—NOTES—

THE PRODUCT OF THREE REFLECTIONS*

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1. Introduction. This paper completes a trilogy, as it is a sequel to Synge's "Reflection in a corner formed by three plane mirrors" and Tuckerman's "Multiple reflections by plane mirrors."[1] However, it is written in such a way as to be self-contained.

The product of reflections in two intersecting planes is easily seen to be a rotation about the line of intersection of the planes through twice the angle between them. Hence the product of three reflections is, in general, a *rotatory reflection*: the product of a rotation and a reflection. We shall see (in §2) that, by a suitable displacement of the mirrors, without altering the final effect, we can arrange for the axis of the rotation to be perpendicular to the plane of the remaining reflection, so that the rotation and reflection are commutative. [An exception arises in the case of three "vertical" mirrors, all perpendicular to one plane. Then the product is a *glide-reflection*: the combination of a translation and a reflection. We shall discuss that case in Sec. 8.]

If the three mirrors have a common point, we represent them by the great circles which they cut out on the unit sphere around the point. Of the eight spherical triangles formed by the three great circles, just one is distinguished by having the three "fronts" of the mirrors all facing inwards. We shall call this the *mirror triangle* and denote it by **ABC**. [Similarly, if the three mirrors are all vertical, we represent them by the sides of the plane triangle **ABC** which they cut out on a horizontal plane.]

Let $\mathbf{AA_1}$, $\mathbf{BB_1}$, $\mathbf{CC_1}$ be the *altitudes* of triangle **ABC**; then $\mathbf{A_1B_1C_1}$ is called the *pedal triangle* (or "orthic triangle") of **ABC**. (See Fig. 1.) We shall find that the product of reflection in the three sides of an acute-angled triangle **ABC**, in any particular order, is a rotatory reflection [or glide-reflection] consisting of the reflection in one side of the pedal triangle combined with a rotation [or translation] along that side, of an amount equal to the perimeter of the pedal triangle. The three sides of the pedal triangle, with two senses along each, account for the six possible orders of the three reflections. We shall reconcile this with the results of Synge and Tuckerman, and finish with a brief statement concerning the analogous problem in m dimensions.

Steiner remarked in 1837 that the pedal triangle of an acute-angled spherical triangle **ABC** is the triangle of minimum perimeter inscribed in **ABC**. The present treatment was suggested by Schwarz's beautiful proof of the corresponding theorem for plane triangles.[2]

For simplicity, all the spherical triangles in the diagrams have been drawn as

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[1] This Quarterly **4**, 166–176 (1946), **5**, 133–148 (1947).

[2] Steiner, *Aufgaben und Lehrsätze*, Gesammelte Werke, vol. 2, Berlin, 1882, p. 45 (No. 7). Schwarz, *Beweis des Satzes, dass unter allen einem spitzwinkligen Dreiecke eingeschriebenen Dreiecken das Dreieck der Höhenfusspunkte den kleinsten Umfang hat*, Gesammelte Mathematische Abhandlungen, vol. 2, Berlin, 1890, pp. 344–345.

plane triangles. This economy is justified by the fact that the results remain valid in the case of three vertical mirrors.

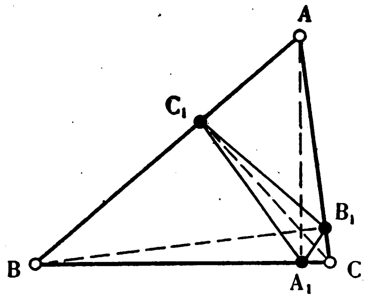

FIG. 1

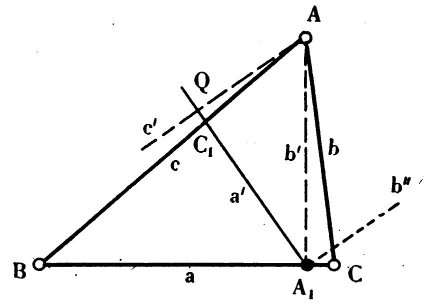

FIG. 2

The author wishes to express his gratitude to Professor Synge for many helpful suggestions in connection with this work.

2. The sides of the pedal triangle. If a congruent transformation of the sphere leaves a great circle invariant (as a whole, but not necessarily point by point) it is either the reflection in the plane of the great circle, or a rotation about the perpendicular line (which we call a rotation *along* the great circle), or the combination of both: a rotatory reflection. In the last case we shall call the great circle the *line of action* of the rotatory reflection. In general it is unique: no other great circle is invariant under the given transformation. The only exception is when the rotation is a half-turn, or rotation through π; then the rotatory reflection reduces to the *central inversion*, which reverses all vectors and transforms each point of the sphere into its antipodal point.

