ROTOR BLADE FLAPPING MOTION*

BY
GABRIEL HORVAY
McDonnell Aircraft Corporation

1. Introduction. In an earlier paper a method was developed for the solution of a class of Hill differential equations.[1] In the present paper the method is applied to the solution of the differential equation of flapping motion of helicopter rotor blades. A brief review of the equation in Sec. 2 is followed by the solution of the homogeneous equation in Sec. 3. In Sec. 4 the stability of blade flapping motion is established for the range of the numerical parameters which usually arise in helicopter theory. The stability of the equation for large values of μ is discussed in Sec. 5. In Sec. 6 the possibility of resonance is investigated.

2. Differential equation of rotor blade flapping motion. The blades of a helicopter rotor are not attached rigidly to the rotor shaft, but are hinged so as to permit the blades to flap out of the rotor plane. This prevents transmission of severe alternating moments from the blades to the rotor shaft. The angle between a blade and the rotor plane (the plane normal to the rotor shaft) is called the flapping angle β; it is counted positive upwards.

At a standstill, the blades of the rotor rest on the droop stops. When the rotor is revolving at the angular velocity ω, the centrifugal force

$$\int_0^R dC = \int_0^R r\omega^2 dm \tag{1a}$$

(dm is the mass of the blade element at distance r from the axis of rotation, R the blade length) lifts the blades off the stops and keeps them in approximately horizontal position. The blades also experience lift forces

$$\int_0^R dL = \frac{1}{2}\int_0^R \rho ca(\varphi + \vartheta) U^2 dr. \tag{1b}$$

Here ρ is the air density, c the chord of the blade profile, a the slope of the lift curve, ϑ the blade pitch angle, $\varphi+\vartheta$ the angle of attack, and U the resultant air velocity, cf. Fig. 1a.

In *vertical flight* the equilibrium angle β of the blade is determined from the condition, cf. Fig. 1b, that the moments of lift, weight and centrifugal forces balance each other:

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* Received Sept. 5, 1946. This paper is based on an investigation which was carried out at the McDonnell Aircraft Corporation under sponsorship of the Bureau of Aeronautics, U. S. Navy Department.

[1] G. Horvay, *Unstable solutions of a class of Hill differential equations*, Quarterly Appl. Math. **000,** 385–396 (1947). In the following, this paper will be referred to as I. (Note that A in Eq. (29) of I should be written with a superscript 2 instead of 0.) The present paper is a continuation of the study, and deals with the practical aspects of the problem. The writer's thanks are due to his colleagues, Elizabeth J. Spitzer, Kathryn Meyer, and Frances Schmitz for checking the derivations and for help with the numerical calculations.

$$\int_0^R r(dL - gdm - \beta dC) = 0.$$

(The drag has a negligible effect on the flapping motion.) It is assumed here as elsewhere that the blade is rigid, and that β is so small that the approximation $\sin \beta \sim \beta$, $\cos \beta \sim 1$ is valid. The same assumption is made also for the angles ϑ and φ.[2]

Let t denote the time and

$$\psi = \omega t \tag{2}$$

the azimuth of the blade as measured from the rearmost position. In the course of *forward flight* at the speed of $\mu\omega R$ (μ, the so-called *advance ratio*, is the ratio of forward

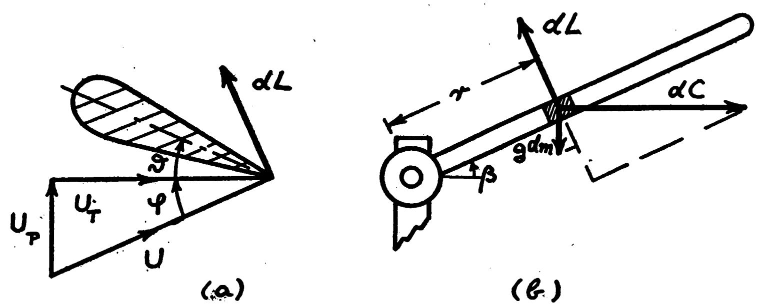

FIG. 1. Velocities and forces at a blade element.

velocity to blade tip speed), an advancing blade element ($\psi = 90°$) moves with the speed $\omega(r + \mu R)$, a retreating element ($\psi = 270°$) moves with the speed $\omega(r - \mu R)$. The variation in the relative air velocity brings forth a variable lift moment, and thus produces a blade flapping which varies with the azimuth ψ. The tangential component of the air velocity at element dm is

$$U_T = r\omega + \mu\omega R \sin \psi, \tag{3a}$$

and the perpendicular component is

$$U_P = \lambda\omega R - r\dot{\beta} - \mu\omega R\beta \cos\psi. \tag{3b}$$

Here $\lambda\omega R$ is the difference between the sinking speed of the helicopter and the induced velocity through the rotor disc. λ is called the *inflow ratio*.

The flapping motion of the blade is governed by the equation

$$I\ddot{\beta} = \int_0^R r(dL - gdm - \beta dC), \tag{4}$$

where

$$I = \int_0^R r^2 dm \tag{4a}$$

is the moment of inertia of the blade with respect to the flapping hinge. Observing that

[2] A brief list of the assumptions involved is given at the end of Section 4. For a more extended discussion cf. J. B. Wheatley, *Aerodynamic analysis of the autogiro rotor*, NACA Technical Report 487, 1934.

$$(\varphi + \vartheta)U^2 \simeq U_T U_P + \vartheta U_T^2 \tag{3c}$$

performing the integration (4a), and dividing by I, one arrives at the *differential equation of flapping motion*

$$\mathcal{J}(\beta) = E(\omega t), \tag{5}$$

where

$$\mathcal{J}(\beta) \equiv \ddot{\beta} + p(t)\dot{\beta} + s(t)\beta \tag{5a}$$

with

$$p(t) \equiv n\omega[1 + \tfrac{4}{3}\mu \sin \omega t], \qquad s(t) \equiv \omega^2[1 + \tfrac{4}{3}n\mu \cos \omega t + n\mu^2 \sin 2\omega t]; \tag{5b}$$

and

$$E(\omega t) \equiv - mgr_{cg}/I + n\omega^2\{\tfrac{4}{3}\lambda + (1 + \mu^2)\vartheta + (2\lambda\mu + \tfrac{8}{3}\mu\vartheta) \sin \omega t - \mu^2\vartheta \cos 2\omega t\}. \tag{5c}$$

The first term of the forcing function $E(\omega t)$ represents the moment of the blade weight (r_{cg} is the c.g. distance from the hinge), and is usually negligibly small. The remaining terms constitute the aerodynamic excitation. Control of the helicopter is effected by cyclic pitch variation:

$$\vartheta = \vartheta_0 + \vartheta_c \cos \omega t + \vartheta_s \sin \omega t. \tag{5d}$$

Here ϑ_0 is the "collective pitch" setting, ϑ_c the lateral, $-\vartheta_s$ the fore-and-aft control setting.

It will be convenient to call the equation

$$\mathcal{J}(\beta) = 0 \tag{6}$$

the "homogenized" flapping equation. The term $\ddot{\beta}$ of $\mathcal{J}(\beta)$ is the inertia term. The expression $p(t)\dot{\beta}$ represents the damping which is produced by the change in the angle of attack caused by flapping. The dimensionless quantity

$$n = \rho c a R^4/8I \tag{7}$$

can be called the *aerodynamic damping coefficient.*[3] The leading member, $\omega^2\beta$, of the spring term $s(t)\beta$ represents the restoring moment of the centrifugal force. The frequency ω of the coefficients $p(t)$ and $s(t)$ can be called the *parametric frequency.*[4] In the present problem this frequency coincides with the frequency impressed by the function $E(\omega t)$.

The solution of Eq. (5) can be written in the form

$$\beta = \beta_0 + h_1\beta_1 + h_2\beta_2 \tag{8}$$

where β_0 is the particular integral and represents the forced (steady state) response; β_1 and β_2 are the two solutions of the homogeneous equation, and represent the natural modes (transients). h_1 and h_2 are numerical coefficients determined by the initial conditions.

Two papers appear in the literature on the solution of the homogeneous equation,

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[3] The Germans refer to $\gamma \equiv 8n$ as the "Locksche Trägheitszahl."

[4] N. Minorsky, *On parametric excitation*, Journal Franklin Institute **250**, 25, 1945.

