ANALYSIS OF THE TURBULENT BOUNDARY LAYER FOR ADVERSE PRESSURE GRADIENTS INVOLVING SEPARATION*

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PART 1.

1. Introduction. In a recent paper by von Doenhoff and Tetervin,[1] and subsequently in another by Garner,[2] a new differential equation of an empirical character has been advanced, whereby the analysis of turbulent boundary layer development, and the prediction of turbulent separation, are greatly facilitated. The present paper deals with the problem theoretically, and, from considerations of the fundamental equations of motion, establishes an analogous expression which, though analytically different from that proposed by the above authors, contains the same parameters. It is further shown how this equation may be solved numerically by an approximate method of a rather complex character.

On comparing results with those obtained from the empirical relations, it appears that there is good agreement when the pressure gradient is small, but that discrepancies are more serious for larger gradients tending to separation. In this respect there is a notable divergence between the results given by von Doenhoff's formula on the one hand, and that of Garner on the other, the former being appreciably smaller at values of H in the region of separation. It appears, however, that Garner's relation involves the three basic parameters of the theoretical equation, whereas, in the case of von Doenhoff and Tetervin, only two are apparent. There is reason to think, therefore, that Garner's treatment may be the more reliable, a conclusion which is supported by the theoretical calculations, in that they are in very much better agreement with Garner's predictions than with those of von Doenhoff and Tetervin. Nevertheless, despite the above discrepancies, step-by-step integration of the empirical equations, in conjunction with the momentum equation, leads to estimates of the boundary layer characteristics in good agreement with experiment, so that the errors appear to be less important in the final result.

So far as the theoretical treatment is concerned, it would seem of value in establishing the essential parameters associated with the new equation, but for ease and rapidity of calculation in practical cases the empirical approach may prove more attractive.

We shall now proceed to elaborate the fundamental arguments leading to the equation concerned.

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[1] A. E. von Doenhoff and N. Tetervin, *Determination of general relations for the behaviour of turbulent boundary layers*, Nat. Ad. Comm. for Aeron. confidential report No. 3G13 (1943). Also reprinted by Aeron. Res. Comm., Report 6845, F.M.597, Ae. 2255 (1943).

[2] H. C. Garner, *The development of turbulent boundary layers*, Aeron. Res. Comm., Report 7814, F.M. 705 (1944).

2. Equations of fully developed, turbulent flow. The Reynolds equations[3] of turbulent motion in a two dimensional field of incompressible, viscous flow take the well known form

$$\frac{D\bar{u}}{Dt} = \frac{1}{\rho}\left[\frac{\partial}{\partial x}(\bar{p}_{xx} - \rho\overline{uu'}) + \frac{\partial}{\partial y}(\bar{p}_{yx} - \rho\overline{u'v'})\right], \tag{2.1}$$

$$\frac{D\bar{v}}{Dt} = \frac{1}{\rho}\left[\frac{\partial}{\partial x}(\bar{p}_{xy} - \rho\overline{u'v'}) + \frac{\partial}{\partial y}(\bar{p}_{yy} - \rho\overline{v'v'})\right]. \tag{2.2}$$

There is also the equation of continuity,

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0, \tag{2.3}$$

where bars denote temporal mean values and primes the fluctuating components.

Adopting Boussinesq's procedure[4] of representing the Reynolds stresses $\rho\overline{u'u'}$, $\rho\overline{v'v'}$, $\rho\overline{u'v'}$ by an apparent increase of viscosity, we replace the natural coefficient of viscosity μ by $(\mu+\rho\epsilon)$, where ϵ is the measure of the momentum interchange due to turbulence, and must therefore be considered to vary with respect to the space co-ordinates. The Reynolds stresses are further treated in exactly the manner prescribed by the general theory of stress. Accordingly,

$$\left.\begin{aligned} \bar{p}_{xx} - \rho\overline{u'u'} &= -\bar{p} + 2(\mu + \rho\epsilon)\frac{\partial \bar{u}}{\partial x} \\ \bar{p}_{yy} - \rho\overline{v'v'} &= -\bar{p} + 2(\mu + \rho\epsilon)\frac{\partial v}{\partial y} \\ \bar{p}_{xy} - \rho\overline{u'v'} = \bar{p}_{yx} - \rho\overline{u'v'} &= (\mu + \rho\epsilon)\left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y}\right) \end{aligned}\right\}. \tag{2.4}$$

and Eqs. (2.1), (2.2) become

$$\frac{D\bar{u}}{Dt} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + (\nu + \epsilon)\nabla^2\bar{u} + 2\frac{\partial \epsilon}{\partial x}\frac{\partial \bar{u}}{\partial x} + \frac{\partial \epsilon}{\partial y}\left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y}\right), \tag{2.5}$$

$$\frac{D\bar{v}}{Dt} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} + (\nu + \epsilon)\nabla^2\bar{v} + 2\frac{\partial \epsilon}{\partial y}\frac{\partial \bar{v}}{\partial y} + \frac{\partial \epsilon}{\partial x}\left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y}\right), \tag{2.6}$$

whence the bars denoting mean quantities are no longer necessary, and will therefore be omitted in what follows.

3. Curved flow. Intrinsic form of the equations. Equations (2.5), (2.6) will now be referred to the curvilinear co-ordinates of Fig. 1. Let AB be a segment of any streamline of the mean flow, and let PN be the orthogonal curve through the point P on the streamline where the resultant velocity is q. Let ds, dn be elements of arc of AB and PN respectively, and finally, let θ be the angle the tangent to PN at P makes with the axis of X.

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[3] O. Reynolds, *On the dynamical theory of incompressible viscuous fluids and the determination of the criterion*, Phil. Trans. (A) **186**, 123–164 (1895).

[4] T. V. Boussinesq, *Essai sur la théorie des eaux courantes*, Mém. Sav. Étrang. **23**, No. 1 (1877).

