MULTIPLE REFLECTIONS BY PLANE MIRRORS*

BY

L. B. TUCKERMAN

National Bureau of Standards

The article by Synge[1] called to my mind a more general treatment of reflections by plane mirrors which I saw when, in 1898, as a sophomore in college, I was studying quaternions. I cannot cite the reference because I took no particular note of it at the time. I never had occasion to use the theory until 1921 when I needed it to determine the errors in my optical lever system,[2] which consists of an autocollimated three-mirror combination.

My attention has since been called to a book by Silberstein[3] which treats the same problem with a modified Gibbs notation. Silberstein's Eq. (25) on p. 22 is identical in form with Synge's Eq. (4.1) but his later treatment is different and he does not note, a Synge does, that the formulae represent rotations.

The quaternion treatment of the problem is so simple that I had no trouble in reconstructing it, at least in its essential features. Comparing it with Synge's and Silberstein's treatments strengthens my feeling that Heaviside, Föppl, Gibbs and their followers did the world a disservice in rejecting Hamilton's simple, concise notation and algorithms for the cumbersome complexity of the vector notations currently taught. It therefore occurred to me that the readers of the Quarterly of Applied Mathematics might be interested in seeing an outline of the quaternion treatment.

When the first draft of this paper was submitted for publication, the reviewer called attention to a recent article by Coxeter.[4] He gives a very general quaternion treatment of reflection and rotation in three and four dimensions but does not discuss special cases in detail.

In his introduction Coxeter states: "Apparently none of these men thought of considering first the simpler operation of *reflection* and deducing a rotation as the product of two reflections."

The article which I read, presented a rotation as the simpler operation and considered a reflection as a rotatory inversion, compounded of a rotation and a central inversion. This is the order used in the following discussion.

The underlying basic theorem of reflection at plane mirrors is not confined to rays of light incident upon the mirror but applies to the whole image space. It may be stated as follows: *The image space formed by a plane mirror can be derived from the object space by rotating the object space 180° about any normal to the plane, followed by a central inversion about the foot of the normal.*

In the quaternion notation the square of a vector is negative instead of positive. This, as Hamilton showed, makes it possible to treat a "quaternion," the sum of a scalar and a vector, as a single quantity in addition, multiplication, division and

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[1] J. L. Synge, *Reflection in a corner formed by three plane mirrors*, this Quarterly **4**, 166–176 (1946).

[2] L. B. Tuckerman, *Optical strain gages and extensometers*, Proc. A.S.T.M. **23**, Part II, 602–610 (1923).

[3] Ludwig Silberstein, *Simplified method of tracing rays through any optical system of lenses, prisms and mirrors*. London, Longmans, Green and Co., 1918.

[4] H. S. M. Coxeter, *Quaternions and reflections*, Amer. Math. Monthly **53**, 36–146 (1946).

differentiation instead of introducing the multiplicity of scalar and vector operations of the notations of Heaviside, Föppl and Gibbs.

Computations with quaternions are thus carried out with the simplicity of ordinary algebraic computations. The only complexity, if it may be called a complexity, is the necessity, common to all vector notations, of remembering that in general $\mathbf{ab} \neq \mathbf{ba}$. The separation of the quaternion results into their scalar and vector parts can be made at any desired stages in the process by the operators, S = scalar part and V = vector part.

The quaternion formula for rotation of a vector $\mathbf{i}$ through an angle ϑ about a unit vector $\mathbf{n}$ as axis is:

$$\mathbf{i}' = q\mathbf{i}q^{-1}, \tag{1}$$

where the quaternion

$$q = \cos \tfrac{1}{2}\vartheta + \mathbf{n} \sin \tfrac{1}{2}\vartheta \tag{2}$$

is the sum of $Sq = \cos \frac{1}{2}\vartheta$ and $Vq = \mathbf{n} \sin \frac{1}{2}\vartheta$.

If $\vartheta = 180°$, this reduces to:

$$\mathbf{i}' = \mathbf{nin}^{-1}. \tag{3}$$

The central inversion merely requires changing the sign of all vectors passing through the foot of the normal, giving for the effect of a single plane mirror

$$\mathbf{i}' = -\mathbf{nin}^{-1} = \mathbf{nin}. \tag{4}$$

Equations (1) and (2) are identical in form with Coxeter's (p. 139) Theorem 3.2 and Eq. (4) with his Theorem 3.1.

The equivalence of Eq. (4) with Synge's Eq. (4.1) may readily be verified by expressing it in quaternion notation. Writing $(\mathbf{n}\cdot\mathbf{i}) = S\mathbf{ni}$ and $\mathbf{n}^2 = -1$, gives:

$$\mathbf{i}' = \mathbf{i} - 2\mathbf{n}(\mathbf{n}\cdot\mathbf{i}) = \mathbf{i} + 2\mathbf{n}S\mathbf{ni} = \mathbf{i} + \mathbf{n}(\mathbf{ni} + \mathbf{in}) = (1 + \mathbf{n}^2)\mathbf{i} + \mathbf{nin} = \mathbf{nin}. \tag{5}$$

The effect of two mirrors is obtained by simple iteration:

$$\mathbf{i}'' = \mathbf{n}_2\mathbf{i}'\mathbf{n}_2 = \mathbf{n}_2\mathbf{n}_1\mathbf{i}\mathbf{n}_1\mathbf{n}_2 = q\mathbf{i}q^{-1}. \tag{6}$$

The second central inversion cancels the first, so the result is a pure rotation. The angle ϑ and the direction of the axis $\mathbf{a}$ are given by

$$\cos \tfrac{1}{2}\vartheta + \mathbf{a} \sin \tfrac{1}{2}\vartheta = q = \mathbf{n}_2\mathbf{n}_1 = S\mathbf{n}_2\mathbf{n}_1 + V\mathbf{n}_2\mathbf{n}_1. \tag{7}$$

Here $S\mathbf{n}_2\mathbf{n}_1 = -\cos(\mathbf{n}_2, \mathbf{n}_1)$, where $(\mathbf{n}_2, \mathbf{n}_1)$ represents the angle between the normals to the two mirrors, and $V\mathbf{n}_2\mathbf{n}_1$ is a vector with a length equal to $\sin(\mathbf{n}_2, \mathbf{n}_1)$ perpendicular to $\mathbf{n}_2$ and $\mathbf{n}_1$, and therefore parallel to the line of intersection of their planes.

Since in the reflection from one mirror we are free to choose *any normal* as the axis of rotation, it is convenient to choose some point on the line of intersection of the two planes as the common foot of the normals to the two planes. With this choice, the equations prove the familiar result: *The image space formed by successive reflections at two plane mirrors can be derived from the object space by rotating the object space through an angle* ϑ *given by* $\cos \frac{1}{2}\vartheta = -\cos(\mathbf{n}_2, \mathbf{n}_1)$ *about the line of intersection of the two planes.*

The effect of three mirrors is obtained by a further simple iteration:

$$\mathbf{i}''' = \mathbf{n}_3\mathbf{i}''\mathbf{n}_3 = \mathbf{n}_3\mathbf{n}_2\mathbf{i}'\mathbf{n}_2\mathbf{n}_3 = \mathbf{n}_3\mathbf{n}_2\mathbf{n}_1\mathbf{i}\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3 = -\, qiq^{-1}. \tag{8}$$