$\mathcal{J}(\beta)=0$. Bennett[5] has investigated the solution for the particular values $n=1.5$, $\mu=1.0$, $\beta(0)=1$, $\dot{\beta}(0)=0$, in the following manner. He determined the coefficients of the expansion

$$\beta(\psi) = \beta(0) + t\dot{\beta}(0)/1! + t^2\ddot{\beta}(0)/2! + \cdots \tag{9}$$

by repeated differentiation of Eq. (6), plotted $\beta(\psi)$ from $\psi=0$ to $\psi=4\pi$, and noted that $\beta(4\pi)/\beta(0)<1$. From this result Bennett concluded that stability is insured for all $\mu<1$. In an earlier paper Glauert and Shone[6] solved Eq. (6) by omitting the $\ddot{\beta}$ term for simplicity, and concluded that the motion is unstable.

Evidently, the above approaches to the stability problem are approximate and inconclusive.

More extended investigations are available for the particular integral. β_0 can be assumed in the form[7]

$$\beta_0 = a_0 + a_1 \cos \omega t + b_1 \sin \omega t + a_2 \cos 2\omega t + b_2 \sin 2\omega t + \cdots . \tag{10}$$

The constant part, a_0, is called the *coning angle* of the blade. The angles a_1 and b_1 determine the tilt of the cone axis (forward and to the port side, respectively, when ω is counterclockwise); the higher terms determine the motion of the blade in and out of the cone. Substituting (10) into Eq. (5), one obtains the infinite system of equations

$$\begin{aligned}
1:&\quad a_0 + \tfrac{1}{2}\mu^2 n b_2 = -\, mgr_{cg}/I\omega^2 + \tfrac{4}{3}n\lambda + (1+\mu^2)n\vartheta_0 + \tfrac{4}{3}\mu\eta\vartheta_s \\
\cos\psi:&\quad \tfrac{4}{3}\mu a_0 + (1+\tfrac{1}{2}\mu^2)b_1 - \tfrac{2}{3}\mu a_2 + \tfrac{1}{2}\mu^2 b_3 = (1+\tfrac{1}{2}\mu^2)\vartheta_c \\
\sin\psi:&\quad -(1-\tfrac{1}{2}\mu^2)a_1 - \tfrac{2}{3}\mu b_2 - \tfrac{1}{2}\mu^2 a_3 = 2\mu\lambda + \tfrac{8}{3}\mu\vartheta_0 + (1+\tfrac{3}{2}\mu^2)\vartheta_s \\
\cos 2\psi:&\quad \tfrac{4}{3}\mu n a_1 - 3a_2 + 2nb_2 - \tfrac{4}{3}\mu n a_3 + \cdots = -\,\mu^2 n\vartheta_0 - \tfrac{4}{3}\mu n\vartheta_s \\
\sin 2\psi:&\quad \mu^2 n a_0 + \tfrac{4}{3}\mu n b_1 - 2na_2 - 3b_2 - \tfrac{4}{3}\mu n b_3 + \cdots = \tfrac{4}{3}\mu n\vartheta_c \\
\cos 3\psi:&\quad \tfrac{1}{2}\mu^2 n b_1 - 2\mu n a_2 + 8a_3 - 3nb_3 + \cdots = -\,\tfrac{1}{2}\mu^2 n\vartheta_c \\
\sin 3\psi:&\quad \tfrac{1}{2}\mu^2 n a_1 + 2\mu n b_2 - 3na_3 - 8b_3 + \cdots = -\,\tfrac{1}{2}\mu^2 n\vartheta_s \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{11}$$

in the flapping coefficients $a_0, a_1, b_1, \cdots$. Placing $a_n=b_n=0$ for $n>N$, one can solve the first $2N+1$ equations for $a_0, a_1, \cdots, a_N, b_N$. For instance, for the numerical values

$$n = 1.7,\ mgr_{cg}/I\omega^2 = 0.03,\ \lambda = -0.10,\ \vartheta_0 = 0.2,\ \vartheta_c = \vartheta_s = 0,\ \mu = 0.34738 \tag{12a}$$

which are representative for a helicopter, one finds that

$$\begin{aligned}\beta_0(\psi) &= 0.124 - 0.125\cos\psi - 0.057\sin\psi - 0.012\cos 2\psi \\ &\quad + 0.007\sin 2\psi - 0.001\cos 3\psi + \cdots\end{aligned} \tag{12b}$$

is the forced response.

Glauert[8] initiated the above method of approach in his fundamental paper on

[5] J. A. J. Bennett, *Rotary-wing aircraft*, Aircraft Engineering, May, 1940. An account of the work of Glauert and Shone[6] is also given in this paper.

[6] Glauert and Shone, R.A.E. Report BA 1080, 1933.

[7] This is a departure from the commonly accepted notation $\beta_0=a_0-a_1\cos\omega t-b_1\sin\omega t-a_2\cos 2\omega t-b_2\sin 2\omega t-\cdots$.

[8] H. A. Glauert, *A general theory of the autogyro*, R. & M. No. 1111, British A.R.C., 1926.

rotor blade theory, and carried the expansion of β_0 to first harmonics. Subsequently Lock,[9] Wheatly[2] and others determined also the second and higher harmonic flapping coefficients. They found from numerous examples (for $\mu \lesssim 0.5$) that the convergence of β_0 is sufficiently rapid to permit termination of the series with the second harmonic terms.

Although a numerical determination of β_0 has thus been accomplished, (in contrast with the lack of similar solutions for β_1 and β_2,) the problem of forced vibrations cannot be considered as completely solved so long as the convergence of series (10) is based solely on an appeal to numerical examples. It will be shown in Section 6 that for definite values of μ and n the expansion (10) diverges. The motion β is then said to be in *resonance* with the excitation $E(\omega t)$.

3. Solution of the homogeneous equation. Determination of the transients is very simple in vertical flight. Placing $\mu=0$ in Eq. (6) one obtains the familiar equation of damped motion

$$d^2\beta/d\psi^2 + nd\beta/d\psi + \beta = 0. \tag{13}$$

This has the solutions

$$\begin{aligned} \beta_1 &= \exp\left(-\tfrac{1}{2}n\psi\right)\cdot\cos\sqrt{1-\tfrac{1}{4}n^2}\,\psi, \\ \beta_2 &= \exp\left(-\tfrac{1}{2}n\psi\right)\cdot\sin\sqrt{1-\tfrac{1}{4}n^2}\,\psi. \end{aligned} \tag{14}$$

For $\mu > 0$ one can write in analogy

$$\begin{aligned} \beta_1 &= e_1(\psi)\mathcal{P}_1(\psi), \\ \beta_2 &= e_2(\psi)\mathcal{P}_2(\psi), \end{aligned} \tag{15a, b}$$

where

$$\begin{aligned} e_1(\psi) &= \exp\left(-\tfrac{1}{2}n_{app_1}\psi\right), \\ e_2(\psi) &= \exp\left(-\tfrac{1}{2}n_{app_2}\psi\right), \end{aligned} \tag{16a, b}$$

and $\mathcal{P}_1(\psi)$ and $\mathcal{P}_2(\psi)$ are oscillatory functions. n_{app} can be called the *apparent damping coefficient.*

The functions e_1, e_2, $\mathcal{P}_1$, $\mathcal{P}_2$ can be determined in the following manner. One substitutes[10,11]

$$\beta(\psi) = v(\psi)\cdot\epsilon(\psi), \tag{17}$$

$$\epsilon(\psi) = \exp\left(-\frac{1}{2}\int^t p(t)dt\right) = \exp\left(-\tfrac{1}{2}n\psi + \tfrac{2}{3}n\mu\cos\psi\right) \tag{17a}$$

into Eq. (6). This leads to an equation for the factor $v(\psi)$ in which the first derivative is absent. Replacing the trigonometric functions by exponentials, the equation can be written in the form

$$d^2v/d\psi^2 + \left[\overset{*}{\theta}_2 e^{-2i\psi} + \overset{*}{\theta}_1 e^{-i\psi} + \theta_0 + \theta_1 e^{i\psi} + \theta_2 e^{2i\psi}\right]v = 0, \tag{18}$$

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[9] C. N. H. Lock, *Further development of autogyro theory,* R. & M. No. 1127, British A.R.C. 1928.

[10] Whittaker and Watson, *Modern analysis,* p. 194, Cambridge, 1927.

[11] The notation $\int^t$ indicates that only the upper limit, t, is to be substituted into the indefinite integral of $p(t)$.

where

$$\begin{aligned} \theta_0 &= 1 - \tfrac{1}{4}n^2 - \tfrac{3}{8}\mu^2 n^2, \\ \theta_1 &= \tfrac{1}{3}\mu n(1 + in), \\ \theta_2 &= \mu^2 n(\tfrac{1}{8}n - \tfrac{1}{2}i), \end{aligned} \tag{19a, b, c}$$

and θ^* denotes the conjugate complex of θ. By writing $v(\psi)$ in the form

$$v(\psi) = e^{\sigma\psi} \sum_{-\infty}^{+\infty} C_k e^{ik\psi} \tag{20}$$

the differential equation (18) is reduced to an infinite system of linear homogeneous equations in the C_k.