Then

$$u = -q\sin\theta, \qquad v = q\cos\theta;$$

$$\frac{\partial}{\partial x} = -\sin\theta\frac{\partial}{\partial s} + \cos\theta\frac{\partial}{\partial n}, \qquad \frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial s} + \sin\theta\frac{\partial}{\partial n};$$

$$\nabla^2 = \frac{\partial^2}{\partial s^2} - \frac{\partial\theta}{\partial n}\frac{\partial}{\partial s} + \frac{\partial\theta}{\partial s}\frac{\partial}{\partial n} + \frac{\partial^2}{\partial n^2},$$

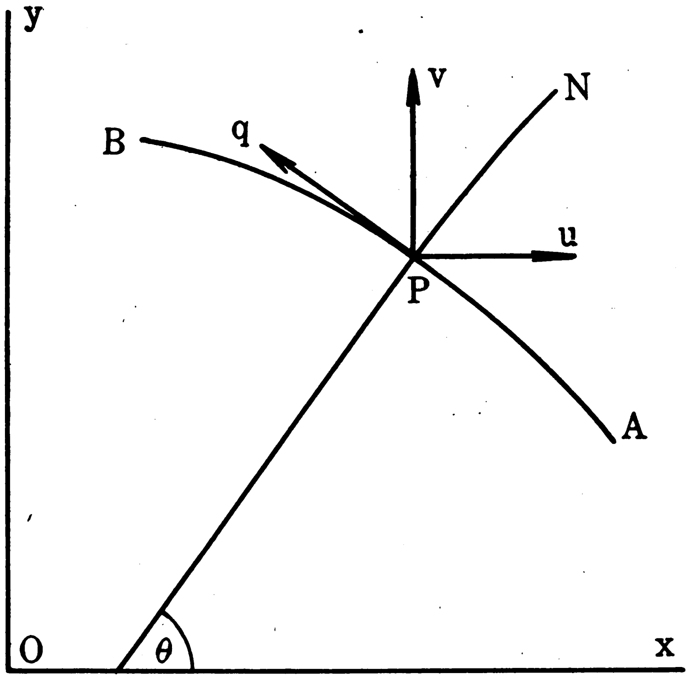

FIG. 1.

and the equations of motion for steady flow take the form

$$q\frac{\partial q}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + (\nu+\epsilon)\left[\frac{\partial^2 q}{\partial s^2} - \frac{\partial q}{\partial s}\frac{\partial\theta}{\partial n} - q\left\{\left(\frac{\partial\theta}{\partial s}\right)^2 + \left(\frac{\partial\theta}{\partial n}\right)^2\right\} + \frac{\partial q}{\partial n}\frac{\partial\theta}{\partial s} + \frac{\partial^2 q}{\partial n^2}\right]$$
$$+ 2\frac{\partial\epsilon}{\partial s}\frac{\partial q}{\partial s} + \frac{\partial\epsilon}{\partial n}\left(\frac{\partial q}{\partial n} - q\frac{\partial\theta}{\partial s}\right), \tag{3.1}$$

$$q^2\frac{\partial\theta}{\partial s} = \frac{1}{\rho}\frac{\partial p}{\partial n} + (\nu+\epsilon)\left[2\frac{\partial q}{\partial s}\frac{\partial\theta}{\partial s} + q\left\{\frac{\partial^2\theta}{\partial s^2} + \frac{\partial^2\theta}{\partial n^2}\right\} + 2\frac{\partial q}{\partial n}\frac{\partial\theta}{\partial n}\right]$$
$$+ 2q\frac{\partial\epsilon}{\partial n}\frac{\partial\theta}{\partial n} - \frac{\partial\epsilon}{\partial s}\left(\frac{\partial q}{\partial n} - q\frac{\partial\theta}{\partial s}\right); \tag{3.2}$$

whilst the equation of continuity leads to

$$\frac{\partial q}{\partial s} - q\frac{\partial\theta}{\partial n} = 0. \tag{3.3}$$

4. Application to the boundary layer. Integral equation for θ. In applying the equations of motion to the boundary layer, we make the usual assumptions that

(a) the fluid is of small viscosity;

(b) the boundary layer thickness δ is small, $O(\nu^{1/5})$;*

(c) the curvatures $\partial\theta/\partial s$, $\partial\theta/\partial n$ are nowhere large, $O(\nu^{-1/2})$.

Then, with the usual Prandtl approximation[5] of neglecting all terms other than those of first order magnitude, Eqs. (3.1), (3.2) reduce to

$$q\frac{\partial q}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + (\nu + \epsilon)\frac{\partial^2 q}{\partial n^2} + \frac{\partial \epsilon}{\partial n}\frac{\partial q}{\partial n}, \tag{4.1}$$

$$q^2\frac{\partial \theta}{\partial s} = \frac{1}{\rho}\frac{\partial p}{\partial n}, \tag{4.2}$$

the equation of continuity remaining as in (3.3).

Further, Eq. (4.2) can generally be neglected, since the total change of pressure along a normal must be $O(\delta^2)$. The derivatives of p with respect to s may accordingly be replaced by the total differentials. It will also be legitimate, on the basis of (b) and (c) above, to regard s as measured along the surface, and n along normals to the surface.

Substitute, now, (3.3) in (4.1) and integrate with respect to n. Then

$$\rho\int_0^n q^2\frac{\partial\theta}{\partial n}dn + n\frac{dp}{ds} + C = (\mu + \rho\epsilon)\frac{\partial q}{\partial n}, \tag{4.3}$$

C being the integration constant.

For brevity, write τ for the total shear stress $(\mu+\rho\epsilon)(\partial q/\partial n)$. Then C is equal to the surface value τ_0, and (4.3) may be expressed as

$$f \equiv \frac{\tau}{\tau_0} = 1 + \omega_\delta\eta + \frac{\rho}{\tau_0}\int_0^\eta q^2\frac{\partial\theta}{\partial\eta}d\eta, \tag{4.4}$$

with $\omega_\delta \equiv \delta dp/\tau_0 ds$, $\eta \equiv n/\delta$. Hence, from (4.4)

$$\frac{\partial\theta}{\partial\eta} = \frac{\tau_0}{\rho q^2}\left(\frac{\partial f}{\partial\eta} - \omega_\delta\right),$$

or putting $X^2 \equiv \tau_0/\rho q_1^2$, where q_1 is the velocity at the outer limit of the boundary layer,

$$\frac{\partial\theta}{\partial\eta} = \frac{X^2}{\left(\frac{q}{q_1}\right)^2}\left(\frac{\partial f}{\partial\eta} - \omega_\delta\right), \tag{4.5}$$

* It is usual in the dimensional analysis of the equations of motion to regard δ as $O(\nu^{1/2})$, which is the case if the viscous terms are taken to be of the same order as the inertia terms. However, it is known in the case of the turbulent boundary layer that δ is proportional to $\nu^{1/5}$. Cf. S. Goldstein, *Modern developments in fluid dynamics*, vol. 2, University Press, Oxford, 1938, p. 362.

