the spheroid will hold also for (say) a cylindrical antenna, a problem much more difficult mathematically because of the boundary conditions at its ends.

Any theory of an antenna leads to the calculation of

$$I(0)/V = \text{driving-point admittance,} \tag{1.5}$$

where $I(0)$ is the current at the center of the gap, if the dimensions of the gap tend to zero. In calculating (1.5) for a sphere or spheroid, SC are led to a series

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[2] E. Hallen, Nova Acta Reg. Soc. Sci. Upsaliensis **11,** No. 4 (1939).

[3] L. V. King, Phil. Trans. Roy. Soc. (A) **236,** 381–422 (1937).

[4] Stratton and Chu, J. of Appl. Phys. **12,** 230 (1941); 236–240 quoted here as SC II; 241–248 quoted here as SC III.

[5] S. A. Schelkunoff, Proc. I.R.E. **29,** 493–521 (1941).

$$I(0)/V = Y(0) = \sum_{n=1}^{\infty} Y_n, \tag{1.6}$$

by which the driving-point admittance is represented. The grave difficulty encountered here is that the series (1.6), or rather its imaginary part, *diverges*. The authors dismiss this difficulty by a remark that this divergence is due to the δ-model of the gap. They write[6] "this difficulty is no fault of the mathematical formulation of the problem but results from our assumption that the voltage is applied across a segment of vanishing length, implying in turn an infinite field intensity and infinite current density at the point of application." The authors claim further that a finite gap will not change the first few modes in (1.6) but will make (1.6) quickly convergent. Thus, according to the authors, we ought to take the first few modes for a finite gap, and neglect others. This sounds very vague. On the one hand if we do this, we take into account the fact that the gap is finite; otherwise (1.6) is meaningless. On the other hand, however, this finite gap will not play any role in the result. Instead of introducing its dimensions we cross out all the modes for which n is greater than a certain M. But how many expressions must we keep and how many must we cross out? How will this procedure depend on the dimensions of the gap? These are the questions which we shall try to clarify later.

The same difficulty appearing in the theories of SC appears also in the theories listed before under 2. The difference is, so far as I know, that the difficulty which came out so clearly in SC, was not even discussed in the theories based on the solutions of the integral equations. But if the driving-point admittance (or rather its imaginary part) becomes infinite for a sphere and a spheroid, then it will also become infinite for a cylinder; this can be deduced from the integral equations of one of the theories[2] and was shown by Prof. Stevenson.[7] But infinite $Y(0)$ means zero impedance, that is zero resistance and zero reactance. Therefore the matching problem cannot be solved for a δ-gap.

The theory gives results agreeing with experiment only because the approximate procedure, owing to the technical difficulties, cannot be pushed too far. It will be obvious from the content of this paper that any procedure which does not introduce the finite dimensions of the gap must lead to infinites; therefore any satisfactory theory must introduce these dimensions from the beginning, or, at least say in what stage of approximation and how they should be introduced.[8]

It is the purpose of this paper to analyze the influence of a finite gap upon the theory of antennas. The case in which this analysis is comparatively simple is that of a sphere. The essential features of the difficulties just discussed come out especially clearly in this case. For this reason most of the paper is devoted to the spherical antenna and only at its end is the spheroidal antenna discussed and then less fully.

This paper is in great part a result of discussions in a research group concerned with the problems of antennas and wave guides. In the group were Professors V. G. Smith, A. F. Stevenson, J. L. Synge and myself. The discussions which we have had

[6] SC III, p. 247.

[7] Privately communicated at our research group.

[8] A general theory (due to J. L. Synge), which takes into account both the dimensions of the gap and the electric field in the gap, will appear shortly.

have been extremely helpful in clearing up many difficulties and a great part of what is written down here is due to my colleagues.

To concentrate only on the logical part of the argument and not to interrupt it by calculations, most of them have been shifted to the Appendices.

2. The admittance of a sphere. A metallic sphere has a radius a. A field E_θ' is applied at the surface of the sphere; we assume that E_θ' is a function of the colatitude θ and of the time t through the factor $e^{i\omega t}$. We write

$$E_\theta' = -\frac{V}{a} f(\theta), \tag{2.1}$$

where

$$\int_0^\pi f(\theta) d\theta = 1, \tag{2.2}$$

and where

$$V = -a \int_0^\pi E_\theta' \, d\theta \tag{2.3}$$

is the applied voltage. Our assumptions here are more general than in SC II in which a special $f(\theta)$ was assumed. In SC II we have:

$$f(\theta) = \delta\left(\frac{\pi}{2} - \theta\right), \tag{2.4}$$

the voltage being applied across a narrow strip along the equator. The admittance for any θ (between 0 and π) is:

$$Y(\theta) = \frac{I(\theta)}{V} = 2\pi \sin\theta \sum_{n=1}^{\infty} \frac{\int_0^\pi f(\theta) P_n'(\cos\theta) \sin\theta d\theta}{Z_n \left[\int_0^\pi (P_n')^2 \sin\theta d\theta\right]} P_n'(\cos\theta). \tag{2.5}$$

This formula, which is basic for our further considerations is not in SC II. It is a generalization of SC II (19) in which $f(\theta) = \delta(\theta)$ and $\theta = \pi/2$ is assumed. However there is no difficulty in deriving (2.5) for anyone who has studied SC II. A few remarks about it are added in Appendix A.

About the notation in (2.5): P_n' are the associated spherical harmonics:

$$P_n' = -d\,P_n/d\theta. \tag{2.6}$$

The Z_n are certain complex factors depending on

$$ak = 2\pi a/\lambda \qquad (\lambda = \text{wave length}, a = \text{radius})$$

and the index n. We shall say more about them later, but for the moment it is sufficient to know that for very great n we have* in m.k.s. units

$$Z_n = -120\pi i n/ak. \tag{2.7}$$

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* Comp. (SC II), equation (11), putting $2\pi\omega\epsilon_2 = 2\pi(\epsilon_2/\mu_2)^{1/2}\omega(\epsilon_2\mu_2)^{1/2} = k/60$ and $k = \omega(\epsilon_2\mu_2)^{1/2}$.