For the values

$$\pm\,\sigma = \frac{i}{\pi} \arctan \frac{q}{\sqrt{1-q^2}} + mi \quad \text{for} \quad -1 \leqq q \leqq 1 \tag{21a}$$

$$= \frac{1}{\pi} \log\,(q + \sqrt{q^2 - 1}) + (m - \tfrac{1}{2})i \quad \text{for} \quad q \leqq -1,\, q \geqq 1 \tag{21b}$$

$$= \frac{1}{\pi} \log\,(q' + \sqrt{q'^2 + 1}) + mi \quad \text{for} \quad q \text{ imaginary} \tag{21c}$$

where

$$m = 0, \pm 1, \pm 2, \cdots \tag{22a}$$

$$q = \sqrt{\mathfrak{D}} \sin \pi\sqrt{\theta_0} \quad \text{or} \quad \sqrt{-\mathfrak{D}} \sinh \pi\sqrt{-\theta_0} \tag{22b}$$

$$q' = \sqrt{-\mathfrak{D}} \sin \pi\sqrt{\theta_0} \quad \text{or} \quad \sqrt{\mathfrak{D}} \sinh \pi\sqrt{-\theta_0} \tag{22c}$$

$$y_k = 1/(\theta_0 - k^2) \tag{22d}$$

and[12]

$$\mathfrak{D} = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \theta_1^* y_2 & \theta_2^* y_2 & 0 & 0 & \cdot \\ \cdot & \theta_1 y_1 & 1 & \theta_1^* y_1 & \theta_2^* y_1 & 0 & \cdot \\ \cdot & \theta_2 y_0 & \theta_1 y_0 & 1 & \theta_1^* y_0 & \theta_2^* y_0 & \cdot \\ \cdot & 0 & \theta_2 y_1 & \theta_1 y_1 & 1 & \theta_1^* y_1 & \cdot \\ \cdot & 0 & 0 & \theta_2 y_2 & \theta_1 y_2 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}, \tag{22e}$$

the system of equations is consistent, and the expansion coefficients C_k can be determined. Two solutions are obtained, $C_k^{(1)}$ and $C_k^{(2)}$. The first set for, say,

$$\sigma = \sigma_1 = \sigma_r + i\sigma_i, \qquad \sigma_r \geqq 0, \tag{23a}$$

[12] It was shown in I that the infinite determinant $\mathfrak{D}$ may be expanded (for small μ) into a power series in $\delta = |\theta_1|^2$, $\epsilon = |\theta_2|^2$ and $\eta = \frac{1}{2}(\theta_1^2\theta_2^* + \theta_1^{*2}\theta_2)$. The leading term of the expansion is 1; the higher coefficients are functions of θ_0 only and may be obtained from the Tables of I.

the second set for

$$\sigma = \sigma_2 = -\sigma_1. \tag{23b}$$

The two solutions are linearly independent, except when $\sigma_r=0$, and $2\sigma_i=$integer.

The trigonometric function $\mathcal{P}_1(\psi)$ is now obtained as the product

$$\mathcal{P}_1(\psi) = \exp\left(\tfrac{2}{3}n\mu\cos\psi\right)\cdot \sum_{k=-\infty}^{+\infty} C_k^{(1)} \exp i(\sigma_i + k)\psi \tag{24a}$$

and the exponential function $e_1(\psi)$ as the product

$$e_1(\psi) = \exp\left(-\tfrac{1}{2}n\psi\right)\cdot\exp\left(\sigma_r\psi\right). \tag{24b}$$

The determination of $\mathcal{P}_2(\psi)$ and $e_2(\psi)$ (call these Eqs. (25a, b)) is similar. It is seen now that e_1 involves the apparent damping coefficient

$$n_{app1} = n - 2\sigma_r \tag{26a}$$

which is smaller than n, and e_2 involves the apparent damping coefficient

$$n_{app_2} = n + 2\sigma_r \tag{26b}$$

which is larger than n. Evidently, n_{app_1} is the critical damping coefficient.[13] It will be written without the subscript 1.

Example. Determine the flapping transients β_1 and β_2 of a helicopter for which $n=1.7$. Choose an advance ratio μ between 0.30 and 0.35 so that the Tables of I (which cover the range $\theta_0=-0.6$ to $+0.5$ in 0.05 steps) be directly applicable.

One finds from (19a) that for

$$n = 1.7 \tag{27a}$$

and

$$\theta_0 = 0.2 \tag{27b}$$

the advance ratio has the value

$$\mu = 0.34738. \tag{27c}$$

This yields, by (19b, c) the values

$$\theta_1 = 0.19685 + 0.33465i, \qquad \theta_2 = 0.03875 - 0.10258i. \tag{27d}$$

The characteristic exponents

$$\sigma_1 = 0.30782 + \tfrac{1}{2}i, \qquad \sigma_2 = -\sigma_1 \tag{28}$$

associated with the numerical values (27), were determined in I, Section 4. The corresponding functions $v_1(\psi)$ and $v_2(\psi)$ were also given there. Eqs. (24) and (25) now yield[14]

[13] $n-2\sigma_r<0$ implies instability. The arbitrariness in the value of m, Eq. (22a), is removed by making C_0 the dominant term of $v(\psi)$.

[14] The normalization is used: the sin $\frac{1}{2}\psi$ term of v_1 and the cos $\frac{1}{2}\psi$ term of v_2 are written with the coefficient 1.

$$e_1(y) = e^{-0.5422\psi}, \tag{29a}$$

$$\begin{aligned} \mathcal{P}_1(y) = {} & - 0.0733 \cos \tfrac{1}{2}\psi + 0.8369 \sin \tfrac{1}{2}\psi + 0.1923 \cos \tfrac{3}{2}\psi + 0.1717 \sin \tfrac{3}{2}\psi \\ & + 0.0268 \cos \tfrac{5}{2}\psi + 0.0131 \sin \tfrac{5}{2}\psi + 0.0023 \cos \tfrac{1}{2}\psi + 0.0023 \sin \tfrac{7}{2}\psi \\ & + 0.0002 \cos \tfrac{9}{2}\psi + 0.0004 \sin \tfrac{9}{2}\psi + \cdots , \end{aligned} \tag{29b}$$

and

$$e_2(\psi) = e^{-1.1578\psi}, \tag{29c}$$

$$\begin{aligned} \mathcal{P}_2(\psi) = {} & 1.2863 \cos \tfrac{1}{2}\psi + 0.3795 \sin \tfrac{1}{2}\psi + 0.4404 \cos \tfrac{3}{2}\psi + 0.1028 \sin \tfrac{3}{2}\psi \\ & + 0.0676 \cos \tfrac{5}{2}\psi + 0.0229 \sin \tfrac{5}{2}\psi + 0.0068 \cos \tfrac{7}{2}\psi + 0.0050 \sin \tfrac{7}{2}\psi \\ & + 0.0005 \cos \tfrac{9}{2}\psi + 0.0006 \sin \tfrac{9}{2}\psi + \cdots . \end{aligned} \tag{29d}$$