The third mirror introduces another central inversion. The negative sign before the last term in Eq. (8) represents the resultant central inversion. In this case we are free to choose the common point of intersection of the three planes as the common foot of the normals to the three planes. We may then state: *The image space formed by successive reflections at three plane mirrors can be derived from the object space by rotating the object through an angle, ϑ about an axis,* **a** *passing through the common point of intersection of the three planes, followed by a central inversion about that point. The angle, ϑ and the direction of the axis,* **a** *of the resultant rotatory inversion are given by*:

$$\cos \tfrac{1}{2}\vartheta + \mathbf{a} \sin \tfrac{1}{2}\vartheta = q = \mathbf{n}_3\mathbf{n}_2\mathbf{n}_1 = S\mathbf{n}_3\mathbf{n}_2\mathbf{n}_1 + V\mathbf{n}_3\mathbf{n}_2\mathbf{n}_1. \tag{9}$$

Now

$$\begin{aligned} \cos \tfrac{1}{2}\vartheta = Sq = S\mathbf{n}_3\mathbf{n}_2\mathbf{n}_1 &= -\, S\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3 = S\mathbf{n}_2\mathbf{n}_1\mathbf{n}_3 = -\, S\mathbf{n}_3\mathbf{n}_1\mathbf{n}_2 \\ &= S\mathbf{n}_1\mathbf{n}_3\mathbf{n}_2 = -\, S\mathbf{n}_2\mathbf{n}_3\mathbf{n}_1. \end{aligned} \tag{10}$$

If then, $\frac{1}{2}\vartheta$ is defined as a positive acute angle, one half the angle of rotation will have one of the four values, $\pm\frac{1}{2}\vartheta$ and $(180° \pm \frac{1}{2}\vartheta)$. The angle of rotation will therefore have one of the two values, $\pm\vartheta$. The magnitude of the angle of rotation is thus the same for all orders of incidence and its sign changes when the cyclic order is reversed.

Synge's convenient choice makes the subscript of each **a** correspond to the subscript of the **n** of the second mirror on which the ray falls. Then by an elementary expansion formula:

$$\mathbf{a}_1 \sin \tfrac{1}{2}\vartheta = V\mathbf{n}_2\mathbf{n}_1\mathbf{n}_3 = V\mathbf{n}_3\mathbf{n}_1\mathbf{n}_2 = -\, \mathbf{n}_1 S\mathbf{n}_2\mathbf{n}_3 + \mathbf{n}_2 S\mathbf{n}_3\mathbf{n}_1 + \mathbf{n}_3 S\mathbf{n}_1\mathbf{n}_2. \tag{11}$$

Letting

$$\begin{aligned} M_1 = -\, S\mathbf{n}_2\mathbf{n}_3 = \cos(\mathbf{n}_2, \mathbf{n}_3), \qquad M_2 = -\, S\mathbf{n}_3\mathbf{n}_1 = \cos(\mathbf{n}_3, \mathbf{n}_1) \\ \text{and} \quad M_3 = -\, S\mathbf{n}_1\mathbf{n}_2 = \cos(\mathbf{n}_1, \mathbf{n}_2), \end{aligned} \tag{12}$$

we have

$$\begin{aligned} -\, \mathbf{a}_1 \sin \tfrac{1}{2}\vartheta &= -\, M_1\mathbf{n}_1 + M_2\mathbf{n}_2 + M_3\mathbf{n}_3, \\ -\, \mathbf{a}_2 \sin \tfrac{1}{2}\vartheta &= +\, M_1\mathbf{n}_1 - M_2\mathbf{n}_2 + M_3\mathbf{n}_3, \\ -\, \mathbf{a}_3 \sin \tfrac{1}{2}\vartheta &= +\, M_1\mathbf{n}_1 + M_2\mathbf{n}_2 - M_3\mathbf{n}_3. \end{aligned} \tag{13}$$

These are obviously equivalent to Synge's Eqs. (3.10) and show that in those equations

$$\pm\, 1/k = \sin \tfrac{1}{2}\vartheta. \tag{14}$$

Adding the second and third of Eqs. (13) gives

$$-\, (\mathbf{a}_1 + \mathbf{a}_3) \sin \tfrac{1}{2}\vartheta = 2M_1\mathbf{n}_1, \tag{15}$$

so that the three vectors $\mathbf{n}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are coplanar. Similarly, so are $\mathbf{n}_2$, $\mathbf{a}_3$ and $\mathbf{a}_1$ and $\mathbf{n}_3$, $\mathbf{a}_1$ and $\mathbf{a}_2$.

The angles which $\mathbf{n}_1$ makes with $\mathbf{a}_2$ and $\mathbf{a}_3$ are given by

$$\begin{aligned} \sin \tfrac{1}{2}\vartheta \cos(\mathbf{n}_1\mathbf{a}_2) &= -\, \sin \tfrac{1}{2}\vartheta S\mathbf{n}_1\mathbf{a}_2 = M_1 + M_2M_3 - M_2M_3 = M_1 \\ &= -\, \sin \tfrac{1}{2}\vartheta S\mathbf{n}_1\mathbf{a}_3 = \sin \tfrac{1}{2}\vartheta \cos(\mathbf{n}_1\mathbf{a}_3). \end{aligned} \tag{16}$$

Thus $\mathbf{a}_2$ and $\mathbf{a}_3$ are vectors coplanar with $\mathbf{n}_1$ making the same angle with it. Since they are not identical they must lie on opposite sides of $\mathbf{n}_1$. This proves that $\mathbf{n}_1$ bisects the side $\overline{\mathbf{a}_2\mathbf{a}_3}$ of the spherical triangle $\overline{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3}$, with similar results for $\mathbf{n}_2$ and $\mathbf{n}_3$.

Squaring the first of Eqs. (13) and remembering that $\mathbf{a}_1^2=\mathbf{n}_1^2=\mathbf{n}_2^2=\mathbf{n}_3^2=-1$, gives:

$$-\sin^2\tfrac{1}{2}\vartheta = -M_1^2 - M_2^2 - M_3^2 + 2M_2M_3S\mathbf{n}_2\mathbf{n}_3 - 2M_3M_1S\mathbf{n}_3\mathbf{n}_1 - 2M_1M_2S\mathbf{n}_1\mathbf{n}_2. \quad (17)$$

Substituting from Eqs. (12) gives:

$$\sin^2\tfrac{1}{2}\vartheta = M_1^2 + M_2^2 + M_3^2 - 2M_1M_2M_3, \quad (18)$$

which is equivalent to Synge's Eq. (3.9).

Since, except for sign, ϑ is known from Eq. (18), the directions of the axes $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are, except for sense, given by Eqs. (13). The sense of each may be arbitrarily chosen. When it is chosen, the sign of $\sin\tfrac{1}{2}\vartheta$ for each axis is given. Combined with the sign of $\cos\tfrac{1}{2}\vartheta$ determined from the order of incidence by Eqs. (10), this determines the sign of ϑ for each of the six orders.

These formulae give in essence all that is contained in Synge's three theorems, and the diagrams he gives can readily be constructed from them. Numerical computations using them are straightforward. They are especially simple when one or more of the angles between the mirror normals differ so little from 90° that small quantities of the second order may be neglected.