[5] L. Prandtl, *Über Flüssigkeitsbeweggungen bei sehr kleiner Reibung*, Verhand. des dritten internat. Math.-Kongress, Heidelberg, pp. 484–491 (1904).

so that

$$\theta = X^2 \int_0^\eta \frac{1}{\left(\frac{q}{q_1}\right)^2} \left(\frac{\partial f}{\partial \eta} - \omega_s\right) d\eta + \text{const.} \tag{4.6}$$

5. Equation of the boundary layer parameter *H*. If the constant of integration in (4.6) is chosen so that θ is zero at the surface ($\eta=0$), it follows that $-\theta$ will be equal to the angle between the tangent of the streamline at the point P (Fig. 1) and the tangent to the surface where the orthogonal through P strikes the surface; or in less precise terms, $-\theta$ measures the divergence of the mean flow with respect to the surface.

Let θ_1 be the value of θ at $\eta=1$, s_1 the distance along the outer edge of the boundary layer, and let the value ψ_1 of the stream function ψ be defined as

$$\psi_1 = \int_0^\delta q dn. \tag{5.1}$$

Then, since

$$\frac{d\psi_1}{ds_1} = \frac{\partial \psi_1}{\partial s}\frac{\partial s}{\partial s_1} + \frac{\partial \psi_1}{\partial n}\frac{\partial n}{\partial s_1},$$

we have, to the order of accuracy of the boundary layer equations ($s_1 \doteq s$),

$$\frac{d\psi_1}{ds} = q_1\left(\theta_1 + \frac{d\delta}{ds}\right),$$

or

$$\psi_1 = \int q_1\left(\theta_1 + \frac{d\delta}{ds}\right) ds + \text{const.} \tag{5.2}$$

Now introduce the displacement length δ_1 which, in terms of δ, may be written non-dimensionally as

$$\frac{\delta_1}{\delta} = \int_0^1 \left(1 - \frac{q}{q_1}\right) d\eta. \tag{5.3}$$

With respect to (5.1) we have, therefore,

$$\psi_1 = q_1(\delta - \delta_1), \tag{5.4}$$

so that from (5.2) and (5.4)

$$\int q_1\left(\theta_1 + \frac{d\delta}{ds}\right) ds = q_1(\delta - \delta_1) + \text{const.}, \tag{5.5}$$

and, on differentiating (5.5) with respect to s and rearranging,

$$\frac{d\delta_1}{ds} + \frac{\delta_1}{q_1}\frac{dq_1}{ds} = \frac{\delta}{q_1}\frac{dq_1}{ds} - \theta_1. \tag{5.6}$$

There is also the momentum equation which takes the well known form

$$\frac{d\vartheta}{ds} + \frac{\vartheta}{q_1}\frac{dq_1}{ds}(H+2) = \frac{\tau_0}{\rho q_1^2} = X^2, \tag{5.7}$$

where the momentum length ϑ is defined as

$$\vartheta = \int_0^\delta \frac{q}{q_1}\left(1 - \frac{q}{q_1}\right) dn,$$

and $H = \delta_1/\vartheta$. Hence

$$\frac{dH}{ds} = \frac{d}{ds}\left(\frac{\delta_1}{\vartheta}\right) = \frac{1}{\vartheta}\left(\frac{d\delta_1}{ds} - \frac{\delta_1}{\vartheta}\frac{d\vartheta}{ds}\right),$$

and therefore

$$\vartheta\frac{dH}{ds} = \frac{d\delta_1}{ds} - H\frac{d\vartheta}{ds}\cdot \tag{5.8}$$

Now substitute for $d\delta_1/ds$ from (5.6) and for $d\vartheta/ds$ from (5.7). It is then found that Eqs. (5.6), (5.7) may be written alternatively as

$$\vartheta\frac{dH}{ds} = -X^2[\omega_\delta + H\{1 + \omega_\vartheta(H+1)\}] - \theta_1, \tag{5.9}$$

$$\frac{d\vartheta}{ds} = X^2[1 + \omega_\vartheta(H+2)], \tag{5.10}$$

in which $\omega_\delta \equiv \delta dp/\tau_0 ds$ and $\omega_\vartheta \equiv \vartheta dp/\tau_0 ds$.

6. Discussion of Eqs. (5.9), (5.10) in relation to the empirical formulae of von Doenhoff, Tetervin and Garner. It is shown in Part 2, Sec. 12, that Eq. (5.9) can be reduced to the form

$$\vartheta\frac{dH}{ds} = F_1(\omega_\vartheta, R_\vartheta, H), \tag{6.1}$$

R_ϑ being the Reynolds number $q_1\vartheta/\nu$. This is of interest in that von Doenhoff and Tetervin[1] recently introduced an empirical equation in terms of ω_ϑ and H, that is to say,

$$\vartheta\frac{dH}{ds} = F_2(\omega_\vartheta, H), \tag{6.2}$$

though analytically their expression differs appreciably from the theoretical relation. Following a similar procedure, Garner[2] has derived another empirical equation which, functionally, is of precisely the same form as the theoretical result (6.1), but which otherwise bears a close resemblance to the formula of von Doenhoff and Tetervin. Garner's solution, therefore, is of particular interest in relation to Eq. (5.9), and it may be useful to summarize briefly these empirical developments. We will begin by considering the original work of von Doenhoff and Tetervin.