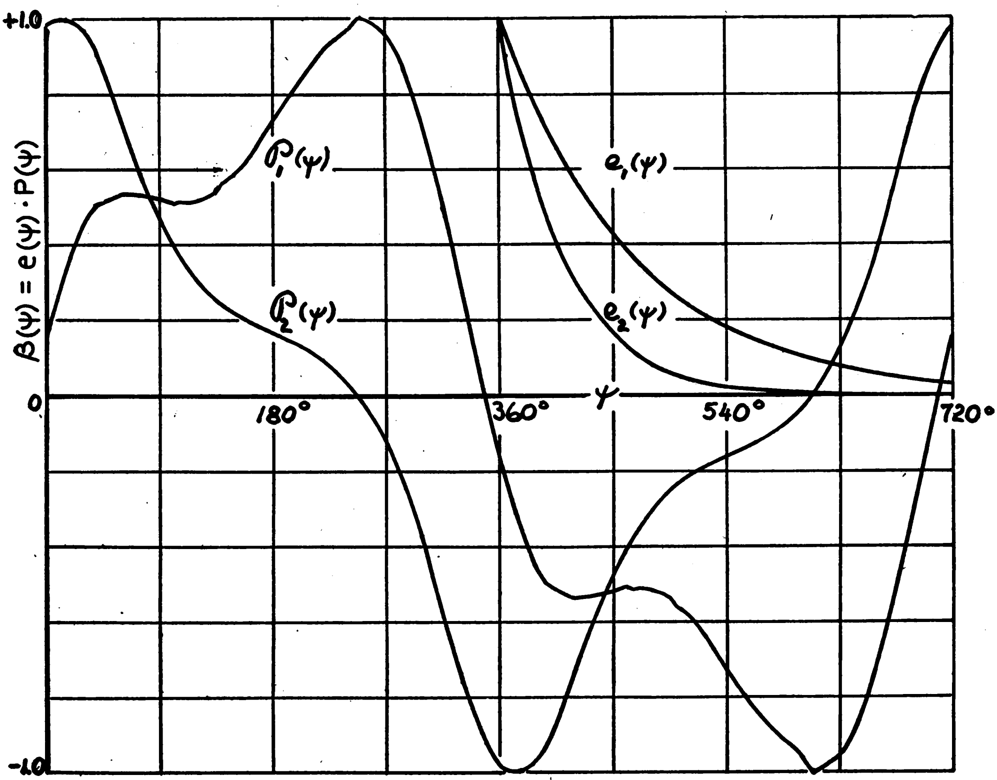

FIG. 2. Transient blade flapping modes $\beta_1(\psi) = e_1(\psi)\mathcal{P}_1(\psi)$ and $\beta_2(\psi) = e_2(\psi)\mathcal{P}_2(\psi)$ for damping coefficient $n = 1.7$ and advance ratio $\mu = 0.34738$.

The factors e_1, $\mathcal{P}_1$ and e_2, $\mathcal{P}_2$ of the transient flapping angles β_1 and β_2 are plotted in Fig. 2 as functions of ψ. $\mathcal{P}_1(\psi)$ and $\mathcal{P}(\psi)$ are drawn normalized to 1 at their maxima, $e_1(\psi)$ and $e_2(\psi)$ are drawn normalized to 1 at $\psi = 360°$. The graphs indicate that at the very large advance ratio $\mu = 0.347$ the flapping motion of the blade, as given by Eq. (6), is still so stable that a transient reduces to less than 4% of its original amplitude within one revolution. The wave form $\mathcal{P}(\psi)$ of the transient contains a pronounced third harmonic component and unimportant higher harmonics. The frequency of the transient (or more exactly, of the dominant term of the transient) is *one half* of the rotor speed.

4. Stability of blade flapping motion. It was seen in the preceding section that the transient β_1 is associated with an apparent damping coefficient n_{app_1} which is less than n when $\sigma_r \neq 0$. σ_r can be called the "absolute degree of destabilization" of the system. For a damped system ($n>0$), like the present one, it is more convenient to characterize destabilization in terms of the ratio

$$2\sigma_r/n = 1 - n_{app}/n. \tag{30}$$

This can be called the "relative degree of destabilization" of the system, or briefly, the *degree of destabilization*. For the example of the preceding section ($n=1.7$, $\theta_0=0.2$, $\mu=0.34738$) the degree of destabilization is $2\sigma_r/n=0.362$.[15]

When

$$2\sigma_r/n = 0, \quad \text{or} \quad n_{app}/n = 1 \tag{31a}$$

a system is *completely stable*. When

$$0 < 2\sigma_r/n < 1, \quad \text{or} \quad 1 > n_{app}/n > 0 \tag{31b}$$

the system becomes partially destabilized (*transition region*), and when

$$2\sigma_r/n > 1, \quad \text{or} \quad n_{app}/n < 0 \tag{31c}$$

the system is *unstable*.

For conventional rotor blades n ranges from 1.4 to 1.9 (at sea level), and $\mu \sim 0.35$ represents the maximum attainable forward speed. By a systematic determination of σ_r associated with the values 1.4, 1.5, $\cdots$, 2.4, 2.5 of n, and the values -1, -0.9, $\cdots$, $+0.4$, $+0.5$ of θ_0 the writer computed the degree of destabilization $2\sigma_r/n$ in a μ, n region which covers the range of practical interest. The plots are shown in Figs. 3a, b. The calculated points are indicated by crosses. The points on the μ-axis (small circles in Fig. 3b) were obtained by means of an auxiliary graph giving the plot of q and q' vs. μ for various n. In Fig. 4 this construction is shown for $n=1.4, \sqrt{3}$, 1.9, 2.0, 2.1, 2.4. The intersections of the curves with horizontal lines of ordinates 0 and 1 yield the μ values for which $\sigma_r=0$.

A replot of Fig. 3, with μ as abscissa, n as ordinate and n_{app}/n as the index of the curves leads to the *stability diagram*, Fig. 5. The diagram indicates two complete stability regions, and two transition regions. The complete stability regions are characterized by $n_{app}/n=1$ and by response frequencies which vary with μ and n (between 0 and $\frac{1}{2}\omega$); the transition regions are characterized by the fixed frequencies 0 and $\frac{1}{2}\omega$, respectively, of the response, and by degrees of stability n_{app}/n which vary with μ and n (between 1 and a minimum).

From Fig. 5 it is clear that the flapping motion is very stable for $\mu<0.5$ at any n. This is in agreement with the conclusions reached in the preceding section for the special case of $n=1.7$.

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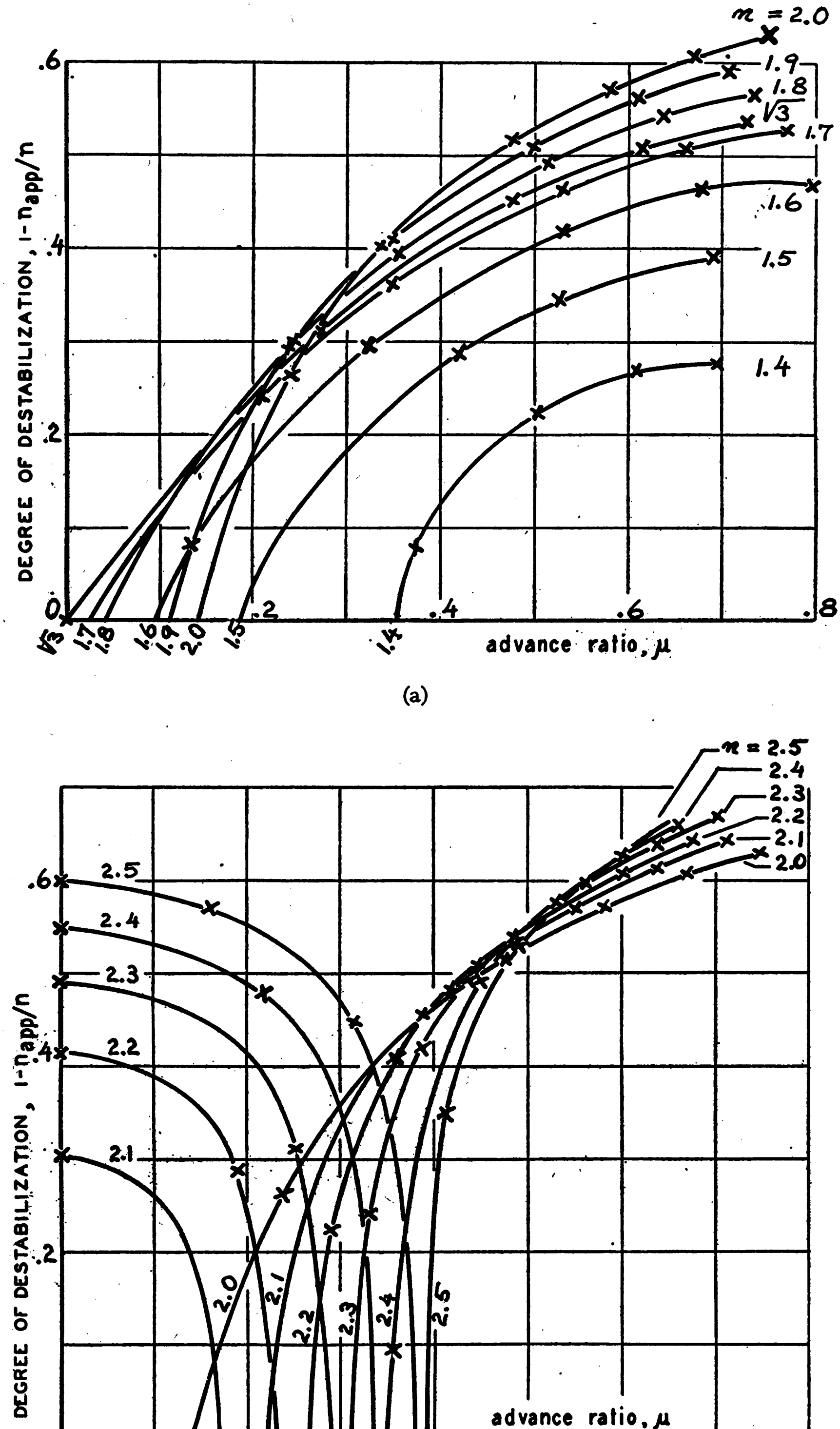
[15] For $n=1.7$, $\theta_0=0$ one finds $\mu=0.65734$, and θ_1 and θ_2 assume the values given in I, Sec. 5. The corresponding degree of destabilization is $2\sigma_r/n=0.510$.