In collimated systems, in which a group of any number of plane mirrors lies wholly within a region of parallel rays, any parallel displacement of the image space of the mirrors with respect to their object space does not affect the relative relation of the final image to the original object. For such systems the effect of any number of plane mirrors may be regarded as a rotation or rotatory inversion in which merely the angle of rotation and the direction of the axis are significant. These can also be found by simple iteration of Equation (4):

$$\mathbf{i}^{(k)} = \mathbf{n}_k\mathbf{n}_{k-1}\cdots\mathbf{n}_2\mathbf{n}_1\mathbf{i}\mathbf{n}_1\mathbf{n}_2\cdots\mathbf{n}_{k-1}\mathbf{n}_k, \quad (19)$$

giving

$$(-1)^kq = (-1)^k(\cos\tfrac{1}{2}\vartheta + \mathbf{a}\sin\tfrac{1}{2}\vartheta) = S\mathbf{n}_k\mathbf{n}_{k-1}\cdots\mathbf{n}_2\mathbf{n}_1 + V\mathbf{n}_k\mathbf{n}_{k-1}\cdots\mathbf{n}_2\mathbf{n}_1, \quad (20)$$

to determine the angle ϑ and the direction of the axis of rotation, $\mathbf{a}$ of the rotation (k, even) or rotatory inversion (k, odd) of the object space into the image space after reflection from k mirrors. The computations are just as simple in form as those for three mirrors but of course the numerical work increases in proportion to the number of mirrors.

An important special case of an even number of mirrors is a nondeviating erecting system, one which turns the image formed by a lens system through 180° without deviating rays parallel to the axis of the lens system. Neglecting, for the time being, the axial and lateral displacement of the image, the condition for this is that $\vartheta=180°$ or

$$\cos\tfrac{1}{2}\vartheta = S\mathbf{n}_k\mathbf{n}_{k-1}\cdots\mathbf{n}_2\mathbf{n}_1 = 0. \quad (21)$$

This may be written:

$$S\mathbf{n}_k(S\mathbf{n}_{k-1}\cdots\mathbf{n}_2\mathbf{n}_1 + V\mathbf{n}_{k-1}\cdots\mathbf{n}_2\mathbf{n}_1) = S\cdot\mathbf{n}_kV\mathbf{n}_{k-1}\cdots\mathbf{n}_2\mathbf{n}_1 = 0. \quad (22)$$

Since $V\mathbf{n}_{k-1} \cdots \mathbf{n}_2\mathbf{n}_1$ is parallel to the axis of the rotatory inversion produced by the first $(k-1)$ mirrors, this merely requires that $\mathbf{n}_k$ shall be perpendicular to that axis. We may therefore state: *Any combination of an odd number of plane mirrors, with their normals arbitrarily oriented, may be converted to a nondeviating erecting system by adding another plane mirror whose normal is perpendicular to the axis of the rotatory inversion produced by the combination.*

An interesting, but impractical, special case is the combination of a cube corner with a fourth mirror in any orientation in which its normal makes an acute angle with the rays emerging from the cube corner. Since the quaternion product of three mutually perpendicular vectors is ± 1, it forms a nondeviating erecting system whose axis is parallel to the normal to the fourth mirror.

The scalar part of the product of any number of quaternions (including vectors) is the same, so long as the cyclic order is maintained. Consequently the angle of rotation produced by any number of mirrors is the same for the same cyclic order of incidence, but the axis will in general be different for different orders. This permits stating a sufficient, but not necessary, special condition:

$$\begin{aligned} 0 &= S\mathbf{n}_k \cdots \mathbf{n}_j \cdots \mathbf{n}_1 \\ &= S\mathbf{n}_1\mathbf{n}_k S\mathbf{n}_{k-1} \cdots \mathbf{n}_j \cdots \mathbf{n}_2 + S\cdot V\mathbf{n}_1\mathbf{n}_k V\mathbf{n}_{k-1} \cdots \mathbf{n}_j \cdots \mathbf{n}_2 \qquad (23) \\ &= S\mathbf{n}_j\mathbf{n}_{j-1} S\mathbf{n}_{j-2} \cdots \mathbf{n}_1\mathbf{n}_k \cdots \mathbf{n}_{j-1} + S\cdot V\mathbf{n}_j\mathbf{n}_{j-1} V\mathbf{n}_{j-2} \cdots \mathbf{n}_1\mathbf{n}_k \cdots \mathbf{n}_{j+1}. \end{aligned}$$

This is obviously satisfied if each term on the right hand side of either of these equations is equal to zero. We may therefore state: *If, in a combination of an even number of plane mirrors, the normals to the first and last or to any two consecutive mirrors are perpendicular to each other and the line of intersection of their planes is perpendicular to the resultant axis of the rotation which would be produced by the other mirrors alone, the combination will form a nondeviating erecting system.*

The condition will obviously be satisfied if all the lines of intersection of the planes of the other mirrors are perpendicular to the line of intersection of the two. This even more special condition is met by all nondeviating erecting systems which I have found described.

It is of course obvious that the mirrors must be so located as to permit unobstructed passage of the light through the combination. If a prism serves as a mirror combination it is further obvious that, to avoid refractive deviation, the exit face must be parallel to the image of the entrance face produced by the prism. This condition is of course never exactly met by any actual prism. One of the problems in prism design is to determine how large tolerances may be permitted in meeting such conditions without significant interference with the performance of an instrument.

The effect of small deviations may be treated as the effect of thin glass wedges cemented to the surface of perfect prisms. Such a wedge will produce a rotation of the image space on the object space represented by

$$q_w \mathbf{x} q_w^{-1} = (\cos \tfrac{1}{2}\vartheta_w + \mathbf{w} \sin \tfrac{1}{2}\vartheta_w)\mathbf{x}(\cos \tfrac{1}{2}\vartheta_w - \mathbf{w} \sin \tfrac{1}{2}\vartheta_w), \qquad (24)$$

where the vector $\mathbf{w}$ is parallel to the edge of the wedge and ϑ_w is the angular deviation produced by it. Since these deviations are small it is usually sufficient to write $q_w = (1 + \frac{1}{2}\vartheta_w \mathbf{w})$. This quaternion is inserted between the proper pair of normals in Eq. (19). The difference between the two results for ϑ_w finite and $\vartheta_w = 0$ gives the effect of the deviation.

Since, in general, four or more planes do not have a common point of intersection further computations are necessary if the relative position of image and object space is needed.

To compute this we choose arbitrarily an origin of coordinates at $\mathbf{d}_{j-1}=0$ and define object and image spaces by vectors $\mathbf{r}_h$ $(h=0, 1, 2, \cdots, j, \cdots, k-1, k)$ drawn from that origin. We represent by $\mathbf{d}_h$ a vector drawn from that origin to some arbitrary point on the h-th mirror. Equation (4) may then be written:

$$\mathbf{r}_h - \mathbf{d}_h = -\mathbf{n}_h(\mathbf{r}_{h-1} - \mathbf{d}_h)\mathbf{n}_h^{-1}. \tag{25}$$

Giving h in succession the values $j, j+1, \cdots, k-1, k, (j=1, \cdots, k)$ and substituting in succession the value of each $\mathbf{r}_{h-1}$ in Eq. (25) the effect of all or a part of the k mirrors is found.