Let A, B, C denote the reflections in the sides **a**, **b**, **c** of the mirror triangle **ABC**. The product BC (first B, then C) is a rotation through $-2A$ about the vertex **A** (or about the line to **A** from the center of the sphere). This can just as well be expressed as the product of reflections in *any* two great circles through **A** making this same angle A, say **b′** and **c′**. Choosing **b′** to be the altitude $\mathbf{AA_1}$, as in Fig. 2, we have $ABC=AB'C'$, where AB' is a half-turn: the product of reflections in two perpendicular great circles through $\mathbf{A_1}$. But this half-turn can just as well be expressed as the product of reflections in *any* two perpendicular great circles through $\mathbf{A_1}$, say **a′** and **b″**. Choosing **a′** to be $\mathbf{A_1Q}$ perpendicular to **c′**, we have $AB'C'=A'B''C'$, where **a′** is perpendicular to both **b″** and **c′**. Thus $ABC=A'B''C'$: the product of the reflection A' and the commutative rotation $B''C'$.

This shows that the line of action of the rotatory reflection ABC passes through $\mathbf{A_1}$. Similarly that of CBA passes through $\mathbf{C_1}$. But ABC and CBA, being inverse transformations, have the same line of action (in opposite senses). Therefore this line of action is the great circle $\mathbf{A_1C_1}$. Moreover, the sense of the rotation $B''C'$ is from $\mathbf{A_1}$ towards $\mathbf{C_1}$. Hence

THEOREM 2.1. *The six rotatory reflections ABC, CBA, BCA, ACB, CAB, BAC consist of reflections in the sides of the pedal triangle* $\mathbf{A_1B_1C_1}$ *combined with rotations along those sides in the senses* $\mathbf{A_1C_1}$, $\mathbf{C_1A_1}$, $\mathbf{B_1A_1}$, $\mathbf{A_1B_1}$, $\mathbf{C_1B_1}$, $\mathbf{B_1C_1}$.

The amount of rotation, being twice the angle between $\mathbf{b}''$ and $\mathbf{c}'$, is equal to $2\mathbf{A}_1\mathbf{Q}$, which is less than π for a sufficiently small triangle **ABC**, and consequently (by continuity) less than π for any triangle except a trirectangular one; for that is the only case where it can attain the value π (which gives the central inversion).

3. Two geometrical theorems. The rearrangement of reflections that led to the equation $ABC = A'B''C'$ provides a kinematical proof for

THEOREM 3.1. *If* $\mathbf{A}_1\mathbf{B}_1\mathbf{C}_1$ *is the pedal triangle of a spherical triangle* **ABC**, *the perpendicular from* **A** *to* $\mathbf{A}_1\mathbf{C}_1$ *makes an angle* A *with the altitude* $\mathbf{AA}_1$.

Similarly the perpendicular from **A** to $\mathbf{A}_1\mathbf{B}_1$ makes this same angle A with $\mathbf{AA}_1$. (See Fig. 3.) Thus $\mathbf{AA}_1$ bisects the angle between the perpendiculars to $\mathbf{A}_1\mathbf{C}_1$ and $\mathbf{A}_1\mathbf{B}_1$. Consequently it also bisects the angle between $\mathbf{A}_1\mathbf{C}_1$ and $\mathbf{A}_1\mathbf{B}_1$ themselves. Thus $\mathbf{AA}_1$ is the internal bisector of $\angle \mathbf{B}_1\mathbf{A}_1\mathbf{C}_1$, and **BC** is the external bisector. Hence

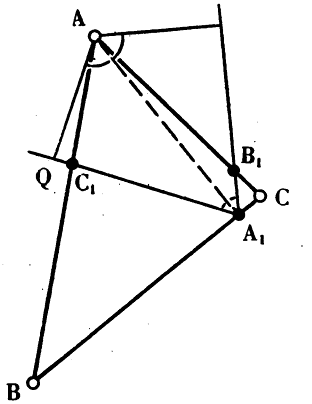

FIG. 3

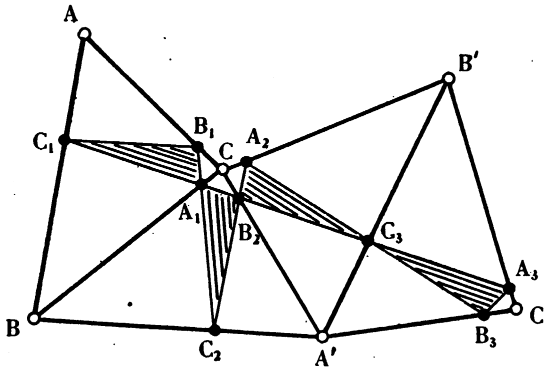

FIG. 4

THEOREM 3.2. *The orthocenter and vertices of a spherical triangle* **ABC** *are the incenter and excenters of its pedal triangle* $\mathbf{A}_1\mathbf{B}_1\mathbf{C}_1$.