(a)

(b)

FIGS. 3a and b. Degree of destabilization of rotor blade vs. advanced ratio at various values of the aerodynamic damping coefficient n.

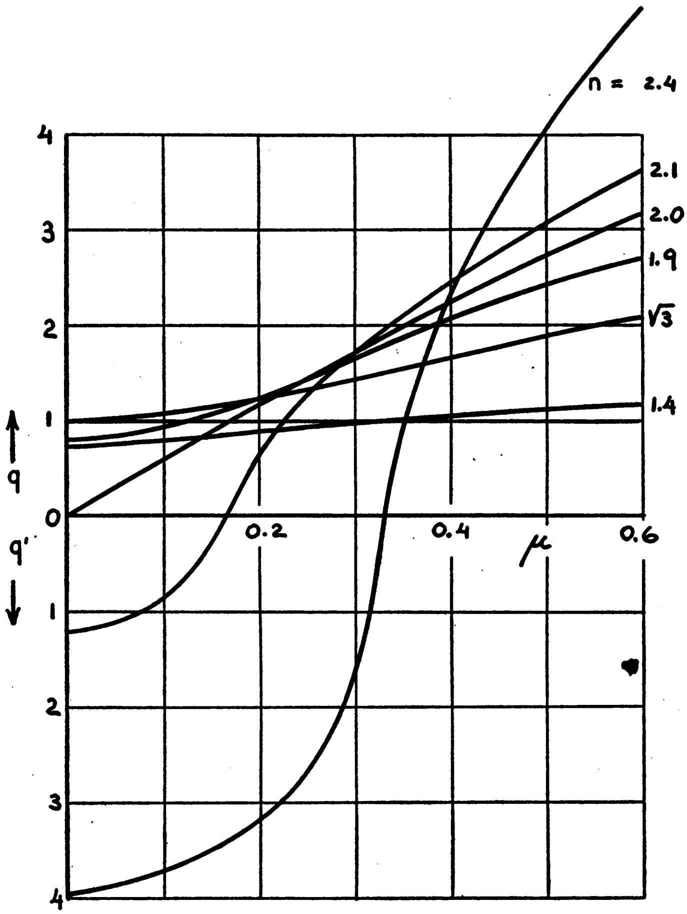


FIG. 4. The functions q and q'.

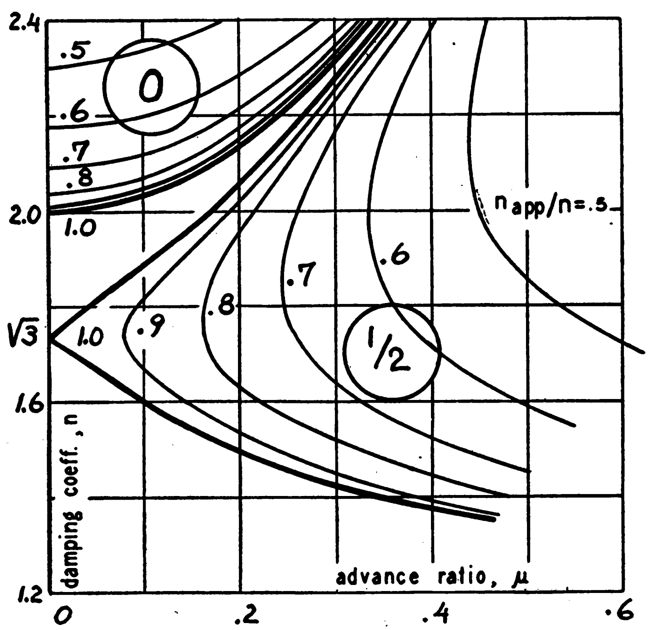


FIG. 5. Rotor blade stability diagram. Level lines indicate the relative degree of stability for various values of the damping coefficient and the advance ratio.

It may be well to recapitulate at this point the various assumptions on the basis of which the stability of flapping motion was established. The assumptions are as follows: The blades are attached to the rotor shaft by means of flapping hinges only (the freedom of oscillation of the blades in the rotor plane, about the lag hinges, is thus disregarded), and the distance of the hinges from the rotor axis is negligible. The hinge axes are perpendicular to the rotor shaft and to the respective blade span axes. The blades are untwisted, and are rigid both in bending and in twist. The chord and the lift coefficient are constant along the blades. The uniform motion of the helicopter is not affected by the oscillations of the blades. Of the aerodynamic

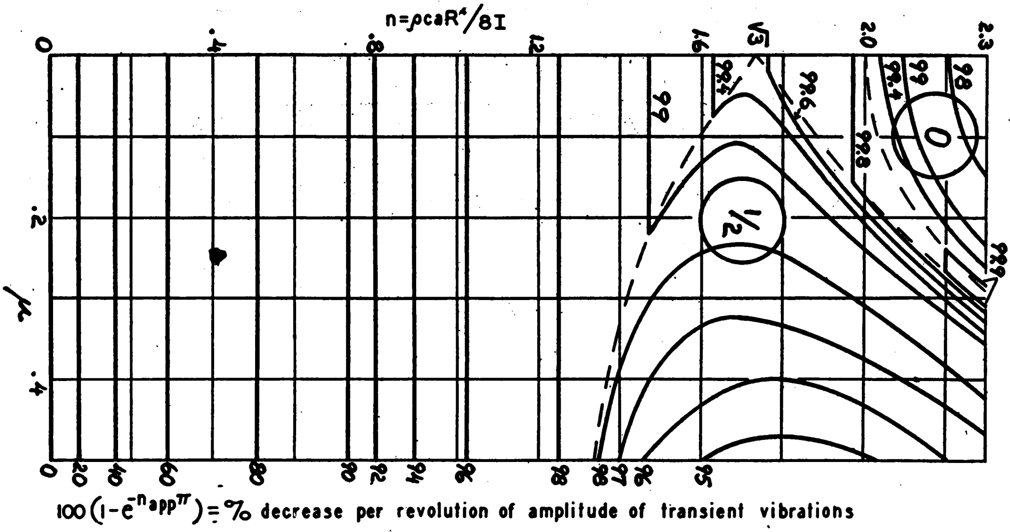


FIG. 6. Stability chart of rotor blade flapping motion.

forces which act on the blades only the velocity square lift forces are taken into account. Air velocities in the direction of the blade span are neglected. The field of induced velocities is assumed as constant over the rotor disc, and interference effects between the various blade elements are disregarded. All angles involved (but ψ) are small, in particular the angle of attack is always less than the stalling angle.

Although the above assumptions are violated by a helicopter in many respects, experience has shown that the steady state solution (10) gives a description of the flapping motion of the blades which, for $\mu \lesssim 0.35$, is in satisfactory agreement with the observations. Thus Eq. (5) can be considered as essentially correct.

From the practical point of view the plots of Fig. 6, giving the percentage decrease per revolution of the amplitude of a transient, constitute the principal result of the present paper. On the one hand the chart establishes the hitherto unsuspected and curious behavior of the flapping transients near $n=\sqrt{3}$ (the stability decreases as μ increases, the natural frequency of the flapping motion is $\frac{1}{2}\omega$ over a wide range of μ's and n's); on the other hand it also focusses attention on the sluggishness of the blades in recovering their steady state motion when the value of n is small. Blades with low n must therefore be avoided.[16]

[16] The widely held belief that "the natural frequency in flapping is equal to the speed of rotation"

The problem of blade response to control deflections leads to considerations which go beyond the question of the rate of extinction of a transient, and will therefore be taken up in a separate paper.