In carrying out these substitutions, it is convenient to use an abbreviated notation due to Hamilton. We define:

$$\phi\mathbf{x} = q\mathbf{x}q^{-1} = (\cos\tfrac{1}{2}\vartheta + \mathbf{a}\sin\tfrac{1}{2}\vartheta)\mathbf{x}(\cos\tfrac{1}{2}\vartheta - \mathbf{a}\sin\tfrac{1}{2}\vartheta) \tag{26}$$

as an abbreviated form of Eqs. (1) and (2). Here ϕ is a "rotor," a special form of linear vector operator which rotates a vector through an angle ϑ about an axis parallel to the unit vector $\mathbf{a}$. Carrying out the operations indicated in Eq. (26) and remembering that

$$\mathbf{x} = -\mathbf{a}V\mathbf{ax} - \mathbf{a}S\mathbf{ax}, \tag{27}$$

where $-\mathbf{a}V\mathbf{ax}$ is the component of $\mathbf{x}$ perpendicular and $-\mathbf{a}S\mathbf{ax}$, the component parallel to $\mathbf{a}$, we find

$$\phi\mathbf{x} = (\cos\vartheta + \mathbf{a}\sin\vartheta)(-\mathbf{a}V\mathbf{ax}) + (-\mathbf{a}S\mathbf{ax}) \tag{28}$$

$$\mathbf{x} - \phi\mathbf{x} = (1-\phi)\mathbf{x} = (1 - \cos\vartheta - \mathbf{a}\sin\vartheta)(-\mathbf{a}V\mathbf{ax}). \tag{29}$$

In the important special case when $\vartheta=0$, we have $q=1$, $\phi\mathbf{x}=\mathbf{x}$, and $(1-\phi)\mathbf{x}$ obviously reduces to zero. Again,

$$\mathbf{x} + \phi\mathbf{x} = (1+\phi)\mathbf{x} = (1 + \cos\vartheta + \mathbf{a}\sin\vartheta)(-\mathbf{a}V\mathbf{ax}) + 2(-\mathbf{a}S\mathbf{ax}). \tag{30}$$

In the important special case when $\vartheta=180°$, we have $q=-1$, and $(1+\phi)\mathbf{x}$ obviously reduces to $2(-\mathbf{a}S\mathbf{ax})$. We will later find it necessary to solve equations of the form of Eqs. (29) and (30) for the components of an unknown vector $\mathbf{x}$. Since

$$(1 - \cos\vartheta + \mathbf{a}\sin\vartheta)(1 - \cos\vartheta - \mathbf{a}\sin\vartheta) = 2(1 - \cos\vartheta), \tag{31}$$

Equation (29) is solved by:

$$-\mathbf{a}V\mathbf{ax} = \frac{1}{2(1-\cos\vartheta)}(1 - \cos\vartheta + \mathbf{a}\sin\vartheta)(1-\phi)\mathbf{x}, \tag{32}$$

and the parallel component $-\mathbf{a}S\mathbf{ax}$ is obviously arbitrary. If $\vartheta=0$, this takes on the indeterminate form 0/0. However, as noted above, in that case, the operator, $\phi=1$, is a pure scalar and represents a parallel displacement as a rotation through a zero angle about an infinitely distant axis. Every vector in the object space is imaged by a parallel vector in the image space. Then, not only the parallel, but the perpendicular component of $\mathbf{x}$ is arbitrary. In this case the solution of Eq. (29) is wholly indeterminate.

Equation (30) must first be solved for $-\mathbf{a}S\mathbf{ax}$ as follows. Remembering that $SV\mathbf{ax}=0$, $S\mathbf{a}V\mathbf{ax}=0$ and $\mathbf{a}^2=-1$, we find

$$-S\mathbf{a}(1+\phi)\mathbf{x} = -S\mathbf{a}(1+\cos\vartheta+\mathbf{a}\sin\vartheta)(-\mathbf{a}V\mathbf{ax}) - 2S\mathbf{a}(-\mathbf{a}S\mathbf{ax}) = -2S\mathbf{ax}. \quad (33)$$

Thus the component $\mathbf{x}$ parallel to $\mathbf{a}$ is one half the component of $(1+\phi)\mathbf{x}$ parallel to the same axis. Then, analogous to the solution of Equation (29) we have

$$-\mathbf{a}V\mathbf{ax} = \frac{1}{2(1+\cos\vartheta)}(1+\cos\vartheta-\mathbf{a}\sin\vartheta)[(1+\phi)\mathbf{x}+\mathbf{a}S\mathbf{a}(1+\phi)\mathbf{x}]. \quad (34)$$

If $\vartheta=180°$ this takes on the indeterminate form 0/0. By returning to the original definition of ϕ in terms of q, it is easy to show that in this case the perpendicular component $-\mathbf{a}V\mathbf{ax}$ is arbitrary and only the parallel component $-\mathbf{a}S\mathbf{ax}$ is defined by Eq. (30). The reflection from a single mirror represents a special case of $\vartheta=180°$. In this case the vector $\mathbf{a}$ is normal to the mirror. Adding to $\mathbf{x}$ any vector in the plane of the mirror merely displaces the foot of the normal which, as noted at the beginning, is arbitrary. If $\vartheta\neq 180°$, both components are determinate and their sum gives the vector $\mathbf{x}$. It is readily seen that:

$$\phi(\mathbf{x}+\mathbf{y}) = \phi\mathbf{x}+\phi\mathbf{y} \quad (35)$$

and

$$\phi_j\phi_h\mathbf{x} = q_j(q_h\mathbf{x}q_h^{-1})q_j^{-1} = q_jq_h\mathbf{x}q_h^{-1}q_j^{-1}. \quad (36)$$

The product of several such operations in sequence is thus evaluated by the multiplication in proper order of the quaternions in terms of which they are defined. We may then define:

$$\phi_{h,i}\mathbf{x} = q_{h,i}\mathbf{x}q_{h,i}^{-1} = \mathbf{n}_h\mathbf{n}_{h-1}\cdots\mathbf{n}_{i+1}\mathbf{n}_i\mathbf{x}\mathbf{n}_i^{-1}\mathbf{n}_{i+1}^{-1}\cdots\mathbf{n}_{h-1}^{-1}\mathbf{n}_h^{-1}. \quad (37)$$

It is frequently convenient to note:

$$\phi_{h,i}\phi_{(i-1),j} = \phi_{h,j}. \quad (38)$$

With this notation Eq. (25) may be written:

$$\mathbf{r}_h = -\phi_{h,h}\mathbf{r}_{h-1}+(1+\phi_{h,h})\mathbf{d}_h. \quad (39)$$

Starting with

$$\mathbf{r}_j = -\phi_{j,j}\mathbf{r}_{j-1}+(1+\phi_{j,j})\mathbf{d}_j, \quad (40)$$

and substituting successively the values of $\mathbf{r}_j$, $\mathbf{r}_{j+1}$, $\cdots$, $\mathbf{r}_{k-1}$ for $\mathbf{r}_{h-1}$ in Eq. (39), we find

$$\mathbf{r}_k = (-1)^{k-j+1}\phi_{k,j}\mathbf{r}_{j-1}+\sum_{h=j}^{h=k}[1+(-1)^{k-h}\phi_{k,h}](\mathbf{d}_h-\mathbf{d}_{h-1}). \quad (41)$$

If $j=1$ and all the $\mathbf{d}_h$'s are zero this obviously reduces to Eq. (19).

The process of computation consists in calculating in order

$$q_{k,k} = \mathbf{n}_k, \cdots, q_{k,h} = q_{k,h+1}\mathbf{n}_h, \cdots, q_{k,1} = q_{k,2}\mathbf{n}_1, \quad (42)$$

from which each of the $\vartheta_{k,h}$'s and $\mathbf{a}_{k,h}$'s are computed by:

$$Sq_{k,h} = \cos \tfrac{1}{2}\vartheta_{k,h}, \qquad \mathbf{a}_{k,h} = Vq_{k,h}/\sin \tfrac{1}{2}\vartheta_{k,h}. \tag{43}$$

The computations may frequently be shortened by using Eq. (38). The term $\phi_{k,j}\mathbf{r}_{j-1}$ is then computed either by Eq. (26) or by Eq. (28) whichever seems easier, and each of the terms in the summation in Eq. (41) by Eq. (29) or (30). They are then added.

Since the origin of coordinates and the points to which the vectors $\mathbf{d}_h$ are drawn are arbitrary, the computations can ordinarily be much simplified by suitably choosing them.