We are interested firstly in the simple case of a gap, resembling the δ-gap, but finite. We assume that $f(\theta)$ is a regular function, such that

$$f\left(\frac{\pi}{2} - \theta\right) = f\left(\frac{\pi}{2} + \theta\right), \tag{2.8}$$

i.e. symmetric with respect to the equatorial plane. In this case only the odd n's give contributions to (2.5). Puttihg $n = 2m+1$, we can write instead of (2.5):

$$Y(\theta) = 2\pi \sin\theta \sum_{m=0}^{\infty} \frac{\int_0^\pi f(\theta) \sin\theta P'_{2m+1} d\theta}{Z_{2m+1}\left[\int_0^\pi (P'_{2m+1})^2 \sin\theta d\theta\right]} \cdot P'_{2m+1}(\cos\theta). \tag{2.9}$$

Furthermore we assume that $f(\theta)$ has its maximum for $\theta = \pi/2$, is always positive and goes to zero for $\theta = 0$ and π. For example, we could choose for $f(\theta)$ the following function:

$$f(\theta) = 2^{-2s-1} \frac{(2s+1)!}{(s!)^2} \sin^{2s+1}\theta, \tag{2.10}$$

where s is an integer. Indeed for great s such a function represents a steep field, concentrated symmetrically around the equatorial plane and is normalized according to (2.2). Though it is only a special case of $f(\theta)$, it will provide us with a convenient example and will allow us to see what happens if $s \to \infty$, that is, if our finite gap becomes a δ-gap. It is, at least, doubtful whether we are allowed to introduce immediately into (2.5) the δ-function and to make use of its properties. With these assumptions we shall calculate the input admittance for an arbitrary θ (in the interval $0, \pi$). This generalization will allow us to study the character of the divergence if $s \to \infty$, $\theta \to \pi/2$.

The calculation of $\int_0^\pi f(\theta) \sin\theta \cdot P'_{2m+1} d\theta$ is done in Appendix B. The result is:

$$\int_0^\pi f(\theta) \sin\theta \cdot P'_{2m+1} \cdot d\theta = 4(-1)^m (s+1) \frac{(2s+1)!(s+m+1)!(2m+1)!}{(2s+2m+3)!(s-m)!(m!)^2} \tag{2.11}$$

if $m \leqq s$, and

$$\int_0^\pi f(\theta) \sin\theta \cdot P'_{2m+1} \cdot d\theta = 0, \tag{2.12}$$

if $m > s$.

Furthermore, because

$$\left[\int_0^\pi P'_{2m+1} \sin\theta d\theta\right]^{-1} = \frac{4m+3}{4(m+1)(2m+1)}, \tag{2.13}$$

we have for the admittance (2.9) as function of θ:

$$Y = 2\pi \sin\theta \sum_{m=0}^{s} (-1)^m (s+1) \frac{(2s+1)!(s+m+1)!2m!(4m+3)}{Z_{2m+1}(2s+2m+3)!(s-m)!m!(m+1)!} P'_{2m+1}(\cos\theta) \tag{2.14}$$

For a finite s, the sum is finite, and therefore the problem of convergence does not appear. Our example, as expressed by the choice of (2.10), is only one of the possible examples. We could have taken a properly normalized step function instead, as we shall do in Appendix E. Our choice (2.10), however, has the advantage that the series by which Y is represented is finite.

Let us write

$$Y = Y(\theta, s), \tag{2.15}$$

indicating the explicit dependence of Y on θ and s. First we wish to calculate:

$$Y(\theta) = \lim_{s\to\infty} Y(\theta, s). \tag{2.16}$$

To do this, we split $Y(\theta, s)$ into two parts:

$$Y(\theta, s) = Y_{0,M}(\theta, s) + Y_{M,s}(\theta, s), \tag{2.17}$$

where:

$$Y_{0,M}(\theta, s) = 2\pi \sin\theta \sum_{m=0}^{M} (-1)^m (s+1) \frac{(2s+1)!(s+m+1)!2m!(4m+3)P'_{2m+1}(\cos\theta)}{Z_{2m+1}(2s+2m+3)!(s-m)!m!(m+1)!} \tag{2.18}$$

$$Y_{M,s}(\theta, s) = 2\pi \sin\theta \sum_{m=M+1}^{s} \text{ as above.} \tag{2.19}$$

Here M is assumed to be an arbitrary but great number, later kept *constant*, while $s\to\infty$ The $Y_{M,s}(\theta, s)$ is calculated approximately by the use of Stirling's formula for $m!$, and by replacing summation by integration. The calculations are performed in Appendix C and the result is

$$Y_{M,s}(\theta, s) = \frac{iak\sqrt{\sin\theta}}{60\pi} \sum_{m=M+1}^{s} (-1)^{m+1} m^{-1} \cos\left[\frac{\pi}{4} + \left(2m + \frac{3}{2}\right)\theta\right] \cdot \exp\left[-s\int_0^{m/s} \log\frac{1+\zeta}{1-\zeta}\, d\zeta\right]. \tag{2.20}$$

Now if $s\to\infty$, we have (App. C),

$$Y_{M,\infty}(\theta) = \frac{iak\sqrt{\sin\theta}}{60\pi} \sum_{m=M+1}^{\infty} (-1)^{m+1} m^{-1} \cos\left[\frac{\pi}{4} + \left(2m + \frac{3}{2}\right)\theta\right]. \tag{2.21}$$

Let us now investigate this expression near the equatorial plane by putting

$$\theta = \frac{\pi}{2} - \frac{d}{2a} \quad \text{or} \quad \theta = \frac{\pi}{2} + \frac{d}{2a}, \tag{2.22}$$

where d is small compared to a. Then we have:

$$Y_{M,\infty}\left(\frac{d}{2a}\right) = \frac{iak}{60\pi}\int_M^{\infty} m^{-1} \cos\left[\left(2m + \frac{3}{2}\right)\frac{d}{2a}\right] dm, \tag{2.23}$$

and, for small d, we finally obtain the leading term:

$$Y_{M,\infty}\left(\frac{d}{2a}\right) = -\frac{iak}{60\pi} Ci\left(\frac{Md}{a}\right). \tag{2.24}$$

We see the character of the divergence of our series if $s \to \infty$, $d \to 0$: $Y_{M,\infty}(\to 0)$ increases to infinity like $\log d/a$:

$$|Y_{s\to\infty}| \sim |\log (d/a)|. \tag{2.25}$$

Therefore we cannot define the driving-point admittance by a δ-gap with $\theta = \pi/2$. Such a driving-point admittance does not exist. There are three possible ways of saving this situation, which we can represent schematically in the following way:

1. $f(\theta) \neq \delta$; $d \neq 0$.
2. $f(\theta) \neq \delta$; $d = 0$.
3. $f(\theta) = \delta$; $d \neq 0$.

In the first case we take a continuous field distribution and Y at the distance $d/2$ from the equatorial plane. The difficulty in accepting this view is that we must know $f(\theta)$ and connect d with $f(\theta)$ in some way. The most obvious way would be to assume that $f(\theta)$ is a step function, and to identify d with the width of this step function on the one hand, and with the dimensions of the gap on the other. This special choice of $f(\theta)$ is discussed in Appendix E. Generally, however, we must introduce three arbitrary factors: (a) the function $f(\theta)$, (b) the dimension of the gap, (c) some connection between d and $f(\theta)$. As long as there is no experimental evidence to guide us in determining all these factors, this view seems to be too arbitrary for a satisfactory theory.