They observe first that the velocity profile determines H, and point out that the converse statement cannot be proved theoretically. They accordingly proceed to subject the hypothesis to experimental test, and, from a considerable volume of experimental data, show that there is very convincing evidence to support the assumption that H uniquely defines the velocity profile. On the strength of this conclusion that the distribution is a uni-parametric function, they then consider how the external forces acting on the boundary layer are related to H. The argument is advanced that the rate of change of H, rather than H itself, is the determining factor, and it is further pointed out that this assumption has the desirable effect of connecting conditions downstream from a point with those upstream of the point. The problem which von Doenhoff and Tetervin investigate, therefore, is the degree of correlation between dH/ds, the pressure gradient dp/ds and the surface friction τ_0. In the first instance they attempt to establish a relation between the gradients of H and p when expressed non-dimensionally as* $\vartheta dH/ds$ and $-\vartheta d(p/\frac{1}{2}\rho q_1^2)/ds$, thus leaving the frictional term as an independent entity. From their analysis, however, the authors conclude that there is no general relationship of this kind, a systematic variation with Reynolds number being noted. They then consider the ratio of the dimensionless pressure gradient, given above, to the friction intensity which they write in the non-dimensional form $\tau_0/\rho q_1^2$. This ratio, it will be seen, is equal to $-2\vartheta dp/\tau_0 ds$. It is therefore proportional to ω_ϑ. In determining the above quantity, von Doenhoff and Tetervin tentatively assume the flat plate skin friction law, as given by Squire and Young,[6] irrespective of the pressure gradient. Allowing for the fact that dH/ds and dp/ds were obtained by graphical differentiation of the experimental results, they conclude that, at a constant value of H, there is an approximately linear relationship between $\vartheta dH/ds$ and $\vartheta dp/\tau_0 ds$, and consequently arrive at the general result indicated by Eq. (6.2). Finally, by analyzing their data at a number of prescribed values of H, they are able to formulate an arbitrary, analytical expression for (6.2) which, in terms of the present notation, is given in section 13, Eq. (13.1).

Garner follows essentially the line of development instigated by von Doenhoff and Tetervin. He expresses the momentum equation in the form first used by Howarth,[7] and, as a result, prefers a power law for the skin friction.[8] He also takes into account the effect of pressure gradients on the skin friction, whereas von Doenhoff and Tetervin are content in general with the plane flow approximation. Using Θ and Γ as basic variables, where** $\Theta \equiv \vartheta R_\vartheta^{1/n}$, and $\Gamma \equiv \Theta dq_1/q_1 ds$, Garner then obtains an equation for $\Theta dH/ds$ which is analytically similar to that of ref. 1. In the existing notation it may be expressed, however, in the alternative form given in section 13, Eq. (13.2), from which it will be seen, since X is a function of R_ϑ, that it is entirely consistent with Eq. (6.1).

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* The presence of the negative sign will be clear when it is recalled that von Doenhoff and Tetervin adopt the alternative form $\vartheta dq/q ds$, where, in their notation, q is the dynamic pressure at the outer limit of the boundary layer.

** The index is subsequently taken as $n=6$ in conformity with the choice of Falkner's equation for the skin friction.

[6] H. B. Squire and A. D. Young, *The calculation of the profile drag of aerofoils*, Tech. Rep. of the Aeron. Res. Comm., R & M. No. 1838 (1938).

[7] L. Howarth, *The theoretical determination of the lift coefficient for a thin elliptic cylinder*, Proc. Roy. Soc. (A) **149**, 574–575 (1935).

[8] V. M. Falkner, *A new law for calculating drag*, Airc. Eng., **15**, 65–69 (1943).

The empirical equation for $\vartheta dH/ds$, the momentum equation and a relation for the skin friction are then sufficient to enable the development of the turbulent boundary layer to be analyzed. The solution depends on a step-by-step integration which Garner elaborates in great detail by the finite difference calculus. It assumes that the static pressure distribution is given, and that the initial values of ϑ and H are known, e.g. at transition. This problem is discussed thoroughly in both refs. 1 and 2, and need not concern us further. The integration then yields ϑ and H at the beginning of each interval. Hence, we seek a form of Eq. (5.9) purely in terms of ϑ, H and p. This is the subject of Part 2.

PART 2.

7. Approximations regarding the turbulent and laminar layers. Boundary conditions. It is inevitable that a theoretical approach to the problem of the turbulent boundary layer must be appreciably more complex than the simple, empirical treatment of von Doenhoff and Tetervin. The difficulty of solving the fundamental equations of motion, even when the flow is laminar, is here increased by the fact that we have to consider an "apparent viscosity" which varies from point to point in the fluid. In order to deal with this feature of the flow, some kind of turbulent mechanism must be specified and incorporated into the basic equations. The very considerable difficulties of a rigorous treatment of this aspect of the problem are well known, and even an approximate and much simplified theory, as adopted in the present paper, leads to a rather complicated solution which does not yield an analytical function for $\vartheta dH/ds$. Indeed, it has not been possible to present the arguments advanced in the following pages in the form of a unified theory. Rather the investigation has been divided into a series of associated problems which are considered individually, and it is then shown how the results may be combined in a numerical solution to provide the data for the step-by-step integration referred to in section 6. As a consequence, it will probably assist the reader to give a statement as to the procedure now to be followed, and the nature of the approximations involved.

In the first place it will be assumed that the flow may be sub-divided into a turbulent layer, where the effects of fluid viscosity are negligible, and a thin surface layer (laminar sub-layer) where viscosity is predominant, the transition from the one flow to the other being regarded as occurring instantaneously, i.e. in zero length η. Neglect of ν when the motion is fully turbulent implies that $\epsilon \gg \nu$ (see Eq. 4.3), an assumption which is valid for sufficiently large Reynolds numbers, and one commonly made in considering this region.[9] It will also be convenient at first to take ω_δ as independent variable, and we will subsequently derive a method of transforming from ω_δ to ω_ϑ which is the variable required in performing the numerical integration of Eqs. (5.9), (5.10).