This theorem can hardly be new, as its plane analogue is so familiar (see Fig. 1); but some of the standard textbooks fail to mention it. We have considered an acute-angled triangle **ABC**; but the effect of an obtuse C is merely to make **C** the incenter of $\mathbf{A}_1\mathbf{B}_1\mathbf{C}_1$ (so that the orthocenter of **ABC** is an excenter of $\mathbf{A}_1\mathbf{B}_1\mathbf{C}_1$).

The converse is, of course, obvious: If the external bisectors of the angles of any triangle $\mathbf{A}_1\mathbf{B}_1\mathbf{C}_1$ form a triangle **ABC**, the internal bisectors are the altitudes of **ABC**.

4. The perimeter of the pedal triangle. (For this part of the work we cannot allow any of the angles A, B, C to be obtuse.) By Theorem 3.2, **A** is the center of an excircle of triangle $\mathbf{A}_1\mathbf{B}_1\mathbf{C}_1$ (viz., the excircle beyond the side $\mathbf{B}_1\mathbf{C}_1$). The arc $\mathbf{A}_1\mathbf{Q}$ (Fig. 3), being one of the tangents from $\mathbf{A}_1$ to this small circle, is equal to s_1, the semiperimeter of triangle $\mathbf{A}_1\mathbf{B}_1\mathbf{C}_1$. As we saw at the end of §2, this arc is half the amount of the rotation $B''C'$. Hence

THEOREM 4.1. *The amount of rotation involved in the product of reflections in the sides of an acute-angled spherical triangle* **ABC**, *in any order, is equal to the perimeter of the pedal triangle* $\mathbf{A}_1\mathbf{B}_1\mathbf{C}_1$.

The right triangle $\mathbf{AA_1Q}$ has side $\mathbf{A_1Q}=s_1$ and hypotenuse $\mathbf{AA_1}=h_a$. Hence[3]

$$\begin{aligned}\sin s_1 &= \sin h_a \sin A\\ &= 2N = -2\frac{\cos S}{\tan R} = \frac{2\sin \Delta/2}{\tan R}\end{aligned} \tag{4.2}$$

where Δ is the area (or spherical excess) of triangle **ABC**, and R is the circumradius.

The equation $\sin \mathbf{A_1Q}=2N$ remains valid when C is obtuse. But then (if A and B are still acute) the excircle of $\mathbf{A_1B_1C_1}$ that has center **A** is the one beyond $\mathbf{C_1A_1}$, so the tangent $\mathbf{A_1Q}$ is not s_1 but s_1-c_1, and the amount of rotation is

$$2\mathbf{A_1Q} = 2(s_1-c_1) = a_1+b_1-c_1.$$

The following alternative proof of Theorem 4.1 is possibly of some interest. Beginning with the acute-angled triangle **ABC**, let us reflect in the side **BC** to obtain **A′BC**, then reflect the latter in its side **CA′** to obtain **A′B′C**, and finally reflect **A′B′C** in **A′B′** to obtain **A′B′C′**, as in Fig. 4. Let the respective pedal triangles be $\mathbf{A_1B_1C_1}$, $\mathbf{A_1B_2C_2}$, $\mathbf{A_2B_2C_3}$, $\mathbf{A_3B_3C_3}$. Since **BC** is the external bisector of $\angle\mathbf{B_1A_1C_1}$, the great circle $\mathbf{C_1A_1}$ contains $\mathbf{B_2}$; similarly it also contains $\mathbf{C_3}$ and $\mathbf{A_3}$. The rotatory reflection CBA, which transforms the triangles **ABC** and $\mathbf{A_1B_1C_1}$ into **A′B′C′** and $\mathbf{A_3B_3C_3}$, is thus clearly exhibited as the reflection in the great circle $\mathbf{C_1A_1}$ combined with a rotation *along* that great circle, of amount

$$\mathbf{C_1C_3} = \mathbf{A_1A_3} = \mathbf{A_1B_2} + \mathbf{B_2C_3} + \mathbf{C_3A_3} = c_1 + a_1 + b_1 = 2s_1.$$