In the second case the driving-point admittance is given by (2.9) for $\theta = \pi/2$. Here only the function $f(\theta)$ is arbitrary and on its choice Y will depend.

In the third case we go back to the δ-function but calculate Y, not in the equatorial plane but at the distance d from it. In this case the definition of the driving-point admittance will be simplest if we can identify d with the physical dimensions of the gap. We shall accept here this point of view, assuming that d—the only new parameter entering the theory—represents the *known* dimension of the gap.

It is possible that one of the other views will turn out to be more convenient and the present discussion tends to show how the frame of the theory of the antenna must be broadened to include the size of the gap.

At present however, we shall accept the simplest view: *δ electric field and finite gap*, corresponding to our *third* case.

For this case $Y_{M,\infty}(d/2a)$ is expressed in (2.24) and the remaining task is to calculate

$$Y_{\substack{0,M\\ s\to\infty}}\left(s, \frac{d}{2a}\right) = Y_{0,M}\left(\frac{d}{2a}\right). \tag{2.26}$$

This is done in Appendix C. According to (C.2), (C.3) and (C.9), we have:

$$\lim_{s\to\infty} \frac{(s+1)(s+m+1)!(2s+1)!}{(s-m)!(2s+2m+3)!} = \frac{1}{2^{2m+2}}. \tag{2.27}$$

Therefore

$$Y\left(\frac{d}{2a}\right) = 2\pi \cos\left(\frac{d}{2a}\right) \sum_{m=0}^{M} (-1)^m \frac{2m!(4m+3)}{2^{2m+2}(m+1)!Z_{2m+1}} P'_{2m+1}\left(\sin\frac{d}{2a}\right)$$
$$- \frac{iak}{60\pi} Ci\left(\frac{Md}{a}\right), \tag{2.28}$$

which we interpret as the final result representing the input admittance at the end of a small but finite gap in the center of a conducting sphere, when the field in the gap is represented by a δ-function.

For $d \rightarrow 0$ we have, because

$$P'_{2m+1}(0) = (-1)^m \frac{(2m+1)!}{(m!)^2 2^{2m}}, \tag{2.29}$$

the following formula for $Y_{0,M}(d \rightarrow 0)$

$$Y_{0,M}(0) = \frac{\pi}{2} \sum_{m=0}^{M} \frac{[P'_{2m+1}(0)]^2(4m+3)}{Z_{2m+1}(m+1)(2m+3)}. \tag{2.30}$$

This is exactly the formula for the driving-point admittance obtained by SC II, (19), if we there replace n by $2m+1$. Though our discussion shows that $Y_{M,\infty}(0)$ and therefore $Y(0)$ becomes infinite at the driving-point, one could argue, as SC does:

In $Y_{0,M}(d/2a)$ the gap d will play a small role and therefore we may assume $d=0$. On the other hand, if d is finite the integral representing $Y_{M,\infty}(d/2a)$ can be neglected and thus we arrive at a formula (2.30) which is convergent and does not contain the dimensions of the gap. But the argument could just as well be reversed. We could say: For small d the end of the series represented by the Ci function becomes essential, whereas a large d will modify even the first few expressions of our series. In a satisfactory theory we must be able to answer the following question: how can we choose M for a given ak, d, $\epsilon > 0$ so that

$$|Y_{M,\infty}| < \epsilon. \tag{2.31}$$

Our discussions provide an answer to this question which was not supplied by SC. Let us take as an example:

$$ka = \sqrt{2}; \qquad d/a = 0.1; \qquad M = 15.$$

Then the absolute value of the first term in the sum in (2.28) is approximately

$$\frac{3\pi}{2} |Z_1^{-1}|.$$

But, as will be seen from equations (3.5) and (3.16),

$$|Z_1^{-1}| = \frac{\sqrt{2}}{2\pi \times 60};$$

therefore the first expression is $3\sqrt{2}/4 \times 60$. On the other hand the absolute value of the integral expression is $(\sqrt{2}/60\pi)Ci(1.5) \sim (\sqrt{2}/2\pi \times 60)$, which is of the same order as the first expression in the series (2.28).

The result as expressed by (2.28) could have been found much more quickly by the use of Dirac's δ-function. Indeed, putting $f(\theta) = \delta[(\pi/2) - \theta]$ in (2.9), we find

$$Y\left(\frac{d}{2a}\right) = 2\pi \cos\left(\frac{d}{2a}\right) \sum_{m=1}^{\infty} \frac{P'_{2m+1}(0) P'_{2m+1}(\sin d/2a)}{Z_{2m+1}[\int_0^\pi (P'_{2m+1})^2 \sin\theta d\theta]}. \tag{2.32}$$

By splitting this into two parts and using the asymptotic formula for P'_{2m+1}, we again obtain (2.28). But in obtaining (2.28) by the transition $s \rightarrow \infty$, we avoided the use of Dirac's δ-function and at the same time gave an example of the input admittance with a given field distribution in the gap. Though the choice of M is arbitrary, as long as we can apply Stirling's formula, our equation (2.28) will be the more exact the greater is the chosen M.

3. The coefficients Z_{2m+1}. The coefficients Z_n are defined in SC II, (8), (6). Assuming σ (conductivity) $\rightarrow \infty$, we have, in m.k.s. units,

$$Z_n = i \cdot 2\pi \cdot 60 \left[\frac{H^{(2)}_{n-1/2}(ka)}{H^{(2)}_{n+1/2}(ka)} - \frac{n}{ka} \right]. \tag{3.1}$$

We shall now show how these expressions can be calculated without the knowledge of Hankel's functions. The reason for doing it is that we intend to apply a similar method later when dealing with the problem of a spheroidal antenna.