When two successive mirrors intersect, choosing a point on their common intersection makes $\mathbf{d}_h - \mathbf{d}_{h-1} = 0$ and the corresponding term in the summation vanishes. When three successive mirrors intersect, choosing their common point of intersection as the common foot of their normals makes $\mathbf{d}_{h-1} = \mathbf{d}_h = \mathbf{d}_{h+1}$ and two successive terms of the summation vanish. If they are the first three and we choose it as the origin of coordinates, we have $\mathbf{d}_{j-1} = \mathbf{d}_j = \mathbf{d}_{j+1} = \mathbf{d}_{j+2} = 0$ and three successive terms vanish. It was this choice which made the treatment of the three mirror problem so much simpler.

In numerical work it is convenient to express all the vectors in terms of a suitably chosen rectangular coordinate system represented by unit vectors, **i**, **j**, and **k**. By substituting for $\mathbf{r}_{j-1}$, **i**, **j**, and **k** in turn and carrying through the computation, the position of the image of any point, $\mathbf{r}_{j-1} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, in the object space may be obtained by simple addition.

Equations (28), (29), (30), (38), (41), (42), and (43) are thus all that are needed to determine the relative position and orientation of object and image after reflection from any number of plane mirrors. It is, however, sometimes more convenient to calculate the parameters of the transformation of the object space into the image space.

The most general rigid displacement of a three dimensional Euclidean configuration is a twist about a screw. The relative position of image and object after reflection from an even number of plane mirrors will be known when the angle of the twist and the pitch and the position of the axis of the resultant screw are known.

A twist through an angle ϑ_k about a screw of pitch a_k/ϑ_k whose axis is parallel to the unit vector $\mathbf{a}_k$ is equivalent to a rotation about that axis followed by a parallel displacement $a_k\mathbf{a}_k$ along the axis. The effect of an even number of mirrors will therefore be given by an equation of the form:

$$\mathbf{r}_k = \phi_k \mathbf{r}_{j-1} + (1 - \phi_k)\mathbf{d}_{k,j} + a_k\mathbf{a}_k, \tag{44}$$

where the vector, $\mathbf{d}_{k,j}$ from the origin to the axis of the twist is given the double subscript to distinguish it from the vector $\mathbf{d}_k$ drawn to a point on the k-th mirror. Comparison with Eq. (41) shows that

$$\phi_k = \phi_{k,j}, \quad \vartheta_k = \vartheta_{k,j} \quad \text{and} \quad \mathbf{a}_k = \mathbf{a}_{k,j}. \tag{45}$$

These are computed by Eqs. (42) and (43) and a_k is given by

$$a_k = -S\mathbf{a}_k \sum_{h=j}^{h=k} [1 + (-1)^{k-h}\phi_{k,h}](\mathbf{d}_h - \mathbf{d}_{h-1}). \tag{46}$$

Then

$$(1 - \phi_k)\mathbf{d}_{k,j} = \sum_{h=j}^{h=k} [1 + (-1)^{k-h}\phi_{k,h}](\mathbf{d}_h - \mathbf{d}_{h-1}) - a_k\mathbf{a}_k. \tag{47}$$

These equations are solved for the perpendicular components of the $\mathbf{d}_{k,j}$'s, by Eq. (32). Their parallel components are arbitrary. The effect of all the even combinations of the k mirrors is then known in terms of the angle of the twist and the pitch and the position of the axis of the resultant screws.

If a twist upon a screw is combined with a central inversion, the result is a rotatory inversion. The relative position of image and object after reflection from an odd number of mirrors will be known when the position of the axis, the location of the center on that axis, and the angle of rotation of the resultant rotatory inversion are known.

A rotatory inversion through an angle ϑ_k about an axis parallel to a unit vector $\mathbf{a}_k$ is readily shown to be given by an equation of the form

$$\mathbf{r}_k = -\phi_k \mathbf{r}_{j-1} + (1 + \phi_k)\mathbf{d}_{k,j}, \tag{48}$$

where $\mathbf{d}_{k,j}$ is a vector drawn from the origin to the center of the resultant inversion. Comparison with Eq. (41) shows that, as in the case of an even number of mirrors.

$$\phi_k = \phi_{k,j}, \quad \vartheta_k = \vartheta_{k,j} \quad \text{and} \quad \mathbf{a}_k = \mathbf{a}_{k,j}. \tag{49}$$

These are computed by Eqs. (42) and (43). Then

$$(1 + \phi_k)\mathbf{d}_{k,j} = \sum_{h=j}^{h=k} [1 + (-1)^{k-h}\phi_{k,h}](\mathbf{d}_h - \mathbf{d}_{h-1}). \tag{50}$$

These equations are solved for the parallel and perpendicular components of the $\mathbf{d}_{k,j}$'s by Eqs. (33) and (34). The effect of all the odd combinations of the k mirrors

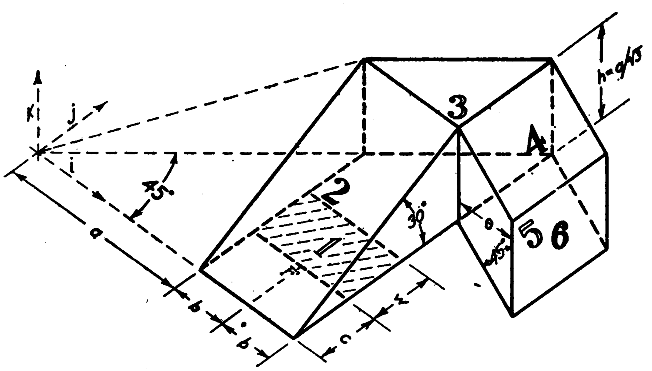

FIG. 1. Range finder five mirror prism combination.

is then known in terms of the position of the axis, the location of the center of inversion on that axis and the angle of rotation of their resultant rotatory inversions.

Although there are necessarily a considerable number of steps in such a computation, the steps are not complicated, consisting merely of simple multiplications and separation of quaternions into their scalar and vector parts.

As an illustration of the straightforwardness of these quaternion computations, suitable dimensions for the five-mirror combination shown in Fig. 1 are computed below. This is part of the prism system used in an Army range finder. The other part is a simple two-mirror system cemented below Face 1.

In use the axial ray from the right hand objective enters through Face 6, is reflected in turn from Faces 5, 4, 3 and 2 in succession and from the middle of the near edge of the silver strip on Face 1 at the point F in Fig. 1, and finally emerges through Face 2, which thus serves both as a mirror and an exit face. The image is focussed on that edge of the silver strip, where it is brought into coincidence with the image from the left hand objective, coming through the lower two-mirror system which is not shown.

We must assure ourselves that the image of the exit Face 2, in the space of the objective, is parallel to the entrance Face 6 and that light from the whole aperture of the objective has unobstructed passage to the edge of the silver strip where the coincidence is observed, and for a reasonable distance around it.

We therefore compute, in the space of the objective, the direction of the image of the normal to Face 2 and the position of the corners of the silver strip and of any prism faces which might obstruct the light. The plane of a mirror and its image in that plane coincide, so that we need only compute the relationship between the space of the objective and every other object space in which the prism faces lie.