On the above basis, we shall then proceed to our first objective, namely the establishment of the velocity distribution in the turbulent and sub-laminar layers, the two cases being considered independently, but in such a manner that the essential boundary conditions are satisfied, and further, that the velocity is continuous at the transition from the laminar to turbulent states. Secondly, the important quantity θ_1 of Eq. (5.9) will be investigated. This, it will be shown, depends primarily on the condi-

[9] See, e.g., S. Goldstein (editor), *Modern developments in fluid dynamics*, vol. 2, University Press, Oxford, 1938, pp. 331–332.

tions in the laminar sub-layer, which cannot be entirely neglected for this reason. To a less, but appreciable degree, it also affects the magnitude of the parameter H. Finally, we develop a method whereby Eq. (5.9) may be evaluated numerically in terms of ϑ, H and p (or q_1, which is related to p by Bernoulli's theorem).

Turning, now, to the approximate nature of the analysis, we have first the simple theory of diffusion upon which the flow in the turbulent layer is based. It is assumed that the intensity components of the turbulent fluctuations do not differ appreciably (or at any rate are proportional to one another), and that the scale may be sufficiently represented by a mean length which is a function of the space co-ordinates. Although the subsequent development of this simplified conception of the turbulent mechanism near a surface in no way depends on a physical model, it is mathematically equivalent, nevertheless, to Prandtl's momentum transfer theory,[10] and is further closely connected with von Kármán's similarity theory[11] which appears as a special case in the present treatment.

Without entering into the details of the argument, it is finally shown that the turbulent velocity distribution may be expressed in terms of two functions f and g, and the surface friction X, namely,

$$\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right) = \frac{X}{\lambda f^{-1/2}},$$

where

$$\lambda f^{-1/2} = -\int_0^\eta g f^{-1/2} d\eta + \text{const.},$$

λ being a function of the correlation between the longitudinal and lateral velocity fluctuations, and the scale of the turbulence.

Strictly, when the natural viscosity of the fluid can not be ignored, f and g are dependent on ω_δ, η and the Reynolds number, but, when the flow is fully trubulent, we neglect, in accordance with our initial assumptions, the viscous terms, and treat both f and g purely as functions of ω_δ and η. The stress function f may then be equated to f_R, where f_R denotes the component due to the Reynolds shear stress. Hence, assuming for the moment that X is known, the problem of calculating the velocity distribution in the turbulent layer reduces to the determination of f_R and g. To this end, we first make use of an argument advanced in section 8 (a), namely that approximate solutions for f_R and g are sufficient provided $\lambda f_R^{-1/2}$ is itself correctly established. Accordingly, we express both f_R and g as a power series in η with coefficients which are to be purely functions of ω_δ. In the case of f_R the coefficients are chosen to satisfy all the known boundary conditions, with the exception of terms dependent on viscosity which are regarded as referring to the laminar sub-layer only. This leads to a solution in quartic form. In addition to the series for g, two particular solutions for λ are considered, viz. (a) when g = function of ω_δ only $= -\Lambda$, (b) when λ = function of ω_δ only $= \lambda_1$. It is then suggested that (a) holds near the surface, while (b) is applicable to-

[10] L. Prandtl, ***Bericht über Untersuchungen zur ausgebildeten Turbulenz***, Zeitschr. angew. Math. Mech. **5**, 136–139 (1925).

[11] T. v. Kármán, ***Mechanische Ähnlichkeit und Turbulenz***, Nachr. Ges. Wiss. Göttingen, Math-Phys. Klasse, 58–76 (1930).

wards the outer edge of the boundary layer. By identifying λ_1 with λ when $\eta=1$, and assuming that $\partial\lambda/\partial\eta$ is then zero, a solution for λ, in terms of Λ and λ_1 may then be obtained, if g is expressed as a quadratic, to satisfy condition (a) when $\eta=0$, and condition (b) when $\eta=1$. The resulting distribution of λ for intermediate values of η is then regarded provisionally as a valid approximation. Further, when $\omega_\delta=0$, it appears that the corresponding value of Λ (designated Λ_0) is then identical to von Kármán's constant K of the similarity theory.[11] Its value is therefore known. In addition, we obtain immediately an integral relation for Λ/Λ_0 purely in terms of f_R. Hence, apart from X, λ_1 is the only remaining unknown. As a first approximation, but nevertheless one which appears to be well substantiated by experiment, we assume that λ_1 is independent of ω_δ. Like Λ_0, it must then be regarded as a fundamental constant to be determined experimentally, the value in the present instance being calculated to give the best agreement between the theoretical and experimental velocity profiles for flow in parallel wall channels. Finally, neglecting second order terms, we obtain a very simple relation for the variation of X with ω_δ, namely $X/X_0=\Lambda/\Lambda_0$, where X_0 refers to the condition $\omega_\delta=0$, and is therefore calculable from either von Kármán's logarithmic skin friction equation,[11,12] or from a power law such as Falkner has published.[8] We thus establish a general solution for the velocity distribution in the turbulent part of the boundary layer.

As regards the treatment of the laminar sub-layer, little need be said. It is essentially in the nature of a linear (double link) interpolation which satisfies the main wall condition, and preserves continuity with the turbulent flow solution at the point of transition. It makes no reference, therefore, to the equations of motion. On the other hand, the approach is justified on the grounds that, when the Reynolds number is moderate or large, the laminar layer is quite thin, and further, that the velocity distribution is then mainly linear throughout the region concerned.

The solution for θ_1, however, requires a little more attention. As already pointed out, it is primarily dependent on conditions in the laminar layer; it is also largely determined by the stress function. Hence, the relation for f_R considered in the study of the turbulent velocity profile is certainly no longer tenable near the surface. We accordingly develop the stress function, which must now be written as f, as a new power series, applicable for small values of η only, and including the effects of viscosity.

The procedure for determining the coefficients of the series is similar to that adopted in dealing with the turbulent layer, except that in this case the boundary conditions are restricted to the surface. The series is also limited to a quartic in view of the fact that η is to be regarded as quite small; at the same time it is desirable on account of the increasingly involved character of the coefficients relating to higher powers of η. The distribution of f in the outer part of the boundary layer is then represented by a second, independent series which satisfies the conditions at $\eta=1$, and is continuous with the distribution of f near the wall. The fact that this "inner and outer" solution is not altogether consistent with the previous solution for f, when viscosity was entirely neglected, is regarded as unimportant, for in the first place both approaches are only approximate, and secondly, so far as θ_1 is concerned, it is the distribution of f in the laminar sub-layer which is critical, whilst the relatively large

[12] T. v. Kármán, *Turbulence and skin friction*, J. Aer. Sci. **1**, 1–20 (1934).

error which probably arises in the turbulent region has only slight significance. In the former case there is reason to believe that, when viscosity is included, the treatment leads to results of good accuracy.