Because of the recurrence formula:

$$H^{(2)}_{2m+1/2} - \frac{2m+1}{ka} H^{(2)}_{2m+3/2} = H^{(2)\prime}_{2m+3/2} + \frac{1}{2ak} H^{(2)}_{2m+3/2}, \tag{3.2}$$

we can write (3.1)

$$Z_{2m+1} = i \cdot 2\pi \cdot 60 \left[\frac{(x^{1/2} H^{(2)}_{2m+3/2})'}{x^{1/2} H^{(2)}_{m+3/2}} \right], \tag{3.3}$$

where we put

$$x = ka. \tag{3.4}$$

But the $x^{1/2} H^{(2)}_{2m+3/2}$ are "densities" corresponding to Hankel's functions of the second kind. We write

$$x^{1/2} H^{(2)}_{2m+3/2} = h_{2m+3/2} = h, \tag{3.5}$$

where $h_{2m+3/2}$ satisfies the equation:

$$h'' + \frac{4m^2 + 6m + 2}{x^2} h + h = 0. \tag{3.6}$$

It is more convenient to calculate Z_{2m+1} directly, and not by means of the h's. Indeed we can easily find the differential equation which Z must satisfy. We write:

$$\zeta_{2m+1} = \frac{h_{2m+3/2}}{h'_{2m+3/2}} \tag{3.7}$$

$$Z_{2m+1} = i \frac{2\pi \times 60}{\zeta_{2m+1}} . \tag{3.8}$$

Then (omitting the indices)

$$\zeta' = 1 - \left(\frac{h}{h'}\right)^2 \frac{h''}{h}, \tag{3.9}$$

which, because of (3.6) and (3.7), becomes

$$\zeta' + \zeta^2\left(\frac{\nu^2}{x^2} - 1\right) = 1, \tag{3.10}$$

where

$$\nu^2 = 4m^2 + 6m + 1. \tag{3.11}$$

Riccati's equation (3.10) can be solved by a recurrence formula. Its derivation is given in Appendix D. This formula is:

$$\zeta_{2m+1} = \frac{\zeta_{2m-1} \times [x^2/(4m+1) - 2m] + x^2}{\zeta_{2m-1}[2m(2m+1) - x^2] + x\{[x^2/(4m+1)] - (2m+1)\}}, \tag{3.12}$$

and it is a purely algebraical formula which does contain derivatives. We start by finding ζ_{-1}. For $m = -1$, equation (3.10) takes the form:

$$\zeta'_{-1} - \zeta^2_{-1} = 1. \tag{3.13}$$

As a solution of (3.13) we choose

$$\zeta_{-1} = i. \tag{3.14}$$

In justifying (3.14) we must go back to (3.5) and (3.7). If

$$\frac{h'}{h} \sim -i \quad \text{for} \quad x \to \infty,$$

then

$$x^{1/2}H^{(2)} \sim h \sim e^{-ix} \quad \text{for} \quad x \to \infty.$$

We can now calculate the first ζ in our series corresponding to $m=0$. We have, from (3.12) and (3.14),

$$\zeta_1 = -\frac{x}{1 - x^2 + x^4} + \frac{ix^4}{1 - x^2 + x^4}, \tag{3.15}$$

or, because of (3.8):

$$\frac{1}{Z_1} = \frac{1}{2\pi \times 60}\left[\frac{x^4}{1 - x^2 + x^4} + \frac{ix}{1 - x^2 + x^4}\right]. \tag{3.16}$$

We see: for $x \to 0$, the real part of ζ_1 goes to zero as x and the imaginary part as x^4. For $x \to \infty$, the real part of ζ_1 goes to zero as $1/x^3$ and the imaginary part goes to i. From the recurrence formula it follows that for *all m* and small $x(x \ll 2m)$ the real part of ζ goes to zero as x, and is negative, whereas the imaginary part goes to zero at least as x^4.

For very great m and very small x, $(x \ll 2m)$ we find the leading term by writing (3.10):

$$\zeta' + \zeta^2 \frac{4m}{x^2} = 1. \tag{3.17}$$

The leading term is:

$$\zeta_{2m+1} = - x/2m, \tag{3.18}$$

in agreement with (2.7), if we take into account (3.8).

4. The spheroidal antenna. We shall now generalize our arguments for the case of a spheroid.

We characterize the points in space by the spheroidal coordinates ξ, η, Φ with:

$$\xi \geqq 1; \qquad -1 \leqq \eta \leqq 1; \qquad 0 \leqq \Phi \leqq 2\pi, \tag{4.1}$$

where ξ, η, Φ are defined in terms of cylindrical coordinates by

$$r^2 = f^2(\xi^2 - 1)(1 - \eta^2); \qquad z = f\xi\eta; \qquad \Phi = \Phi, \tag{4.2}$$

and in terms of polar coordinates by

$$R^2 = f^2(\xi^2 + \eta^2 - 1); \qquad \cos\theta = \frac{\xi\eta}{(\xi^2 + \eta^2 - 1)^{1/2}}, \qquad \Phi = \Phi. \tag{4.3}$$

Here $2f$ is the distance between the foci.* A spheroid is characterized by a *constant* coordinate $\xi = \xi_0$. Knowing ξ_0 and f we can find the half-axes a, b:

$$a = f\xi_0; \qquad b = f\sqrt{\xi_0^2 - 1}. \tag{4.4}$$

In CS III, the input admittance of such a spheroidal antenna is calculated. Again, as in the case of a spheroidal antenna, the δ-model of the gap is assumed and the imaginary part of the input admittance is infinite.

We start by generalizing CS III in two respects: *first*, we do not take the δ-model of a gap but an arbitrary field; *secondly* we write down the admittance not for $\eta = 0$, but for any η ($|\eta| \leqq 1$). The argument is very similar to that in the case of a sphere and hardly worth repeating. The result of these generalizations leads us to the following formula corresponding to (2.5) in the case of a sphere:

$$Y = 2\pi(1 - \eta^2) \sum_{l=0}^{\infty} \frac{\int_{-1}^{+1}(1 - \eta^2) f(\eta) Se_l^1(\eta) d\eta}{Z_l \int_{-1}^{+1}(1 - \eta^2)[Se_l^1(\eta)]^2 d\eta} Se_l^1(\eta). \tag{4.5}$$

It is written in such a way as to expose an analogy with (2.5) and, at the same time, to use the notation in SC III. To establish a connection with SC III one must substitute in (4.5)

$$\frac{2\pi}{Z_l} = -\frac{ikf}{60} \frac{(\xi_0^2 - 1) Re_l^4(\xi_0)}{d/d\xi_0[(\xi_0^2 - 1) Re_l^4(\xi_0)]}. \tag{4.6}$$

Equations (4.5) and (4.6) are, for $\eta = 0$ and $f(\eta) = \delta(\eta)$, identical with SC III (2.8). For the sake of completeness we shall explain the notation in (4.5) and (4.6): The function $f(\eta)$ satisfies the condition

$$\int_{-1}^{+1} f(\eta) d\eta = 1, \tag{4.7}$$

* This f will hardly be confused with $f(\eta)$ characterizing the electric field and always written with its argument η.

and in the case of a symmetrical field:

$$f(\eta) = f(-\eta). \tag{4.8}$$

The functions Se_l^1 and Re_l^4 are solutions of the equation:

$$(1 - z^2)W'' - 4zW' + (\lambda - f^2k^2z^2)W = 0 \tag{4.9}$$

for $z=\eta$ and $z=\xi$ respectively. The eigenvalues of λ_l are determined by the condition that Se_l^1 remains finite at the poles $\eta = \pm 1$. The function Re_l^4 for given eigenvalues of λ_l is so chosen that the field at large distances from the center reduces to a wave travelling radially outward. The functions Se_l^1 and Re_l^4 are related to the spherical harmonics and Hankel's functions through coefficients which must be determined numerically. However, for our argument the knowledge of these relations is not essential.