Since the planes of Faces 1, 2 and 3 produced, intersect in a point, it is convenient to choose that point as the origin of co-ordinates. Choosing axes $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ and dimensions as shown, and remembering that $\cos 60° = \sin 30° = \frac{1}{2}$, $\sin 60° = \cos 30° = \sqrt{3}/2$, and $\cos 45° = \sin 45° = 1/\sqrt{2}$, we have:

$$\mathbf{n}_1 = \mathbf{k}, \quad \mathbf{d}_1 = 0; \qquad \mathbf{n}_2 = \tfrac{1}{2}(\mathbf{j} - \sqrt{3}\,\mathbf{k}), \quad \mathbf{d}_2 = 0; \qquad \mathbf{n}_3 = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j}), \quad \mathbf{d}_3 = 0;$$

$$\mathbf{n}_4 = -\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k}), \quad \mathbf{d}_4 = (a + 2b)\mathbf{i} + h\mathbf{k}; \qquad \mathbf{n}_5 = +\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k}), \quad \mathbf{d}_5 = (a + 2b)\mathbf{i},$$

where $h = a \tan 30° = a/\sqrt{3}$.

Then from Eq. (41),

$$\mathbf{r}_5 - \phi_{5,4}\mathbf{r}_3 = \mathbf{r}_5 + \phi_{5,3}\mathbf{r}_2 = \mathbf{r}_5 + \phi_{5,1}\mathbf{r}_0 = (1 + \phi_{5,5})(\mathbf{d}_5 - \mathbf{d}_4) + (1 - \phi_{5,4})(\mathbf{d}_4 - \mathbf{d}_3).$$

By Eqs. (42) and (43),

$$q_{5,5} = \mathbf{n}_5 = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k}), \quad \cos \tfrac{1}{2}\vartheta_{5,5} = 0, \quad \vartheta_{5,5} = 180°, \quad \mathbf{a}_{5,5} = \mathbf{n}_5 = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k}).$$

Since $\vartheta_{5,5} = 180°$, we have by Eq. (30).

$$(1 + \phi_{5,5})(\mathbf{d}_5 - \mathbf{d}_4) = 2[-\mathbf{a}_{5,5}S\mathbf{a}_{5,5}(\mathbf{d}_5 - \mathbf{d}_4)] = 2\left[-\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k})S\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k})(-h\mathbf{k})\right]$$

$$= -h(\mathbf{i} + \mathbf{k}).$$

It may be interesting to note again here that, since the position of the foot of the normal is arbitrary, $\mathbf{d}_4$ and $\mathbf{d}_5$ could have been written more generally as $\mathbf{d}_4 = (a + 2b + h - z)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{d}_5 = (a + 2b - z')\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$, and $(1 + \phi_{5,5})(\mathbf{d}_5 - \mathbf{d}_4)$ would, as the result of more complicated computations, have come out the same.

$$q_{5,4} = q_{5,5}\mathbf{n}_4 = -\tfrac{1}{2}(\mathbf{i} + \mathbf{k})(\mathbf{i} + \mathbf{k}) = 1.$$

Then according to Eq. (29), $\phi_{5,4}\mathbf{r}_3=\mathbf{r}_3$ and $(1-\phi_{5,4})(\mathbf{d}_4-\mathbf{d}_3)=0$ and Eq. (41) reduces to

$$\mathbf{r}_5 = \mathbf{r}_3 - h(\mathbf{i}+\mathbf{k}) = -\phi_{5,3}\mathbf{r}_2 - h(\mathbf{i}+\mathbf{k}) = -\phi_{5,1}\mathbf{r}_0 - h(\mathbf{i}+\mathbf{k}).$$

$$q_{5,3} = q_{5,4}\mathbf{n}_3 = \frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j}).$$

By Eq. (26),

$$-\phi_{5,3}(x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}) = \tfrac{1}{2}(\mathbf{i}+\mathbf{j})(x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k})(\mathbf{i}-\mathbf{j}) = y_2\mathbf{i}+x_2\mathbf{j}+z_2\mathbf{k}$$

$$q_{5,1} = q_{5,3}\mathbf{n}_2\mathbf{n}_1 = \frac{1}{2\sqrt{2}}(\mathbf{i}-\mathbf{j})(\mathbf{j}-\sqrt{3}\,\mathbf{k})\mathbf{k} = \frac{1}{\sqrt{8}}(-1+\sqrt{3}\,\mathbf{i}-\sqrt{3}\,\mathbf{j}+\mathbf{k})$$

$$\begin{aligned}-\phi_{5,1}(x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k}) &= -\tfrac{1}{8}(-1+\sqrt{3}\,\mathbf{i}-\sqrt{3}\,\mathbf{j}+\mathbf{k})(x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k})\\ &\qquad\cdot(-1-\sqrt{3}\,\mathbf{i}+\sqrt{3}\,\mathbf{j}-\mathbf{k})\\ &= \tfrac{1}{2}(y_0-\sqrt{3}\,z_0)\mathbf{i}+x_0\mathbf{j}+\tfrac{1}{2}(z_0+\sqrt{3}\,y_0)\mathbf{k}.\end{aligned}$$

A unit vector normal to the exit Face 2 of the prism is $\mathbf{n}_2=\tfrac{1}{2}(\mathbf{j}-\sqrt{3}\mathbf{k})$ giving $x_0=0$, $y_0=\tfrac{1}{2}$ and $z_0=-\sqrt{3}/2$. Its image in the space of the objective is therefore

$$-\phi_{5,1}\mathbf{n}_2 = \frac{1}{2}\left[\frac{1}{2}-\sqrt{3}\left(-\frac{\sqrt{3}}{2}\right)\right]\mathbf{i}+0\mathbf{j}+\frac{1}{2}\left[-\frac{\sqrt{3}}{2}+\frac{1}{2}\sqrt{3}\right]\mathbf{k} = \mathbf{i},$$

which is normal to the entrance Face 6 of the prism. If the angles of the prism combination are correct there will thus be no prismatic deviation.

The corners of the silver strip also lie in the space of $\mathbf{r}_0$. Their images are therefore given by

$$\begin{aligned}&-\phi_{5,1}[a\mathbf{i}+(c+w)\mathbf{j}]-h(\mathbf{i}+\mathbf{k}), &&-\phi_{5,1}[(a+2b)\mathbf{i}+(c+w)\mathbf{j}]-h(\mathbf{i}+\mathbf{k}),\\ &-\phi_{5,1}[a\mathbf{i}+c\mathbf{j}]-h(\mathbf{i}+\mathbf{k}), &&-\phi_{5,1}[(a+2b)\mathbf{i}+c\mathbf{j}]-h(\mathbf{i}+\mathbf{k}).\end{aligned}$$

Carrying out the computations we find

$$\begin{gathered}\tfrac{1}{2}(c+w-2h)\mathbf{i}+a\mathbf{j}+\tfrac{1}{2}[\sqrt{3}\,(c+w)-2h]\mathbf{k},\\ \tfrac{1}{2}(c+w-2h)\mathbf{i}+(a+2b)\mathbf{j}+\tfrac{1}{2}[\sqrt{3}\,(c+w)-2h]\mathbf{k},\\ \tfrac{1}{2}(c-2h)\mathbf{i}+a\mathbf{j}+\tfrac{1}{2}[\sqrt{3}\,c-2h]\mathbf{k},\\ \tfrac{1}{2}(c-2h)\mathbf{i}+(a+2b)\mathbf{j}+\tfrac{1}{2}[\sqrt{3}\,c-2h]\mathbf{k}.\end{gathered}$$

These give the location, in the space of the objective, of the "air image" of the silver strip, i.e., the location the image would have if the prism faces were front surface mirrors in air.