On the other hand, when C is obtuse (while A and B are acute) the point $\mathbf{B_2}$ is situated between $\mathbf{C_1}$ and $\mathbf{A_1}$, so we have

$$\mathbf{C_1C_3} = \mathbf{A_1A_3} = -\mathbf{B_2A_1} + \mathbf{B_2C_3} + \mathbf{C_3A_3} = -c_1 + a_1 + b_1 = 2(s_1-c_1).$$

5. The principle of transformation. Let S denote the sphere in any particular position. Let E and F be two congruent transformations which change S into S^E and S^F (without moving the center). Then F changes S and S^E into S^F and S^{EF}. Since $S^{EF}=(S^F)^{F^{-1}EF}$, this means that F *transforms* the operation E into $F^{-1}EF$. Whatever kind of operation E may be, $F^{-1}EF$ is the same kind; e.g., if E is a rotation through a certain angle about an axis l, $F^{-1}EF$ will be a rotation through the same angle about the transformed axis l^F.

Now, since the reflection A transforms the rotatory reflection ABC into $A^{-1}\cdot ABC\cdot A=BCA$, it follows that A reflects the line of action $\mathbf{A_1C_1}$ of the former into the line of action $\mathbf{B_1A_1}$ of the latter; so we have an alternative proof that **BC** is the external bisector of the angle $\mathbf{B_1A_1C_1}$.

The same principle would have enabled us to foresee that the amount of rotation involved in ABC is the same as in BCA or CAB. (The remaining products CBA, ACB, BAC obviously involve the same amount of rotation, because they are the inverse operations.)

6. The case of a right triangle. When two of the mirrors are perpendicular, the pedal triangle collapses.

[3] For this use of the letters N and S, see Todhunter, *Spherical trigonometry*, London, 1914, pp. 29, 33. 92, 109; or McClelland and Preston, *A treatise on spherical trigonometry*, London, 1886, Part I, p. 55, and Part II, p. 11.

If only one right angle occurs, one line of action is the altitude to the hypotenuse; to be precise, if C is a right angle (so that $\mathbf{A}_1$ and $\mathbf{B}_1$ coincide with $\mathbf{C}$, and $AB=BA$), then $\mathbf{CC}_1$ is the line of action of $ABC=BAC$ (or, in the opposite sense, of $CBA=CAB$). By the principle of transformation, the only other line of action (viz., that of ACB and its inverse BCA) is the image of $\mathbf{CC}_1$ by reflection in either of the two perpendicular mirrors. The amount of rotation is equal to the perimeter of the collapsed pedal triangle, viz., $2h_c$.

If two right angles occur, there are only two inverse rotatory reflections, and the unique line of action is provided by that mirror which is perpendicular to both the others. But if all three angles are right angles, the product is the central inversion, and the line of action is indeterminate.

To sum up: the number of distinct lines of action, regardless of sense, is equal to the number of oblique angles.

7. Comparison with the work of Synge and Tuckerman. Synge represents the three mirrors by respectively perpendicular unit vectors $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$, and thence by points (having the same names) on the unit sphere. He shows that the product of the three reflections is a *rotatory inversion*, i.e., a rotation about a certain axis combined with the central inversion (p. 168). The amount of the rotation, denoted by θ, is a certain function of the vectors $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$. The six possible permutations of the three mirrors lead to rotations through $\pm\theta$ about three axes, each combined with the central inversion. These three "optic axes" are represented by unit vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$; and he shows very elegantly that the points $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ are the midpoints of the sides of the spherical triangle $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$ (p. 170) and that $\pi+\theta$ is equal to the spherical excess of that triangle (p. 172).

Since the central inversion may be regarded as a rotatory reflection involving rotation through π, a rotatory inversion involving rotation through θ is the same as a rotatory reflection involving rotation through $\pi+\theta$ (or through $\pi-\theta$ in the opposite sense). The axis of the rotatory inversion joins the poles of the line of action of the rotatory reflection.

Thus Synge's $\mathbf{N}_1\mathbf{N}_2\mathbf{N}_3$ and $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$ are the *polar triangles* of our $\mathbf{ABC}$ and $\mathbf{A}_1\mathbf{B}_1\mathbf{C}_1$. His statement that $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ are the midpoints of the sides of $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$ is simply the dual of our statement that each side of triangle $\mathbf{ABC}$ makes equal angles with the two non-corresponding sides of triangle $\mathbf{A}_1\mathbf{B}_1\mathbf{C}_1$. (Theorem 3.2.) Similarly, dualizing the statement that $\mathbf{AB}$ is perpendicular to $\mathbf{CC}_1$, we find[4] that $\mathbf{N}_3$ is distant $\pi/2$ from the point of intersection $(\mathbf{N}_1\mathbf{N}_2\cdot\mathbf{A}_1\mathbf{A}_2)$.