The prolate spheroid has a gap across the equator $\eta = 0$. By a method similar to that of Section 2, we shall now calculate the input admittance at the points

$$z = \pm d/2, \tag{4.10}$$

which, because of the second equation in (4.2) means:

$$\eta = \pm d/2f\xi_0, \tag{4.11}$$

or, finally, because of (4.4):

$$\eta = \pm d/2a, \tag{4.12}$$

where a is here half the long axis. For $f(\eta)$ we shall take the δ-function:

$$f(\eta) = \delta(\eta), \tag{4.13}$$

and our problem will be to find what happens to (4.5) if $d \to 0$. It would have been more rigorous to take, say:

$$f(\eta) \sim (1 - \eta^2)^s \tag{4.14}$$

and then go to the limit $s \to \infty$, as we did in Section 2. However it will be much quicker to use the δ-functions instead and we have seen in Section 2 that the result is the same, which is also true in the case of a spheroid.

We now rewrite (4.5), introducing $f(\eta) = \delta$ and $\eta = d/2a$. Because $Se_l^1 = 0$ if l is odd, we have:

$$Y\left(\frac{d}{2a}\right) = 2\pi[1 - (d/2a)^2]\sum_{m=0}^{\infty} \frac{Se_{2m}^1(0)Se_{2m}^1(d/2a)}{Z_{2m}\int_{-1}^{+1}(1 - \eta^2)(Se_{2m}^1)^2 d\eta} \cdot \tag{4.15}$$

Now again we write (4.15) in the form

$$Y = Y_{0,M} + Y_{M,\infty}, \tag{4.16}$$

and shall proceed to calculate $Y_{M,\infty}$ approximately for sufficiently great M. For the moment we shall ignore Z_{2m} and shall calculate all other expressions appearing in (4.15).

The functions Se_{2m}^1 must be finite and satisfy (4.9) for $z=\eta$. But if λ_{2m} is very great, which we assume, $f^2k^2\eta$ can be neglected in (4.9), because $|\eta| \leq 1$. Therefore, for great λ_{2m} we can write (4.9):

$$(1 - \eta^2)Se_{2m}^{1\prime\prime} - 4\eta Se_{2m}^{1\prime} + \lambda_{2m}Se_{2m}^{1} = 0. \tag{4.17}$$

Therefore, Se_{2m}^1 will not depend on f for large λ_{2m}. Substituting in (4.17):

$$Se_{2m}^{1} = (1 - \eta^2)^{-1/2}P, \tag{4.18}$$

we can write instead of (4.17):

$$P'' + 2\eta P' - \frac{\eta^2}{1 - \eta^2}P + \lambda_{2m}P = 0. \tag{4.19}$$

But this is exactly the equation for P'_{2m+1} for which an everywhere finite solution exists if

$$\lambda_{2m} = 2m(2m + 3) \sim 4m^2, \tag{4.20}$$

and m is very large. The approximate expression for P'_{2m+1} with large m is:

$$P'_{2m+1}\left(\frac{d}{2a}\right) = 2(-1)^m\sqrt{\frac{m}{\pi \cos(d/2a)}}\cos\left[\left(2m + \frac{3}{2}\right)\frac{d}{2a}\right], \tag{4.21}$$

therefore, because of (4.18),

$$Se_{2m}^{1}(0)Se_{2m}^{1}\left(\frac{d}{2a}\right)\left[1 - \left(\frac{d}{2a}\right)^2\right]$$

$$= \frac{4m}{\pi}\left[1 - \left(\frac{d}{2a}\right)^2\right]^{1/2}\cos\left[\left(2m + \frac{3}{2}\right)\frac{d}{2a}\right]. \tag{4.22}$$

The next step is to calculate the integral in the denominator of (4.15). Again because of (4.18), we have:

$$\int_{-1}^{+1}(1 - \eta^2)(Se_{2m}^{1})^2 d\eta = \int_{-1}^{+1}(P'_{2m+1})^2 d\eta = \frac{4(m + 1)(2m + 1)}{4m + 3} \sim 2m. \tag{4.23}$$

We can, therefore, write for small d and sufficiently great M:

$$Y_{M,\infty} = 2\pi\left[\frac{2}{\pi}\sum_{m=M+1}^{\infty}\frac{\cos\left[(2m + \frac{3}{2})(d/2a)\right]}{Z_{2m}}\right]. \tag{4.24}$$

The last step will be to find Z_{2m}.

5. The coefficients Z_{2m} for a spheroid. It will be a little more convenient to introduce, instead of Re_{2m}^4, the corresponding density-functions, as we did in Section 3. The equation which Re_{2m}^4 satisfies is:

$$(\xi^2 - 1)Re_{2m}^{4\prime\prime} + 4\xi Re_{2m}^{4\prime} - (\lambda_{2m} - f^2k^2\xi^2)Re_{2m}^{4} = 0. \tag{5.1}$$

We now introduce

$$(\xi^2 - 1)Re_{2m}^{4} = \mathrm{P}, \tag{5.2}$$

and (5.1) becomes, because of (4.20),

$$(\xi^2 - 1)\mathrm{P}'' - [(4m^2 + 6m + 2) - f^2k^2\xi^2]\mathrm{P} = 0. \tag{5.3}$$

Because of (4.6) we have:

$$\frac{2\pi}{Z_{2m}} = -\frac{ikf}{60}\,\mathrm{P/P'} = -\frac{ikf}{60}\zeta_{2m}, \tag{5.4}$$

where

$$\mathrm{P/P'} = \zeta_{2m} = \zeta.$$

We have:

$$\zeta' = 1 - \zeta^2\mathrm{P''/P}. \tag{5.5}$$

Putting

$$4m^2 + 6m + 2 = \nu^2; \qquad f^2k^2 = c^2, \tag{5.6}$$

we get from (5.5) because of (5.3)

$$\zeta' + \zeta^2\left(\frac{\nu^2}{\xi^2 - 1} - \frac{c^2\xi^2}{\xi^2 - 1}\right) = 1, \tag{5.7}$$

which is Riccati's equation for ζ. Introducing now:

$$\bar{\nu}^2 = \nu^2 - c^2, \tag{5.8}$$

we can write (5.7):

$$\zeta' + \zeta^2\left(\frac{\bar{\nu}^2}{\xi^2 - 1} - c^2\right) = 1. \tag{5.9}$$

The leading term of the solution for every great $\bar{\nu}$ and small ξ (so that $\bar{\nu}^2/\xi^2 - 1 \gg c^2$) is

$$\frac{\mathrm{P}}{\mathrm{P'}} = \zeta_{2m} \sim \pm\frac{1}{\bar{\nu}}\sqrt{\xi^2 - 1} \sim \pm\frac{1}{2m}\sqrt{\xi^2 - 1}. \tag{5.10}$$

The proper sign will be selected by the condition that for $\xi \to 0$ and $f \to 0$, our solution goes over into that of a spherical antenna.