Finally, having established θ_1, numerical estimates of $\vartheta dH/ds$ may be made from Eq. (5.9) in terms of ω_δ. An additional equation relating ω_δ and ω_ϑ is also given. Hence Eq. (5.9) may be integrated step-by-step according to ref. (1) or (2).

8. (a) Turbulent diffusion in the boundary layer. The first step towards the transformation of Eq. (5.9) is to develop a theory of turbulent diffusion, valid for the boundary layer. This is greatly complicated by the fact that the flow is anisotropic. Nevertheless, an approximate treatment, which has yielded valuable results in the case of turbulent flow over plane surfaces and in pipes, rests on the assumption that the intensity components $\sqrt{\overline{u'^2}}$, $\sqrt{\overline{v'^2}}$ do not differ appreciably, and further that the correlation may still be represented by a single scalar function. Extending this hypothesis to the case of flow in the presence of pressure gradients generally, we write

$$\epsilon \doteqdot l\sqrt{\overline{v'^2}}, \tag{8.1}$$

where l is the length defining the average scale of the turbulence at any point. To the order of accuracy of the boundary layer equation $\rho\epsilon\partial q/\partial n$ is equal to the original Reynolds stress $-\rho\overline{u'v'}$. Denote this component by τ_R. Then

$$\frac{\tau_R}{\rho} = \epsilon\frac{\partial q}{\partial n} = -\overline{u'v'} = R_{uv}\sqrt{\overline{u'^2}}\sqrt{\overline{v'^2}}, \tag{8.2}$$

where the correlation coefficient R_{uv} is here defined as

$$R_{uv} = -\frac{\overline{u'v'}}{\sqrt{\overline{u'^2}}\sqrt{\overline{v'^2}}} = -\frac{\overline{u'v'}}{\overline{v'^2}},$$

since, by hypothesis, $\overline{u'^2} \doteqdot \overline{v'^2}$.

From (8.2) we then have

$$\overline{v'^2} = \frac{\epsilon}{R_{uv}}\frac{\partial q}{\partial n}, \tag{8.3}$$

and combining (8.3) with (8.1),

$$\epsilon = \frac{l^2}{R_{uv}}\frac{\partial q}{\partial n}, \tag{8.4}$$

so that the Reynolds stress may be written non-dimensionally as

$$\frac{\tau_R}{\rho q_1^2} = \frac{1}{R_{uv}}\left(\frac{l}{\delta}\right)^2\left|\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right)\right|\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right) = \lambda^2\left|\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right)\right|\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right), \tag{8.5}$$

the function $\lambda \equiv R_{uv}^{-1/2}(l/\delta)$ being, in general, a function of η, ω_δ and the Reynolds number, since the turbulent mechanism is influenced by viscosity near the surface. It follows that

$$f_R \equiv \frac{\tau_R}{\tau_0} = \frac{\lambda^2}{X^2}\left|\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right)\right|\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right). \tag{8.6}$$

We now seek to express λ in terms of the stress function f_R. First differentiate (8.6) with respect to η. Then

$$\frac{\partial\lambda}{\partial\eta}+\left[\frac{\partial^2 q}{\partial\eta^2}\Big/\frac{\partial q}{\partial\eta}-\frac{1}{2f_R}\frac{\partial f_R}{\partial\eta}\right]\lambda=0. \tag{8.7}$$

Without loss of generality we may also write

$$\lambda=g\frac{\partial q}{\partial\eta}\Big/\frac{\partial^2 q}{\partial\eta^2}, \tag{8.8}$$

g being also a function of η, ω_δ and the Reynolds number. Hence, combining (8.7), (8.8)

$$\frac{\partial\lambda}{\partial\eta}-\frac{1}{2f_R}\frac{\partial f_R}{\partial\eta}\lambda=-g, \tag{8.9}$$

which on integration yields

$$\lambda=-f_R^{1/2}\left[\int_0^\eta g f_R^{-1/2}d\eta+\text{const.}\right], \tag{8.10}$$

or writing $I_1\equiv-\int_0^\eta g f_R^{-1/2}d\eta+\text{const.}$,

$$\lambda=f_R^{1/2}I_1. \tag{8.11}$$

Ignoring for the present the integration constant in (8.10), consider, now, the solution when g is a function only of ω_δ and Reynolds number, so that with respect to η

$$g=\text{constant}=-\Lambda. \tag{8.12}$$

Then

$$\lambda=\Lambda f_R^{1/2}\int_0^\eta f_R^{-1/2}d\eta, \tag{8.13}$$

for which there is a maximum at $\eta=\eta_1$ such that

$$f_{R\eta_1}^{-1/2}\left(\frac{\partial f_R}{\partial\eta}\right)_{\eta_1}\int_0^{\eta_1}f_R^{-1/2}d\eta=-2. \tag{8.14}$$

Again, the solution which makes

$$\lambda=\text{constant}=\lambda_1 \tag{8.15}$$

for all values of η is given by

$$g=\frac{1}{2f_R}\frac{\partial f_R}{\partial\eta}\lambda_1. \tag{8.16}$$

Now assign to λ_1 the value of λ when $\eta=1$. If, further, we make the reasonable assumption $\partial\lambda/\partial\eta\to0$ as $\eta\to1$, then a solution for λ is obtained, which satisfies the above conditions, by equating (8.11) to λ_1 when $\eta=\eta_1$, with $\lambda=\lambda_1$ in the range $\eta_1<\eta<1$.