Face 5 lies in the space of the objective. Its corners and their "air images" are given by

$$\begin{aligned}&(a+2b)\mathbf{i}+a\mathbf{j}, &&(a+2b)\mathbf{i}+(a+2b)\mathbf{j},\\ &(a+2b+e)\mathbf{i}+a\mathbf{j}-e\mathbf{k}, &&(a+2b+e)\mathbf{i}+(a+2b)\mathbf{j}-e\mathbf{k}.\end{aligned}$$

If the diameter d, of the objective is greater than $2b$, which is the case in all practical constructions, and light from the whole aperture of the objective is to reach all parts of the silver strip, it is obvious, since the light is converging to a focus, that the **j** and **k** co-ordinates of the image of the silver strip must lie somewhat within the limits of the corresponding corners of Face 5. Since their **j** co-ordinates are equal it is obvious that there will necessarily be some vignetting of the image at the ends of the silver strip, the amount depending upon the aperture and focal length of the objective and the size of the prisms.

For the **k** co-ordinate of the back edge of this strip we have:

$$\tfrac{1}{2}[\sqrt{3}\,(c + w) - 2h] < 0$$

or

$$(c + w) < 2h/\sqrt{3} = 2a/3.$$

Allowing for some vignetting at the rear edge of the silver strip, which will do no harm, we may set $(c+w)=2a/3$. To allow an equal area for the matching image from the other objective we may set $c=a/3$.

For economy of glass it is obviously desirable to make the viewing aperture approximately square. The projection of $(c+w)=2a/3$ on Face 2 is $\sqrt{3}\cdot 2a/3=a/\sqrt{3}$. For economy of glass we should therefore make $b=a/2\sqrt{3}$.

The bottom edge of Face 5 coincides with the bottom edge of Face 6. The image is focussed on the bottom edge of the silver strip. The length of the path of the axial ray in the glass is therefore the difference of their **i** co-ordinates:

$$(a + 2b + e) - \tfrac{1}{2}(c - 2h) = (5 + 4\sqrt{3})a/6 + e.$$

Because of the index of refraction μ of the glass the image of the focal plane as seen from the lens in air will be displaced in the $+\mathbf{i}$ direction by $(\mu-1)/\mu$ times this length. The effective image of the focal plane, in the space of the objective, lies therefore at a distance

$$[1 - (\mu - 1)/\mu][(5 + 4\sqrt{3})a/6 + e] = [(5 + 4\sqrt{3})a/6 + e]/\mu$$

in the $-\mathbf{i}$ direction back of Face 6. The back principal plane of the objective must be placed a distance equal to its focal length f further in the $+\mathbf{i}$ direction.

At the front edge of the silver strip, where the images coincide, vignetting would be intolerable, so we must have

$$\tfrac{1}{2}[\sqrt{3}\,c - 2h] > -e$$

or

$$e > h - \sqrt{3}\,c/2 = h/2 = a/2\sqrt{3} = \sqrt{3}\,a/6.$$

The thickness e of the rhomb forming Faces 4 and 5 of the mirror combination must, therefore, be somewhat greater than one half the height of the rest of the combination.

The convergence of the rays in the glass is determined by the distance between the focal plane and the "glass image" of the lens, i.e., the distance the back principal plane would have if the light path to it lay wholly in glass. This distance is μf. If d is the diameter of the lens the theoretically necessary value is given by;

$$e = \sqrt{3}\,a/6 + [(5 + 4\sqrt{3})a/6 + e]d/2\mu f$$

or

$$e = [\sqrt{3} + (5 + 4\sqrt{3})d/2\mu f]a/6(1 - d/2\mu f).$$

In practice e is made slightly larger to provide against small misalignments.

In this prism combination, computation of the location of the images of the corners of the other prism faces is not necessary, because simple geometrical considerations show that Face 5 forms the limiting aperture. However, they could readily be computed if desired.

To complete the design it would, of course, be necessary to compute the dimensions and shape of the lower two-mirror prism so that the axial ray from the left hand objective lies in the same line,

$$x\mathbf{i} + (a + b)\mathbf{j} + (\sqrt{3}\, c/2 - h)\mathbf{k},$$

as the axial ray from the right hand objective, after two reflections in the prism has

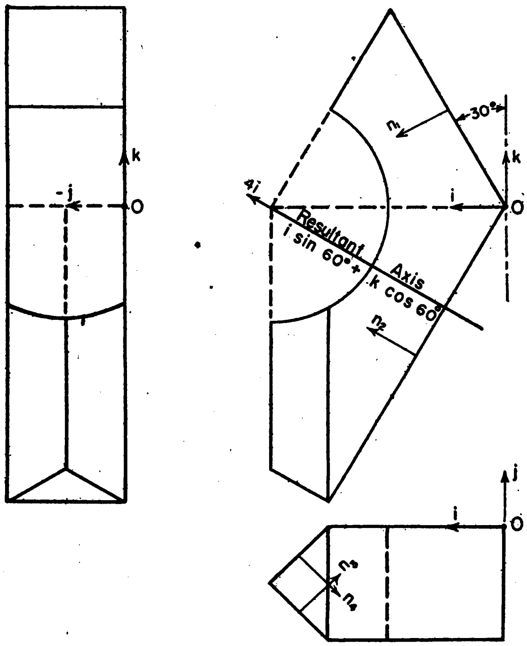

FIG. 2. Leman-Sprenger nondeviating erecting prism.

the direction, $-\mathbf{n}_2$ and is focussed on the same point F, giving $\mathbf{r}_2' = (a+b)\mathbf{i}+c\mathbf{j}$. These computations would obviously be much shorter than the previous ones.

For the purpose of establishing tolerances, computation of the effect of small deviations of the faces from their correct orientation is necessarily longer and more tedious, but is just as straightforward.

These computations of suitable dimensions of this prism combination could, of course, have been made by determining the parameters, the axis and angle of the rotatory inversion produced by Faces 1, 2 and 3, using Eqs. (10), (13) and (18), and combining their effect with the parallel displacement produced by Faces 4 and 5. The computations would, however, have been much longer.

As an illustration of the computation of the parameters of a transformation of an object space into an image space, the effect of a Leman-Sprenger nondeviating erecting prism is calculated below. Axes **i**, **j** and **k** and distances are chosen as shown in Fig. 2. Remembering that:

$$\cos 60^\circ = \sin 30^\circ = 1/2, \quad \sin 60^\circ = \cos 30^\circ = \sqrt{3}/2 \quad \text{and}$$
$$\sin 45^\circ = \cos 45^\circ = 1/\sqrt{2},$$

we have:

$$\mathbf{n}_1 = \tfrac{1}{2}(\sqrt{3}\,\mathbf{i} - \mathbf{k}), \quad \mathbf{d}_1 = 0; \qquad \mathbf{n}_3 = \frac{1}{\sqrt{2}}(-\,\mathbf{i} + \mathbf{j}), \quad \mathbf{d}_3 = 4\mathbf{i} - \mathbf{j},$$

$$\mathbf{n}_2 = \tfrac{1}{2}(\sqrt{3}\,\mathbf{i} + \mathbf{k}), \quad \mathbf{d}_2 = 0; \qquad \mathbf{n}_4 = \frac{1}{\sqrt{2}}(-\,\mathbf{i} - \mathbf{j}), \quad \mathbf{d}_4 = 4\mathbf{i} - \mathbf{j}.$$

The combination of Eqs. (41) and (44) reduces to:

$$\mathbf{r}_4 - \phi_4\mathbf{r}_0 = r_4 - \phi_{4,1}\mathbf{r}_0 = (1 - \phi_{4,3})(\mathbf{d}_3 - \mathbf{d}_2) = (1 - \phi_4)\mathbf{d}_{4,1} + a_4\mathbf{a}_4.$$