Introducing (5.10) into (5.4) and into (4.24) we have:

$$Y_{M,\infty}\left(\frac{d}{2a}\right) = \mp\frac{2ikf}{60\pi}\sum_{m=M+1}^{\infty}\frac{\sqrt{\xi_0^2 - 1}}{2m}\cos\left[\left(2m + \frac{3}{2}\right)\frac{d}{2a}\right]. \tag{5.11}$$

Changing the summation into an integration we have, because of (4.4)

$$Y_{M,\infty}\left(\frac{d}{2a}\right) = \mp\frac{ikb}{60\pi}\int_M^{\infty}\frac{\cos\left[(2m + \frac{3}{2})(d/2a)\right]}{m}\,dm. \tag{5.12}$$

But for $b = a$ this is *exactly* (2.23) if we take the *lower* sign in (5.10). Therefore finally:

$$Y_{M,\infty}\left(\frac{d}{2a}\right) = -\frac{ibk}{60\pi}\,Ci\left(\frac{Md}{a}\right), \tag{5.13}$$

and because of (4.15):

$$Y\left(\frac{d}{2a}\right) = 2\pi\left[1 - \left(\frac{d}{2a}\right)^2\right] \sum_{m=0}^{M} \frac{\overset{1}{Se}_{2m}(0)\overset{1}{Se}_{2m}(d/2a)}{Z_{2m}\int_{-1}^{+1}(1 - \eta^2)(Se^1_{2m})^2 d\eta} - \frac{ibk}{60\pi} Ci\left(\frac{Md}{a}\right). \tag{5.14}$$

As we see, the only difference between (5.13) and (2.24) is that in the case of the spheroid we have as a factor the *small* axis. Thus $Y_{M,\infty}(d\to 0)$ tends to infinity like

$$b \log (d/a) \tag{5.15}$$

and can be made to be always finite if $b \log d/a$ is finite.

For thin antennas and finite d, $Y_{M,\infty}$ is much smaller than in the case of a sphere with radius a; therefore the neglect of $Y_{M,\infty}$ is much more justified. This is however not true for thick antennas. But the essential result is the same as in the case of a spherical antenna: the dimensions of the gap must be involved in the theory of the antenna, otherwise the input admittance becomes infinite. But infinite driving-point admittance means zero driving-point impedance. In view of this result it is difficult to see what the curves 3 and 4 in SC III mean. They seem to represent the resistance and reactance of a spheroidal antenna and a current distribution which does not become infinite for $\eta = 0$. They may refer to a finite gap and to an electric field in the gap, but in that case we would like to know to what field and to what gap they refer. It is plausible that neither the field nor the size of the gap matters much if the antenna is thin and if the size of the gap is greater than the thickness of the antenna and much smaller than its length. But such a statement ought to be deduced from the theory or at least formulated explicitly as an assumption.

APPENDIX A. ON THE DERIVATION OF (2.5)

Equations (6) and (8) in SC II give

$$H_{\Phi|R=a} = -\sum_{n=1}^{\infty} \frac{A_n}{Z_n} e^{i\omega t} P_n'(\cos\theta), \tag{A.1}$$

where A_n is here used instead of $(\omega\mu_1/i\sigma_1)^{1/2}C_n$ in SC's notation. On the other hand SC II (7) is:

$$-\frac{V}{a} f(\theta) = E_\theta' = e^{i\omega t}\sum_{n=1}^{\infty} A_n P_n'(\cos\theta). \tag{A.2}$$

From (A.2) it follows because of the orthogonal properties of P_n', that:

$$A_n = -\frac{2n+1}{2n(n+1)}\left[\int_0^\pi f(\theta)(P_n')\sin\theta d\theta\right] e^{-i\omega t}\frac{V}{a}\cdot \tag{A.3}$$

Putting (A.3) into (A.1) we have:

$$\begin{aligned} H_{\Phi|R=a} &= \sum_{n=1}^{\infty} \frac{V}{a}\, \frac{2n+1}{2n(n+1)}\left[\int_0^\pi f(\theta)(P_n')\sin\theta d\theta\right]\frac{P_n'}{Z_n} \\ &= \frac{V}{a}\sum_{n=1}^{\infty} \frac{\int_0^\pi f(\theta)(P_n')\sin\theta d\theta}{Z_n\int_0^\pi (P_n')^2 \sin\theta d\theta} P_n'(\cos\theta). \end{aligned} \tag{A.4}$$

Finally from SC II (13) we see:

$$I = 2\pi a \sin \theta H_{\Phi|R=a};$$

therefore

$$Y = \frac{I}{V} = 2\pi \sin\theta \sum_{n=1}^{\infty} \frac{\int_0^\pi f(\theta)(P_n') \sin\theta d\theta}{Z_n \int_0^\pi (P_n')^2 \sin\theta d\theta} P_n'(\cos\theta) \tag{A.5}$$

which is exactly (2.5).

APPENDIX B. THE CALCULATION OF

$$B = 2^{-2s-1} \frac{(2s+1)!}{(s!)^2} \int_0^\pi \sin^{2s+2}\theta P'_{2m+1} d\theta. \tag{B.1}$$

Because of

$$P'_{2m+1} = -dP_{2m+1}/d\theta \tag{B.2}$$

and the recursion formula

$$\cos\theta P_{2m+1} = \frac{2(m+1)}{4m+3} P_{2(m+1)} + \frac{2m+1}{4m+3} P_{2m}, \tag{B.3}$$

we have, introducing $\mu = \cos\theta$:

$$\begin{aligned}\int_0^\pi P'_{2m+1} \sin^{2s+2}\theta d\theta &= 2(s+1) \int_1^{+1} P_{2m+1}(1-\mu^2)^s \mu d\mu \\ &= \frac{2(s+1)}{4m+3}\Big\{2(m+1)\int_{-1}^{+1}(1-\mu^2)^s P_{2(m+1)} d\mu \\ &\quad + (2m+1)\int_{-1}^{+1}(1-\mu^2)^s P_{2m} d\mu\Big\}.\end{aligned} \tag{B.4}$$