5. The functional relation $\sigma_r + i\sigma_i = \sigma(\mu, n)$. The characteristic exponent σ is a simple expression, (21), in the combination q (or q') of θ_0 and $\mathcal{D}$. θ_0 and $\mathcal{D}$ in turn are functions of the advance ratio μ and the damping coefficient n. As μ increases, θ_0, Eq. (19a), decreases monotonically. The variation of $\mathcal{D}$ with μ is more complicated. $\mathcal{D}$ becomes infinite for $\theta_0 = 0^2, 1^2, 2^2, \cdots$; for the flapping problem ($\theta_0 < 1$) only for $\theta_0 = 0$. Elsewhere $\mathcal{D}$ fluctuates between positive maxima and negative minima.[18] Correspondingly q maxima (σ_r maxima) alternate with q' maxima (σ_r maxima). (Cf. Fig. 4 which shows the $q(\mu)$, $q'(\mu)$ variation near the origin.) Complete stability exists only in the very narrow strips where $0 \leqq q \leqq 1$. In these regions the ratio n_{app}/n is constant at the value 1, while the frequency varies across the region by $\frac{1}{2}\omega$. In the remainder of the μ, n diagram the dominant frequency of the transient is an integral multiple of $\frac{1}{2}\omega$, and one has either $0 < 2\sigma_r/n < 1$ (transition region), or $2\sigma_r/n > 1$ (instability region).

The relationships $\sigma_r + i\sigma_i = \sigma(\mu, n)$ can be best appraised by considering the variation of σ with μ for one particular value of n. Such a survey is carried out below, first for $n = 2.4$ in the μ-range 0 to 0.5 with the aid of Fig. 5; then by an approximate method for $n = \sqrt{3}$ in the μ-range 0 to 8.0 wth the aid of Figs. 7, 8, 9. Although Eq. (5) ceases to give an approximate description of blade flapping when the stalled areas of the blades are large (when $\mu \gtrsim 0.5$), the equation is of interest, per se, also in this case, and an investigation of the behavior of the solutions of (6) for large values of μ aids in the formation of a clear picture of the variation of σ with μ and n.

For $n = 2.4$, $\mu = 0$, the transients β_1 and β_2 are solutions of Eq. (13), and thus can be written as

$$\beta_1 = \exp\left(-\tfrac{1}{2}n + \sqrt{\tfrac{1}{4}n^2 - 1}\right)\psi = \exp\left(-1.2 + 0.663\right)\psi, \tag{32a}$$

$$\beta_2 = \exp\left(-\tfrac{1}{2}n - \sqrt{\tfrac{1}{4}n^2 - 1}\right)\psi = \exp\left(-1.2 - 0.663\right)\psi. \tag{32b}$$

Therefore $n_{app}/n \doteq 0.447$ and $\sigma_i = 0$. The same result is obtained also from the general theory. Since for $\mu = 0$ one has $\mathcal{D} = 1$, therefore

$$q' = \sinh \pi\sqrt{\tfrac{1}{4}n^2 - 1} = 3.95, \tag{32c}$$

$$\sigma = \frac{1}{\pi}\log\left(q' + \sqrt{q'^2 + 1}\right) = 0.663, \tag{32d}$$

and again

$$1 - 2\sigma_r/n = 0.447, \qquad \sigma_i = 0. \tag{32e}$$

(quotation from Bennett's Princeton University Lecture Notes *On the physical principles of the helicopter*, p. 1, 1944) is largely the result of an unfortunate misuse of the expression "natural frequency" for the frequency at which the phase displacement between impressed force and response is 90°. For many slightly damped systems with constant parameters this frequency nearly coincides with the frequency of the transient,[17] but sometimes, as in the present case, there is considerable divergence between the two.

[17] J. P. Den Hartog, *Mechanical vibrations*, p. 63, McGraw-Hill, 1940.

[18] Oscillation theorem, cf. M. J. O. Strutt, *Lamésche, Mathieusche und verwandte Funktionen in Physik und Technik*, p. 15, Springer, 1933.

As μ increases, q' decreases until it reaches zero at $\mu \sim 0.334$. Correspondingly σ_r also decreases to 0, the *characteristic curve* (the common boundary of the transition and the complete stability regions) is breached, q' changes into q, and the complete stability region is entered. Between $\mu \sim 0.334$ and $\mu \sim 0.352$, q varies from 0 to 1. Correspondingly $\sigma_r = 0$, while σ_i varies from 0 to $\frac{1}{2}$. In the transition region that follows $\sigma_i = \frac{1}{2}$, and σ_r varies from 0 to a maximum, then back to zero. (The diagram does not extend sufficiently to the right to show this variation.)

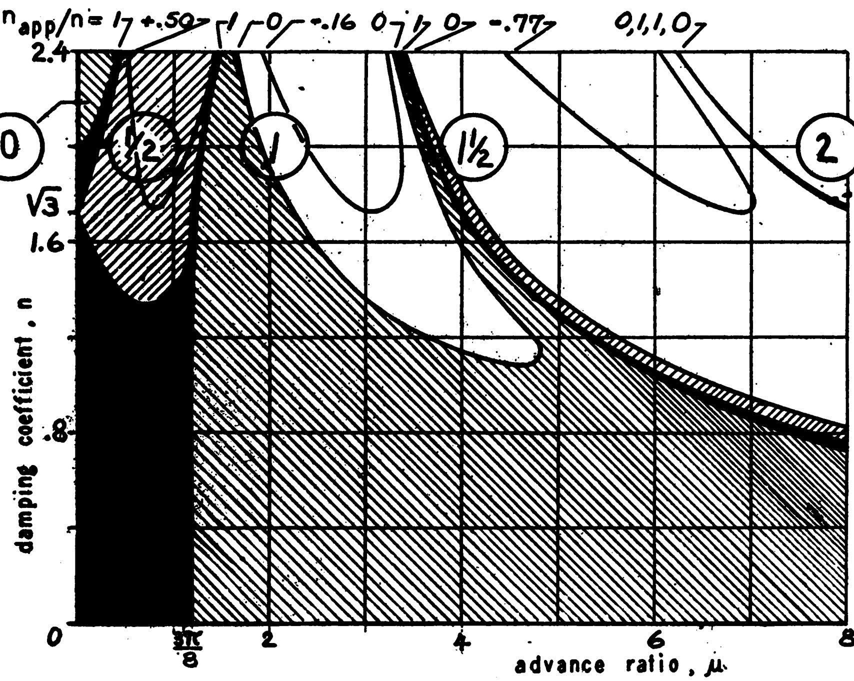


FIG. 7. Stability of the equation $\ddot{\beta}+n\omega\dot{\beta}+\omega^2(1+\frac{4}{3}n\mu\cos\omega t)\beta=0$ by the "twin-ripple method" of approximation. Complete stability regions are blackened, transition regions (bounded by the curves $n_{app}/n=1$ and $n_{app}/n=0$) are lightly shaded, and unstable regions are left blank. The encircled numbers indicate the dominant frequency of the transient (in units of ω) in the respective instability and transition regicns.

Fig. 7 shows qualitatively the conditions which hold for Eq. (6) at large values of μ. This diagram was obtained by replacing the variable coefficients of $\dot{\beta}$ and β in Eq. (6) by their constant averages first in the $\psi = -\pi/2$ to $+\pi/2$ interval, then in the $\psi = \pi/2$ to $3\pi/2$ interval. The resulting pair of differential equations was solved by the method of Meissner[19] (better known as the Van der Pol-Strutt "rectangular ripple" method[20,21]) which was adapted to damped systems by Shao Wen Yuan and the writer.[22]

[19] Strutt, ibid., p. 39.

[20] Den Hartog, loc. cit., p. 389.

[21] S. Timoshenko, *Vibration problems in engineering*, p. 164, D. Van Nostrand, 1937.

[22] G. Horvay and S. W. Yuan, *Stability of rotor blade flapping motion when the hinges are tilted. Generalization of the "rectangular ripple method" of solution*, to be published soon. The procedures used in obtaining diagrams like Figs. 7, 8, 9 are also discussed in this paper.

In Fig. 7 the regions of complete stability are blackened, the transition regions are lightly shaded, and the regions of instability are left blank. Consider the ordinate $n=\sqrt{3}$. For $\mu=0$ one has the solution

$$\beta_1 = \cos \tfrac{1}{2}\psi, \qquad \beta_2 = \sin \tfrac{1}{2}\psi, \qquad n_{app}/n = 1; \tag{33}$$

the transition region (½) extends into the point $(\mu, n)=(0, \sqrt{3})$ of the ordinate axis. As μ increases, the stability decreases until a minimum of $n_{app}/n \sim 0.50$ is reached around $\mu \sim 0.85$. Thereafter the stability again increases.