Since the angles of Synge's triangle $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$ are supplementary to the sides of its polar triangle (our $\mathbf{A}_1\mathbf{B}_1\mathbf{C}_1$), the spherical excess of the former is equal to

$$(\pi - a_1) + (\pi - b_1) + (\pi - c_1) - \pi = 2\pi - (a_1 + b_1 + c_1) = 2\pi - 2s_1.$$

Thus $\pi-\theta=2s$, in agreement with our Theorem 4.1.

Tuckerman obtains the same results again, representing the vectors by pure quaternions. His method shows very simply that $\pm\cos\theta/2$ is equal to the scalar triple product of the vectors $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ (which Synge denotes by P). Consequently

$$\cos\theta = 2P^2 - 1 = 8\sin\sigma\sin(\sigma-\alpha_1)\sin(\sigma-\alpha_2)\sin(\sigma-\alpha_3) - 1,$$

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[4] Todhunter, *op. cit.*, p. 114.

where α_1, α_2, α_3, σ are the sides and semiperimeter of the spherical triangle $\mathbf{N_1N_2N_3}$. Comparing Synge's (3.10) with Tuckerman's analogous Eqs. (13), we see that

$$\sin^2 \theta/2 = k^{-2} = \cos^2 \alpha_1 + \cos^2 \alpha_2 + \cos^2 \alpha_3 - 2 \cos \alpha_1 \cos \alpha_2 \cos \alpha_3.$$

In terms of the mirror triangle **ABC** (with $2s_1 = \pi - \theta$), this takes the form

$$\cos^2 s_1 = \cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C. \tag{7.1}$$

Thus $\sin s_1 = P = 2N$, in agreement with (4.2).

8. Three mirrors forming a prism. When the common point of the three mirrors recedes to infinity, so that the mirrors are all perpendicular to one plane, we have essentially a two-dimensional problem: the product of reflections in the three sides of a plane triangle **ABC**. Work analogous to §2 gives $ABC = A'B''C'$, where A' is the reflection in a line $\mathbf{a}'$ perpendicular to both $\mathbf{b}''$ and $\mathbf{c}'$. We now have a glide-reflection: the product of the reflection A' and the translation $B''C'$ along the "line of action" $\mathbf{a}'$. Theorems 2.1 and 4.1 remain valid with the word "rotation" changed to "translation." The plane version of Theorem 3.1 is closely connected with the fact that the pedal triangle has angles $\pi - 2A$, $\pi - 2B$, $\pi - 2C$. Finally, (4.2) reduces to the well known formula

$$s_1 = \Delta/R$$

for the semiperimeter of the pedal triangle of a plane triangle.

9. The product of m reflections in m dimensions. Since the product of two reflections is a rotation, the product of m is equivalent to $[m/2]$ rotations (with an extra reflection if m is odd), and it is possible to arrange for the planes of the rotations to be all completely orthogonal, so that the rotations (and the extra reflection when it occurs) are commutative. Let the amounts of the rotations be $2s_1, 2s_2, \cdots, 2s_{[m/2]}$. We wish to express them in terms of the angles between pairs of mirrors. Let c_{jk} denote the cosine of the internal angle between the jth and kth mirrors. Then[5] $e^{\pm 2s_1 i}, e^{\pm 2s_2 i}, \cdots$ (and -1 if m is odd) are the roots of the equation

$$\begin{vmatrix} x+1 & -2c_{12}x & -2c_{13}x & \cdots & -2c_{1m}x \\ -2c_{21} & x+1 & -2c_{23}x & \cdots & -2c_{2m}x \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -2c_{m1} & -2c_{m2} & -2c_{m3} & \cdots & x+1 \end{vmatrix} = 0.$$

When $m = 3$, this reduces to

$$(x+1)^3 - 4(c_{23}^2 + c_{31}^2 + c_{12}^2 + 2c_{23}c_{31}c_{12})x(x+1) = 0;$$

so we have

$$(2 \cos s_1)^2 = (x^{1/2} + x^{-1/2})^2 = 4(c_{23}^2 + c_{31}^2 + c_{12}^2 + 2c_{23}c_{31}c_{12}),$$

in agreement with (7.1).

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[5] For a proof, see Coxeter, *Lösung der Aufgabe* 245, Jahresbericht der Deutschen Mathematiker-Vereinigung, **49**, *4–6* (1939).