Therefore, the problem is reduced to finding an integral of the type

$$T_n^s = \int_{-1}^{+1} (1-\mu^2)^s P_{2n} d\mu. \tag{B.5}$$

To find T_n we must combine the results of Examples 1 and 6 on pp. 310, 311, in Whittaker and Watson's *Modern Analysis*, Fourth Edition, 1940. The result is:

$$\begin{aligned} T_n^s &= (-1)^n 2^{2s+1}\left(\frac{s!}{n!}\right)^2 \frac{(2n)!(s+n)!}{(s-n)!(2s+2n+1)!} \quad \text{if} \quad s \geqq n, \\ T_n^s &= 0 \quad \text{if} \quad s < n. \end{aligned} \tag{B.6}$$

The combination of (B.1) and (B.4), (B.5), (B.6) gives:

$$B = 2^{-2s} \frac{(s+1)(2s+1)!}{(4m+3)(s!)^2} [2(m+1)T^s_{m+1} + (2m+1)T^s_m], \tag{B.7}$$

which, because of (B.6) and some simple algebra, gives the results as expressed in (2.11) and (2.12).

APPENDIX C. CALCULATION OF $Y_{M,s}(\theta, s)$

Because of (2.7) we can write (2.19)

$$\sum_{m=M+1}^{s} C_m^s = \frac{Y \cdot 60}{iak \sin\theta}$$

$$= \sum_{m=M+1}^{s} (-1)^m (s+1) \frac{(2s+1)!(s+m+1)!2m!(4m+3)}{(2s+2m+3)!(s-m)!m!(m+1)!(2m+1)} P'_{2m+1}(\cos\theta). \quad \text{(C.1)}$$

First, we calculate the expressions in which s appears. These are:

$$\frac{(s+1)[1\cdot 2\cdot 3 \cdots (s+m+1)][1\cdot 2 \cdots (2s+1)]}{[1\cdot 2 \cdots (s-m)][1\cdot 2 \cdots (2s+1)(2s+2) \cdots (2s+2m+3)]}$$

$$= \frac{[(s+1-m) \cdots (s+1+m-m)][(s+1)(s+2) \cdots (s+m+1)]}{2^{2m+2}(s+1)(s+1+\frac{1}{2})(s+2)(s+2+\frac{1}{2}) \cdots (s+m+1)(s+m+1+\frac{1}{2})}$$

$$= \frac{1}{2^{2m+2}} \prod_{\alpha=0}^{m} \frac{[1-\alpha/(s+1)]}{[1+(\alpha+\frac{1}{2})/(s+1)]}. \quad \text{(C.2)}$$

Assuming that s is great, taking log of the above expression and changing the summation into integration, we have:

$$\frac{1}{2^{2m+2}} \prod_{\alpha=0}^{m} \frac{[1-\alpha/(s+1)]}{[1+(\alpha+\frac{1}{2})/(s+1)]} \sim \frac{1}{2^{2m+2}} \exp\left[-s\int_0^x \log\frac{1+\xi}{1-\xi}\,d\xi\right], \quad \text{(C.3)}$$

where

$$x = m/s\,. \quad \text{(C.4)}$$

If for the expressions depending on m, we use Stirling's formula, we obtain:

$$C_m^s = \frac{1}{2m\sqrt{\pi m}}(-1)^m \exp\left[-s\int_0^x \log\frac{1+\xi}{1-\xi}\,d\xi\right] P'_{2m+1}(\cos\theta). \quad \text{(C.5)}$$

Now for P'_{2m+1} with great m we have approximately:

$$P'_{2m+1} = -2\sqrt{\frac{m}{\pi\sin\theta}}\cos\left(\frac{\pi}{4}+\left(2m+\frac{3}{2}\right)\theta\right), \quad \text{(C.6)}$$

therefore

$$C_m^s = \frac{(-1)^{m+1}}{m\pi\sqrt{\sin\theta}} \exp\left[-s\int_0^{m/s} \log\frac{1+\xi}{1-\xi}\,d\xi\right] \cos\left[\frac{\pi}{4}+\left(2m+\frac{3}{2}\right)\theta\right], \quad \text{(C.7)}$$

and because of (C.1) we have (2.20):

$$Y_{M,S} = \frac{iak\sqrt{\sin\theta}}{60\pi} \sum_{m=M+1}^{s} (-1)^{m+1} \frac{1}{m}$$
$$\exp\left[-s\int_0^{m/s} \log\frac{1+\xi}{1-\xi}\,d\xi\right] \cos\left[\frac{\pi}{4} + \left(2m+\frac{3}{2}\right)\theta\right]. \quad \text{(C.8)}$$

If $s \to \infty$ the expression in the exponent is:

$$\lim_{s\to\infty}\left[-s\int_0^{m/s} \log\frac{1+\xi}{1-\xi}\,d\xi\right] = -\lim_{\alpha\to 0}\left[\frac{1}{\alpha}\int_0^{m\alpha} \log\frac{1+\xi}{1-\xi}\,d\xi\right] = 0. \quad \text{(C.9)}$$

Therefore (2.21) follows from (C.8) and (C.9).

APPENDIX D. THE RECURSION FORMULA (3.12)

We can deduce the recursion formula (3.12) for any cylindrical function density z_p satisfying the equation:

$$z_p'' - \frac{p^2 - \frac{1}{4}}{x^2} z_p + z_p = 0. \quad \text{(D.1)}$$

The known recursion formula for z_p is:

$$\frac{p - \frac{1}{2}}{x} z_{p-1} - z'_{p-1} = z_p. \quad \text{(D.2)}$$

On the other hand we have, because of (D.2),

$$\frac{z_p'}{z_p} = \frac{-(p-\frac{1}{2})x^{-2}z_{p-1} + (p-\frac{1}{2})x^{-1}z'_{p-1} - z''_{p-1}}{(p-\frac{1}{2})x^{-1}z_{p-1} - z'_{p-1}}. \quad \text{(D.3)}$$

This gives, because of (D.1):

$$\frac{z_p'}{z_p} = \frac{-(p-\frac{1}{2})x^{-2}z_{p-1} + (p-\frac{1}{2})x^{-1}z'_{p-1} - \left\{[(p-1)^2 - \frac{1}{4}]x^{-2} - 1\right\}z_{p-1}}{(p-\frac{1}{2})x^{-1}z_{p-1} - z'_{p-1}}. \quad \text{(D.4)}$$

Putting

$$z_p'/z_p = y_p, \quad \text{(D.5)}$$

we have

$$y_p = \frac{1 + (p-\frac{1}{2})x^{-1}y_{p-1} - (p-\frac{1}{2})^2x^{-2}}{(p-\frac{1}{2})x^{-1} - y_{p-1}}. \quad \text{(D.6)}$$