In the present approximation of Eq. (6) complete stability ($\sigma_r=0$) sets in at $\mu \sim 1.24$ and ceases at $\mu \sim 1.28$. In the meantime σ_i has increased from $\frac{1}{2}$ to 1. In the region $\sigma_i=$ (1) (the dominant frequency of the transient is now ω) σ_r varies from 0 at $\mu \sim 1.28$ to a maximum of ~ 1.0 at $\mu \sim 3.2$ (yielding $n_{app}/n \sim -0.16$), and then back to zero at $\mu \sim 4.02$. First comes the transition interval ($1>n_{app}/n>0$) from $\mu \sim 1.28$ to $\mu \sim 2.27$. This is followed by the instability interval ($n_{app}/n<0$) from $\mu \sim 2.27$ to $\mu \sim 3.83$. A second transition interval ($1>n_{app}/n>0$) from $\mu \sim 3.83$ to $\mu \sim 4.02$ completes the (1) interval. There follows now a very narrow complete stability interval, $\mu \sim 4.02$ to $\mu \sim 4.03$, in which $\sigma_r=0$ while σ_i varies from 1 to $1\frac{1}{2}$. This is followed by the (1½) transition interval, this by the (1½) instability interval; and so on, indefinitely. Complete stability, transition, instability, transition, complete stability regions alternate, ad infinitum.

The stability regions, associated with $\sigma_r=0$ and varying σ_i, become ever narrower, the instability regions, associated with $\sigma_i=$const. and varying σ_r become wider and graver. The orders (0), (½), (1), $\cdots$ of the transition and instability regions are marked in the diagram. The regions (0), (1), (2), $\cdots$ can be called *even ordered regions* (the dominant frequency of the transient is $0, 2, 4, \cdots \times \frac{1}{2}\omega$); the regions (½), (1½), $\cdots$ can be called *odd-ordered regions* (the dominant frequency of the transient is $1, 3, \cdots \times \frac{1}{2}\omega$). The boundary curves $n_{app}/n=1$ and 0 are also indicated in Fig. 7. Level curves which touch the $n=\sqrt{3}$ horizontal line complete the picture.

It is rather interesting to note that although Eq. (6) reduces to a damped Mathieu equation when μ is very small ($\mu^2 \sim 0$), Fig. 7 shows a departure from the conventional stability diagram given for these equations. In the first place, the present diagram plots n vs. μ, whereas conventionally one plots θ_0 vs. μ. More fundamental is the change in behavior near the origin. Since the oscillating terms of the spring force $s(t)\beta$ tend to zero as $n \to 0$, no transition region, like the one at $n=\sqrt{3}$, $\mu=0$, $\theta_0=\frac{1}{2}$, wedges into the origin $n=\mu=0$, $\theta_0=1$. A second comment is also in order. The twin-ripple approximation used in obtaining Fig. 7 accounts in an approximate manner for the stiffness fluctuations produced by the $\frac{4}{3}\omega^2\mu n \cos \omega t \cdot \beta$ spring term of Eq. (6). In the process of averaging the coefficients of Eq. (6) both the $\frac{4}{3}\omega\mu n \sin \omega t \cdot \dot{\beta}$ damping term and the $\dot{\omega}^2\mu^2 n \sin 2\omega t \cdot \beta$ spring term are eliminated. However for large μ, the μ^2 term of $s(t)$ has the principal effect. This effect can also be evaluated by a "twin-ripple" approximation, provided one first omits the two μ-proportional terms of $\mathcal{J}(\beta)$, and then averages the μ^2 term in the intervals 0 to $\pi/2$ and $\pi/2$ to π. The resulting stability conditions are indicated in Fig. 8 for the value $n=\sqrt{3}$.

The diagram indicates that in the absence of the terms $\frac{4}{3}\omega\mu n \sin \psi\dot{\beta}$ and $\frac{4}{3}\omega^2\mu n \cos \psi\beta$ of $\mathcal{J}(\beta)$ the region of complete stability extends from $\mu=0$ to about $\mu \sim 1$. For $\mu>1$ the system is mostly unstable. Comparison of Figs. 7 and 8 indicates that the high order instability regions crowd much closer to the origin than was sur-

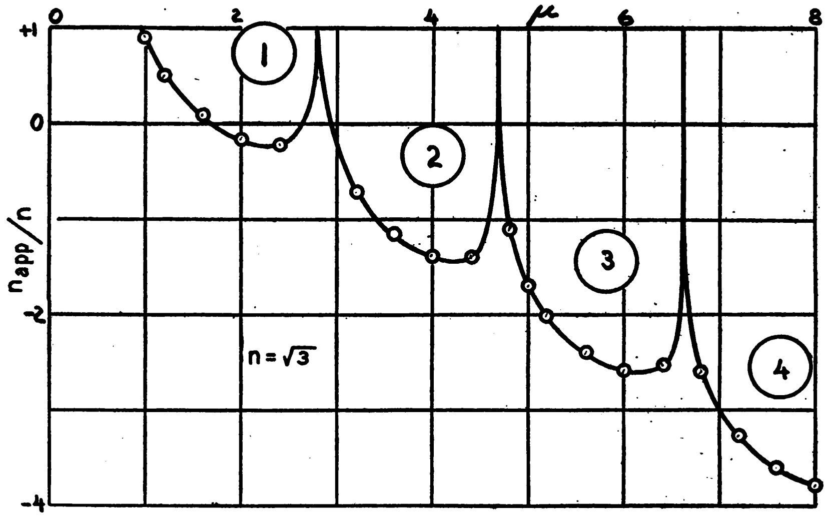


FIG. 8. Instability troughs of the equation $\ddot{\beta}+n\omega\dot{\beta}+\omega^2(1+n\mu^2 \sin 2\omega t)\beta=0$, $(n=\sqrt{3})$, by the twin-ripple method of approximation; $n_{app}/n=1$ represents complete stability, $0<n_{app}\ n<1$ represents transition, $n_{app}/n<0$ represents instability. The calculated points are indicated by small circles. The encircled numbers indicate the "orders" of the regions.

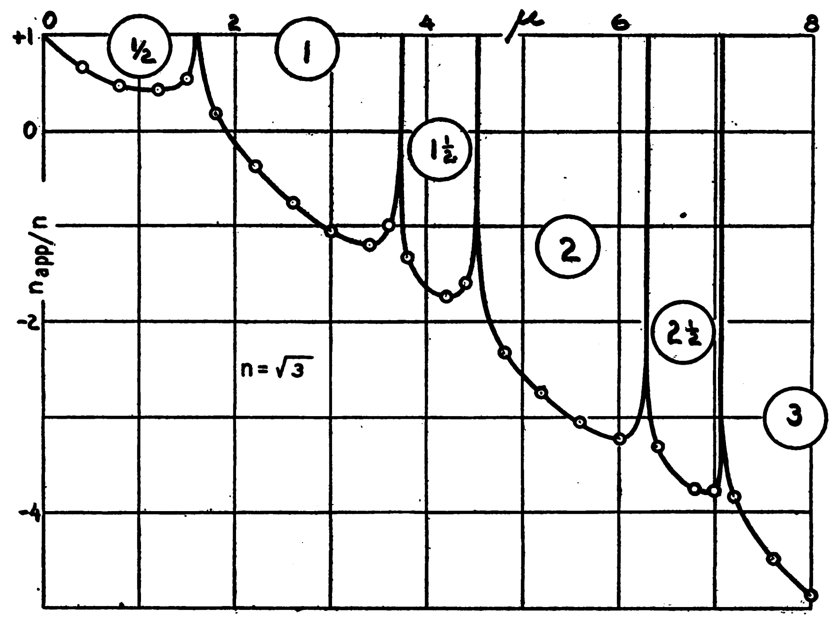


FIG. 9. Instability troughs of Eq. (6) for $n=\sqrt{3}$ by a "4-ripple approximation." The calculated points are indicated by small circles. The encircled numbers indicate the "orders" of the regions.

mised on the basis of Fig. 7. Moreover the instability caused by the μ^2 spring term is much graver than that caused by the first power μ term.

It is to be expected that in an exact solution of Eq. (6) the first power μ terms determine the stability conditions for $\mu<1$, whereas the μ^2 term is principally responsible for the stability conditions at $\mu>1$. This is born out by Fig. 9. Fig. 9 was obtained by use of a "4-ripple" approximation. (The interval 0 to 2π is divided into 4 equal parts and the coefficients $p(t)$ of $\dot\beta$ and $s(t)$ of β are replaced by their averages in each interval.) This approximation takes into account the effect of all the oscillating coefficients of Eq. (6). The figure shows that for $\mu>1$ the odd ordered instability regions (which are produced solely by the μ-proportional terms) are of minor importance as compared with the even ordered instability regions.