Whether or not this solution for λ leads to boundary layer characteristics in accordance with fact can only be decided by test, but for the present we will regard it provisionally as satisfactory. Hence the necessary and sufficient conditions are

$$\left.\begin{array}{lll} \eta = 0, & \lambda = 0, & g = -\Lambda \\ \eta = \eta_1, & \lambda = \lambda_1, & g = \dfrac{1}{2f_{R\eta_1}}\left(\dfrac{\partial f_R}{\partial \eta}\right)_{\eta_1}\lambda_1 \end{array}\right\}. \tag{8.17}$$

In general write

$$g = -\Lambda\left(1 + \sum_{n=1} \Lambda_n \eta^n\right), \tag{8.18}$$

where Λ_n are arbitrary constants. Then (8.17) will be satisfied by the series

$$g = -\Lambda(1 + \Lambda_1\eta + \Lambda_2\eta^2), \tag{8.19}$$

so that from (8.10)

$$\lambda = \Lambda f_R^{1/2}\left[\int_0^\eta f_R^{-1/2} d\eta + \Lambda_1 \int_0^\eta \eta f_R^{-1/2} d\eta + \Lambda_2 \int_0^\eta \eta^2 f_R^{-1/2} d\eta\right], \tag{8.20}$$

and

$$\frac{\partial \lambda}{\partial \eta} = \Lambda\left[(1 + \Lambda_1\eta + \Lambda_2\eta^2) + \tfrac{1}{2} f_R^{-1/2} \frac{\partial f_R}{\partial \eta}\left(\int_0^\eta f_R^{-1/2} d\eta + \Lambda_1 \int_0^\eta \eta f_R^{-1/2} d\eta + \Lambda_2 \int_0^\eta \eta^2 f_R^{-1/2} d\eta\right)\right]. \tag{8.21}$$

Let

$$G_0 \equiv f_{R\eta_1}^{1/2} \int_0^{\eta_1} f_R^{-1/2} d\eta, \qquad G_1 \equiv f_{R\eta_1}^{1/2} \int_0^{\eta_1} \eta f_R^{-1/2} d\eta,$$

$$G_2 \equiv f_{R\eta_1}^{1/2} \int_0^{\eta_1} \eta^2 f_R^{-1/2} d\eta,$$

and

$$H_0 \equiv 1 + \tfrac{1}{2} f_{R\eta_1}^{-1/2} \left(\frac{\partial f_R}{\partial \eta}\right)_{\eta_1} \int_0^{\eta_1} f_R^{-1/2} d\eta,$$

$$H_1 \equiv \eta_1 + \tfrac{1}{2} f_{R\eta_1}^{-1/2} \left(\frac{\partial f_R}{\partial \eta}\right)_{\eta_1} \int_0^{\eta_1} \eta f_R^{-1/2} d\eta,$$

$$H_2 \equiv \eta_1^2 + \tfrac{1}{2} f_{R\eta_1}^{-1/2} \left(\frac{\partial f_R}{\partial \eta}\right)_{\eta_1} \int_0^{\eta_1} \eta^2 f_R^{-1/2} d\eta.$$

Then (8.14) makes

$$H_0 = 0, \tag{8.22}$$

and, to satisfy (8.17), we have from (8.20), (8.21)

$$\left.\begin{aligned} G_0 + G_1\Lambda_1 + G_2\Lambda_2 &= \frac{\lambda_1}{\Lambda} \\ H_1\Lambda_1 + H_2\Lambda_2 &= 0 \end{aligned}\right\}. \tag{8.23}$$

Therefore,

$$\Lambda_1 = \frac{H_2\left(\dfrac{\lambda_1}{\Lambda} - G_0\right)}{(G_1H_2 - G_2H_1)}, \tag{8.24}$$

$$\Lambda_2 = -\frac{H_1}{H_2}\Lambda_1. \tag{8.25}$$

Hence, by the definition of I_1, when $0<\eta<\eta_1$,

$$I_1 = \Lambda\int_0^\eta (1 + \Lambda_1\eta + \Lambda_2\eta^2)f_R^{-1/2}d\eta + \text{const.}; \tag{8.26}$$

and when $\eta_1<\eta<1$,

$$I_1 = \lambda_1 f_R^{-1/2} + \text{const.} \tag{8.27}$$

It follows from Eqs. (8.6), (8.11) that

$$\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right) = \frac{X}{I_1}, \tag{8.28}$$

whence, integrating along normals from the outer edge of the boundary layer towards the surface, and allowing for the condition $q\to q_1$ as $n\to\delta$,

$$\frac{q}{q_1} = 1 + I_2X, \tag{8.29}$$

where $I_2 \equiv \int_1^\eta I_1^{-1}d\eta$. Hence, the velocity profile is determined primarily, not by the individual values of f_R and λ, but only by the product $\lambda f_R^{-1/2}$, for which there will be a unique distribution across any section of the boundary layer, depending on the flow conditions. We may therefore seek approximate solutions, of an arbitrary character, for f_R and λ, provided that, when combined in the above manner, they yield a solution of (8.29) in accordance with the physical facts.

In the next section we shall make use of this argument to obtain a general solution of the velocity distribution in the turbulent layer.

8. (b) Velocity distribution in the turbulent layer. It has been shown experimentally[13,14,15] that, for moderate or large Reynolds numbers, fully developed turbulent flow in the absence of a pressure gradient, or when the gradient is small (as in pipe

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[13] H. Darcy, *Recherches expérimentales relatives au mouvement de l'eau dans les tuyaux*, Mém. Sav. Étrang. **15**, 141–403 (1858).

[14] T. Stanton, *The mechanical viscosity of fluids*, Proc. Roy. Soc. (A) **85**, 366–376, (1911).