By Eq. (37),

$$q_{4,3} = \mathbf{n}_4\mathbf{n}_3 = \tfrac{1}{2}(-\,\mathbf{i} - \mathbf{j})(-\,\mathbf{i} + \mathbf{j}) = -\,\mathbf{k},$$
$$q_{2,1} = \mathbf{n}_2\mathbf{n}_1 = \tfrac{1}{4}(\sqrt{3}\,\mathbf{i} + \mathbf{k})(\sqrt{3}\,\mathbf{i} - \mathbf{k}) = \tfrac{1}{2}(-\,1 + \sqrt{3}\,\mathbf{j}).$$

By Eq. (38)

$$q_4 = q_{4,1} = q_{4,3}q_{2,1} = -\,\tfrac{1}{2}\mathbf{k}(-\,1 + \sqrt{3}\,\mathbf{j}) = \tfrac{1}{2}(\sqrt{3}\,\mathbf{i} + \mathbf{k}),$$

giving:

$$\cos \tfrac{1}{2}\vartheta_4 = -\,Sq_4 = 0, \qquad \cos \tfrac{1}{2}\vartheta_{4,3} = -\,Sq_{4,3} = 0,$$
$$\sin \tfrac{1}{2}\vartheta_4 = \sin \tfrac{1}{2}\vartheta_{4,3} = 1, \qquad \cos \vartheta_4 = \cos \vartheta_{4,3} = -\,1, \qquad \sin \vartheta_4 = \sin \vartheta_{4,3} = 0.$$

By Eq. (43),

$$\mathbf{a}_4 = Vq_4/\sin \tfrac{1}{2}\vartheta_4 = \tfrac{1}{2}(\sqrt{3}\,\mathbf{i} + \mathbf{k}), \qquad \mathbf{a}_{4,3} = Vq_{4,3}/\sin \tfrac{1}{2}\vartheta_{4,3} = -\,\mathbf{k}.$$

Then

$$-\,\mathbf{a}_{4,3}V\mathbf{a}_{4,3}(\mathbf{d}_3 - \mathbf{d}_2) = -\,(-\,\mathbf{k})V(-\,\mathbf{k})(4\mathbf{i} - \mathbf{j}) = \mathbf{k}(-\,4\mathbf{j} - \mathbf{i}) = (4\mathbf{i} - \mathbf{j}).$$

By Eq. (29),

$$(1 - \phi_{4,3})(\mathbf{d}_3 - \mathbf{d}_2) = (1 - \cos \vartheta_{4,3} + \mathbf{a}_{4,3} \sin \vartheta_{4,3})(4\mathbf{i} - \mathbf{j})$$
$$= 2(4\mathbf{i} - \mathbf{j}) = 8\mathbf{i} - 2\mathbf{j} = (1 - \phi_4)\mathbf{d}_{4,1} + a_4\mathbf{a}_4.$$

By Eq. (46),

$$a_4 = -\,S\mathbf{a}_4(1 - \phi_{4,3})(\mathbf{d}_3 - \mathbf{d}_2) = \tfrac{1}{2}S(\sqrt{3}\,\mathbf{i} + \mathbf{k})(8\mathbf{i} - 2\mathbf{j}) = 4\sqrt{3}.$$

By Eq. (47),

$$(1 - \phi_4)\mathbf{d}_{4,1} = (1 - \phi_{4,3})(\mathbf{d}_3 - \mathbf{d}_2) - a_4\mathbf{a}_4$$
$$= (8\mathbf{i} - 2\mathbf{j}) - 2\sqrt{3}\,(\sqrt{3}\,\mathbf{i} + \mathbf{k}) = 2(\mathbf{i} - \mathbf{j} - \sqrt{3}\,\mathbf{k}).$$

By Eq. (32),

$$-\mathbf{a}_4 V \mathbf{a}_4 \mathbf{d}_{4,1} = \frac{1}{2(1-\cos\vartheta_4)}(1-\cos\vartheta_4+\mathbf{a}_4\sin\vartheta_4)(1-\phi_4)\mathbf{d}_{4,1} = \mathbf{i}-\mathbf{j}-2\sqrt{3}\,\mathbf{k}.$$

Since the component of $\mathbf{d}_{4,1}$ parallel to the axis $\mathbf{a}_4$ is arbitrary it is convenient to add to this normal component a parallel component

$$2\sqrt{3}\,\mathbf{a}_4 = 3\mathbf{i} + 2\sqrt{3}\,\mathbf{k},$$

giving for $\mathbf{d}_{4,1}$ the value

$$\mathbf{d}_{4\cdot 1} = 4\mathbf{i} - \mathbf{j}.$$

The resultant axis of rotation, shown in Fig. 2, thus intersects the prolongation of the roof angle formed by $\mathbf{n}_3$ and $\mathbf{n}_4$ at the point $\mathbf{r}=4\mathbf{i}-\mathbf{j}$ and is perpendicular to the entance face of the prism.

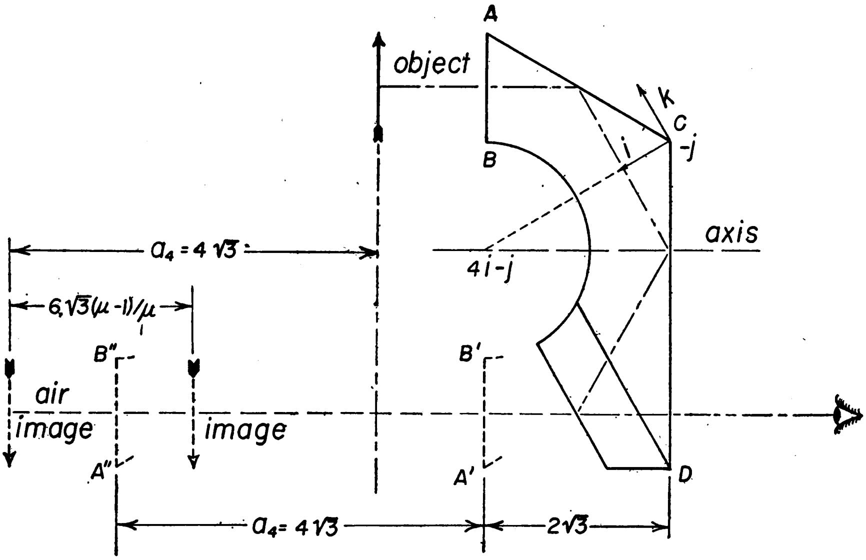

FIG. 3. Leman-Sprenger nondeviating erecting prism.

If the erecting system consisted of front surface mirrors, the image, besides being rotated through 180° about the axis would be displaced backward through the distance $a_4=4\sqrt{3}$. Since, however, the path through the prism is in glass, this backward displacement is partially compensated for by a forward displacement equal to $(\mu-1)/\mu$ times the length of the glass path, where μ is the index of refraction of the glass. The glass path is readily determined by rotating the entrance face AB (Fig. 3) about the axis to $A'B'$ and displacing it backward through the distance $a_4=4\sqrt{3}$ to $A''B''$. The distance of this "air image" of the entrance face from the exit face CD is the length of the glass path and is readily seen to be $6\sqrt{3}$. Assuming $\mu=1.5$, $6\sqrt{3}(\mu-1)/\mu=2\sqrt{3}$. The relative positions of object, "air image" and image are shown in Fig. 3.

The effect of a perfect prism of this type could obviously be determined much more easily by a simple geometrical construction. Further, the quaternion computations have obviously been carried through in needless detail for a perfect prism. This has been done to show the steps which would be necessary for the purpose of establishing tolerances in determining the effect of small deviations of the faces from their correct orientation. In such computations, the scalar part of the quaternions $q_{4,3}$ and q_4 would not be zero, but small quantities representing the effect of the deviation of the prism faces from their proper orientation, making necessary the computation of the perpendicular and parallel components of vectors which, in the example given, could have been written down by inspection.