Now, by employing this formula twice, we have:

$$y_p = \frac{y_{p-2}\,x(x^2/(2p-2) - (p-\frac{1}{2})) + (p-3/2)(p-\frac{1}{2}) - x^2}{y_{p-2}x^2 + x(x^2/(2p-2) - (p-3/2))}.$$

Let us now introduce

$$p = 2m + 3/2, \quad \text{(D.7)}$$

as dedicated by the comparison of z_p'/z_p with (3.4). Thus we have:

$$y_{2m+3/2} = \frac{x(x^2/(4m+1) - (2m+1))y_{2m-1/2} + 2m(2m+1) - x^2}{x^2 y_{2m-1/2} + x(x^2/(4m+1) - 2m)} \cdot \tag{D.8}$$

Finally, introducing:

$$\zeta_{2m+1} = \frac{1}{y_{2m+3/2}}, \tag{D.9}$$

we have our recursion formula (3.12).

APPENDIX E. THE DRIVING-POINT ADMITTANCE IF $f(\theta)$ IS A STEP FUNCTION

We shall calculate the driving-point admittance for a spherical antenna, assuming for $f(\theta)$ a step function defined by:

$$\begin{aligned} f(\theta) &= \frac{a}{d} \quad \text{for} \quad \frac{\pi}{2} + \frac{d}{2a} \geqq \theta \geqq \frac{\pi}{2} - \frac{d}{2a}, \\ f(\theta) &= 0 \quad \text{for} \quad \theta > \frac{\pi}{2} + \frac{d}{2a} \quad \text{and} \quad \theta < \frac{\pi}{2} - \frac{d}{2a} \cdot \end{aligned} \tag{E.1}$$

We assume that d is small and we shall calculate the admittance for

$$\cos\theta_0 = \frac{d}{2a}, \quad \text{that is} \quad \theta_0 = \frac{\pi}{2} - \frac{d}{2a} \cdot \tag{E.2}$$

We introduce (E.1) into (2.9). We have, because of (2.6) and because d is small:

$$\int_0^\pi f(\theta)P'_{2m+1} \sin\theta d\theta = 2P_{2m+1}\left(\frac{d}{2a}\right)\frac{a}{d} \cdot \tag{E.3}$$

Therefore (2.9) can be written for our choice of $f(\theta)$:

$$Y\left(\frac{d}{2a}\right) = \frac{4\pi a}{d}\sum_{m=0}^{\infty} \frac{P_{2m+1}(d/2a)P'_{2m+1}(d/2a)}{[\int_0^\pi (P'_{2m+1})^2 \sin\theta d\theta]Z_{2m+1}} \cdot \tag{E.4}$$

We are interested in calculating $Y_{M,\infty}(d/2a)$ defined by:

$$Y_{M,\infty}\left(\frac{d}{2a}\right) = \frac{4\pi a}{d}\sum_{m=M+1}^{\infty} \frac{P_{2m+1}(d/2a)P'_{2m+1}(d/2a)}{Z_{2m+1}[\int_0^\pi (P'_{2m+1})^2 \sin\theta d\theta]} \cdot \tag{E.5}$$

For great m we have:

$$\begin{aligned} P_{2m+1}\left(\frac{d}{2a}\right) &= (-1)^m\sqrt{\frac{1}{\pi m}} \sin\left[\left(2m + \frac{3}{2}\right)\frac{d}{2a}\right] \\ P'_{2m+1}\left(\frac{d}{2a}\right) &= (-1)^m 2\sqrt{\frac{m}{\pi}} \cos\left[\left(2m + \frac{3}{2}\right)\frac{d}{2a}\right] \\ \left[\int_0^\pi (P'_{2m+1})^2 \sin\theta d\theta\right]^{-1} &= \frac{1}{2m} \end{aligned} \tag{E.6}$$

$$\frac{1}{Z_{2m+1}} = \frac{iak}{4\pi \times 60m}.$$

Therefore

$$Y_{M,\infty}\left(\frac{d}{2a}\right) = \frac{iak}{60 \times \pi} \sum_{M+1}^{\infty} \frac{\sin\left[(2m+\frac{1}{2})d/a\right]}{2m^2 d/a}. \tag{E.7}$$

Changing summation into integration we have:

$$Y_{M,\infty}\left(\frac{d}{2a}\right) = \frac{iak}{\pi \times 60} \int_{2M(d/a)}^{\infty} \frac{\sin x}{x^2}\,dx, \tag{E.8}$$

and the leading part of (E.8) is for sufficiently small d:

$$Y_{M,\infty}\left(\frac{d}{2a}\right) = -\frac{iak}{\pi \times 60}\,Ci\left(\frac{2Md}{a}\right). \tag{E.9}$$

It is interesting to compare this result with (2.24). The only difference is in the argument of the Ci function, which matters very little, if the argument is small.

One would be tempted to calculate the driving-point admittance by taking $\theta = \pi/2$ and not, as we did, $\theta = \pi/2 - d/2a$. This means that in (E.4) and (E.5) we would have to substitute zero instead of $d/2a$ as an argument in P'_{2m+1}. Then for $Y_{M,\infty}(0)$ we easily find:

$$Y_{M,\infty}(0) = \frac{iak}{\pi \times 60} \sum_{M+1}^{\infty} \frac{\sin\left[(2m+3/2)d/2a\right]}{m^2 d/a}, \tag{E.10}$$

and changing summation into integration we have

$$Y_{M,\infty}(0) = -\frac{iak}{\pi \times 60}\,Ci\left(\frac{Md}{a}\right),$$

that is the same result as in (2.24).

The argument will be very similar in the case of a spheroid. Thus we can collect all the results concerning $Y_{M,\infty}$ in the following two tables:

Sphere

$f(\theta)$	θ	$Y_{M,\infty}$
Dirac's function	$\frac{\pi}{2}$	∞
	$\frac{\pi}{2} \pm \frac{d}{2a}$	$-\frac{iak}{60\pi}\,Ci\left(\frac{Md}{a}\right)$
Step function	$\frac{\pi}{2}$	
	$\frac{\pi}{2} \pm \frac{d}{2a}$	$-\frac{iak}{60\pi}\,Ci\left(\frac{2Md}{a}\right)$

Spheroid

$f(\eta)$	η	$Y_{M,\infty}$
Dirac's function	0	∞
	$\pm\frac{d}{2a}$	$-\frac{ibk}{60\pi}\,Ci\left(\frac{Md}{a}\right)$
Step function	0	
	$\pm\frac{d}{2a}$	$-\frac{ibk}{60\pi}\,Ci\left(\frac{2Md}{a}\right)$