6. Resonance.[23] The preceding sections were concerned mainly with the transient solutions of Eq. (5). The present section deals with the steady state response.

In general, the assumption (10) leads to an equation system (11) which can be solved for the Fourier coefficients $a_0, a_1, \cdots, a_N, b_N, \cdots$. When however the determinant Δ of the system vanishes, then the Ansatz (10) fails. In this case

$$\begin{aligned}\beta_0 &= a_0 + a_1 \cos \omega t + b_1 \sin \omega t + a_2 \cos 2\omega t + \cdots \\ &\quad + t[c_0 + c_1 \cos \omega t + d_1 \sin \omega t + c_2 \cos 2\omega t + \cdots]\end{aligned} \tag{34}$$

is the correct form for the solution of the equation. Note that *the response becomes infinite even though the system is damped.*

What are the conditions for $\Delta=0$? Evidently, $\Delta=0$ implies that the homogeneous equation (6) has a solution which is of the form (10). This in turn implies

$$\sigma = \tfrac{1}{2}n \pm mi \qquad (m = 0, 1, 2, \cdots). \tag{35}$$

Thus, *resonance occurs on the $n_{app}/n=0$ boundary lines of the even ordered transition and instability regions.* From Fig. 5 it is seen that for helicopters resonance excitation is just as remote as instability.[24]

There are two more interesting questions which arise in connection with resonance. (1) How would conditions change if the exciting force on the right side of Eq. (5) were of the form

$$E(t) = \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} \nu t, \quad \text{where} \quad \nu \neq \omega, 2\omega, \cdots ? \tag{36}$$

(2) How would conditions change if the system were undamped, i.e., if the $p(t)\dot\beta$ term were absent from Eq. (5), but the $s(t)\beta$ expression still contained oscillating terms of frequency ω?

If $\nu=\frac{1}{2}\omega$ or an odd multiple of it, then placement of

$$\beta_0 = a_1 \cos \tfrac{1}{2}\omega t + b_1 \sin \tfrac{1}{2}\omega t + a_3 \cos \tfrac{3}{2}\omega t + b_3 \sin \tfrac{3}{2}\omega t + \cdots \tag{10'}$$

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[23] This section is based on the paper by G. Kotowski, *Lösungen der inhomogenen Mathieuschen Differentialgleichung mit periodischer Störfunktion beliebiger Frequenz,* Z. angew. Math. Mech. **25**, 213, 1943. Kotowski credits introduction of the method to A. Erdelyi, Arch. Elektrotechn. **29**, 473, 1935.

[24] Since the response amplitudes are large near a resonance frequency the following question is of interest in many problems: how deep do the regions of large amplifications extend into the transition regions or perhaps stability regions? Kotowski answers the question for the equation $y''+(\lambda+\cos x)y=a_0+a_1 \cos x+a_2 \cos 2x$, by plotting the resonance curve. (λ is the variable parameter.) In the present problem the great distance of the helicopter operating range, $\mu\lesssim 0.35$, from the nearest $n_{app}/n=0$ curve eliminates the necessity for such an investigation.

into $\mathcal{J}(\beta) = E(t)$ leads to an equation system (11′), with a determinant Δ' which in general does not vanish;[25] Δ' vanishes only when the homogeneous Eq. (6) has a solution of the form (10′), i.e., when

$$\sigma = \tfrac{1}{2}n + (\tfrac{1}{2} + m)i \qquad (m = 0, \pm 1, \pm 2, \cdots) \tag{35′}$$

Thus, *for $\nu =$ odd multiple of $\frac{1}{2}\omega$, resonance, occurs on the $n_{app}/n = 0$ boundary lines of the odd ordered transition and instability regions.* In this case expression (10′) plus t times a similar expression is the correct "Ansatz" for the forced response.

When ν is a fractional multiple of $\frac{1}{2}\omega$, then

$$\beta_0 = \cos \nu t[a_0 + a_1 \cos \omega t + b_1 \sin \omega t + a_2 \cos 2\omega t + \cdots] + \sin \nu t[c_0 + c_1 \cos \omega t + d_1 \sin \omega t + c_2 \cos 2\omega t + \cdots] \tag{10″}$$

is the correct assumption for the particular integral. The associated equation system (11″) has a determinant Δ'' which vanishes only when the homogeneous equation (6) has a solution of the form (10″), i.e., when

$$\sigma = \tfrac{1}{2}n + \left(\frac{\nu}{\omega} + m\right)i \qquad (m = 0, \pm 1, \pm 2, \cdots). \tag{35″}$$

But from Eq. (21) it is obvious that σ is never of this form, unless $n = 0$. Thus, *when 2ν is a fractional multiple of the parametric frequency ω, resonance can occur only when the system is undamped.* In such a case the *response occurs along the lines $\sigma_i = m + \nu/\omega$ of the complete stability regions,* and (10″) plus t times an expression of type (10″) is the correct form for the solution.

It is to be noted that a single frequency ν excites an infinity of response frequencies ν, $\nu \pm \omega$, $\nu \pm 2\omega$, $\cdots$, and thus in a system with variable spring and damping coefficients one obtains an infinity of resonance peaks in contrast with a system characterized by a 2nd order differential equation with constant coefficients which shows only one resonance peak.

The absence of damping also modifies the results for the cases when ν is an even or odd multiple of $\frac{1}{2}\omega$. The $n_{app}/n = 0$ curves now coincide with the characteristic curves of the stability diagram, and

$$\sum_0^\infty [(a_k + b_k t + c_k t^2) \cos k\omega t + (d_k + e_k t + f_k t^2) \sin k\omega t] \tag{37a}$$

and

$$\sum_0^\infty [(a_k + b_k t + c_k t^2) \cos (2k + 1)\omega t + (d_k + e_k t + f_k t^2) \sin (2k + 1)\omega t, \tag{37b}$$

respectively, are the proper forms for the resonant response.

The proofs of the expansions (34), etc. are based on the well-known formula[26]

$$J\epsilon^{-1}(\psi)\beta_0(\psi) = v_2(\psi) \int^\psi v_1(x)\epsilon^{-1}(x)E(x)dx - v_1(\psi) \int^\psi v_2(x)\epsilon^{-1}(x)E(x)dx \tag{38a}$$

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[25] There is no need to write out the equation system.

[26] Frank-Mises, *Differential und Integralgleichungen der Physik*, Vol. I, p. 300, Fr. Vieweg u. Sohn, 1935.

where $\epsilon(\psi)$ is given by Eq. (17a); $v_1(\psi)$, $v_2(\psi)$ are the two linearly independent solutions of Eq. (18);

$$J = v_1 v_2' - v_2 v_1' = \text{const.}; \tag{38b}$$

and $E(\psi)$ is the exciting function.

When v_2 is of the form

$$e^{-n\psi/2} \sum g_k e^{ik\psi} \tag{39a}$$

and the exciting function is of the form

$$E(\psi) = \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} j\psi \qquad (j = 0, 1, 2, \cdots) \tag{39b}$$

then the integrand

$$v_2(x)\epsilon^{-1}(x)E(x)$$

is a series which contains a constant term. This term integrates into ψ, and, multiplied by $-\epsilon(\psi)v_1(\psi)/J$, yields the non-periodic

$$t\sum [c_k \cos k\psi + d_k \sin k\psi]$$

terms of the expansion (34). The remainder of (38a) yields the periodic terms of (34). This proves (34). The proof of the other expansions is similar.[27]

When the exciting function is of the form

$$E(\psi) = \psi \cos \psi \tag{40}$$

then, by (38), the response is of the form

$$\beta_0(\psi) = \sum_0^\infty [(a_k + c_k\psi) \cos k\psi + (b_k + d_k\psi) \sin k\psi] \tag{41}$$

Substitution of (41) into the non-homogeneous equation $\mathcal{J}(\beta) = E(t)$, and a comparison of the coefficients of $\cos k\psi$, $\sin k\psi$, $\psi \cos k\psi$, $\psi \sin k\psi$ on right and left, yields an infinite system of equations (11''') for the coefficients a_k, b_k, c_k, d_k which can be solved in the manner of Eqs. (11). Use of the expansion (41) will be made in the study mentioned at the end of Sec. 4.

[27] The details (for Mathieu's equation) are given in Kotowski's paper. Kotowski also points out that there are certain exceptional cases. For instance, when a homogeneous Hill equation has a solution of the form $\sum C_k \cos k\psi$ but none of the form $\sum C_k \sin k\psi$, then the latter expression is admissible as forced response to an excitation $E(\psi) = \sin \psi$.