[15] W. Fritsch, *Der Einfluss der Wandrauhigkeitsverteilung in Rinnen*, Zeitschr. angew. Math. Mech. **8**, 199–216 (1928).

flow), does not depend on viscosity. This fact is expressed quantitatively by the well known velocity defect law which, in the present notation, becomes*

$$-I_2 = \frac{1}{X}\left(1 - \frac{q}{q_1}\right) = \phi(\eta), \tag{8.30}$$

or, extending (8.30) to include the case when the pressure gradient is not negligibly small ($\omega_\delta \neq 0$),

$$-I_2 = \frac{1}{X}\left(1 - \frac{q}{q_1}\right) = \phi(\omega_\delta, \eta). \tag{8.31}$$

Hence,

$$\frac{\partial I_2}{\partial \eta} = \frac{1}{I_1} = \frac{f_R^{1/2}}{\lambda} = \phi'(\omega_\delta, \eta). \tag{8.32}$$

But, from the preceding section, we have obtained λ, and therefore I_1, in terms of f_R and the variables Λ and λ_1 which are purely functions of ω_δ. Hence, it follows that in the turbulent region, for which Eqs. (8.31), (8.32) are valid, f_R must be a function of ω_δ and η only. Moreover, we may then write $f \doteq f_R$, and therefore f is also simply a function of ω_δ and η.

We will now obtain an expression for I_1, which satisfies the turbulent velocity distribution generally, by the argument advanced in the last section. This, it will be recalled, depends only on the approximate determination of both f_R and λ, provided that $\lambda f_R^{-1/2}$ is correctly established. First, therefore, express f_R in an arbitrary form. The problem then reduces to the determination of Λ and λ_1 so as to satisfy the above conditions for I_1. Accordingly, proceeding to the approximate development of f_R, we note first that Eq. (4.4) has the alternative form

$$f = 1 + \omega_\delta \eta + \frac{\delta}{X^2}\int_0^\eta \left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\left(\frac{q}{q_1}\right) d\eta. \tag{8.33}$$

Successive differentiation of (8.33) also gives

$$\frac{\partial f}{\partial \eta} = \omega_\delta + \frac{\delta}{X^2}\left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\left(\frac{q}{q_1}\right),$$

$$\frac{\partial^2 f}{\partial \eta^2} = \frac{\delta}{X^2}\left[\frac{\partial}{\partial \eta}\left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\left(\frac{q}{q_1}\right) + \left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\frac{\partial}{\partial \eta}\left(\frac{q}{q_1}\right)\right],$$

$$\frac{\partial^3 f}{\partial \eta^3} = \frac{\delta}{X^2}\left[\frac{\partial^2}{\partial \eta^2}\left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\left(\frac{q}{q_1}\right) + 2\frac{\partial}{\partial \eta}\left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\frac{\partial}{\partial \eta}\left(\frac{q}{q_1}\right) + \left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\frac{\partial^2}{\partial \eta^2}\left(\frac{q}{q_1}\right)\right],$$

etc.

Consider, next, the known boundary conditions. They are:

* This equation is commonly written in the form $(U_c - U)/U_T = f(y/h)$.

$$\eta = 0, \quad \frac{q}{q_1} = 0, \quad \frac{\partial}{\partial s}\left(\frac{q}{q_1}\right) = 0, \quad \frac{\partial}{\partial \eta}\left(\frac{q}{q_1}\right) = \frac{q_1\delta}{\nu} X^2 = R_\delta X^2;$$

$$\eta = 1, \quad \frac{q}{q_1} = 1, \quad q_1 \frac{dq_1}{ds} = -\frac{1}{\rho}\frac{dp}{ds}, \quad \frac{\partial}{\partial \eta}\left(\frac{q}{q_1}\right) = 0.$$

Hence, for f and the known derivatives, we have

$$\eta = 0, \quad f = 1, \quad \frac{\partial f}{\partial \eta} = \omega_\delta, \quad \frac{\partial^2 f}{\partial \eta^2} = 0,$$

$$\frac{\partial^3 f}{\partial \eta^3} = 4XR_\delta^2\delta\left[\frac{dX}{ds} + \frac{X}{q_1}\frac{dq_1}{ds}\right],$$

and when

$$\eta = 1, \quad f = 0, \quad \frac{\partial f}{\partial \eta} = 0.$$

These conditions must be satisfied in addition to that discussed earlier, namely, for fully developed turbulence, the flow is independent of the Reynolds number when moderate or large. It also follows that f_R may then be taken equal to f. Express, therefore, f_R as a power series in η, with coefficients which are purely functions of ω_δ. If, in addition, f_R is to satisfy the boundary conditions, we must ignore those in which the Reynolds number occurs explicitly, the argument implying that such terms are only important in the laminar sub-layer. This leaves five boundary conditions to be satisfied. Accordingly, we write*

$$f_R = A_0 + A_1\eta + A_2\eta^2 + A_3\eta^3 + A_4\eta^4. \tag{8.34}$$

Hence,

$$A_0 = 1, \quad A_1 = \omega_\delta, \quad A_2 = 0, \tag{8.35}$$

and

$$\left.\begin{aligned} A_0 + A_1 + A_2 + A_3 + A_4 &= 0 \\ A_1 + 2A_2 + 3A_3 + 4A_4 &= 0 \end{aligned}\right\}$$

which on solution yield

$$\left.\begin{aligned} A_3 &= -4 - 3\omega_\delta \\ A_4 &= 3 + 2\omega_\delta \end{aligned}\right\}. \tag{8.36}$$

We must now consider the boundary functions Λ and λ_1. First, with respect to λ_1, let us provisionally assume that it has a unique value for the fully developed turbulent boundary layer (i.e. it is constant for all positive values of ω_δ). It then follows from (8.13) that

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* The above series, to fit the same boundary conditions, was first suggested by Fediaevsky. See K. Fediaevsky, *Turbulent boundary layer of an aerofoil*, J. Aer. Sci. **4**, 491–498 (1937).

$$\frac{\Lambda}{\Lambda_0} = \frac{\left[f_{R\eta_1}^{1/2} \int_0^{\eta_1} f_R^{-1/2} d\eta \right]_0}{f_{R\eta_1}^{1/2} \int_0^{\eta_1} f_R^{-1/2} d\eta}, \tag{8.37}$$

the suffix 0 to Λ and the bracket of the numerator on the right hand side of the equation denoting the condition $\omega_\delta = 0$. Hence, it remains to determine Λ_0 and λ_1. From Eqs. (8.8), (8.19) we see that, when $\eta = 0$,

$$\lambda = -\Lambda \frac{\partial q}{\partial \eta} \Big/ \frac{\partial^2 q}{\partial \eta^2}, \tag{8.38}$$

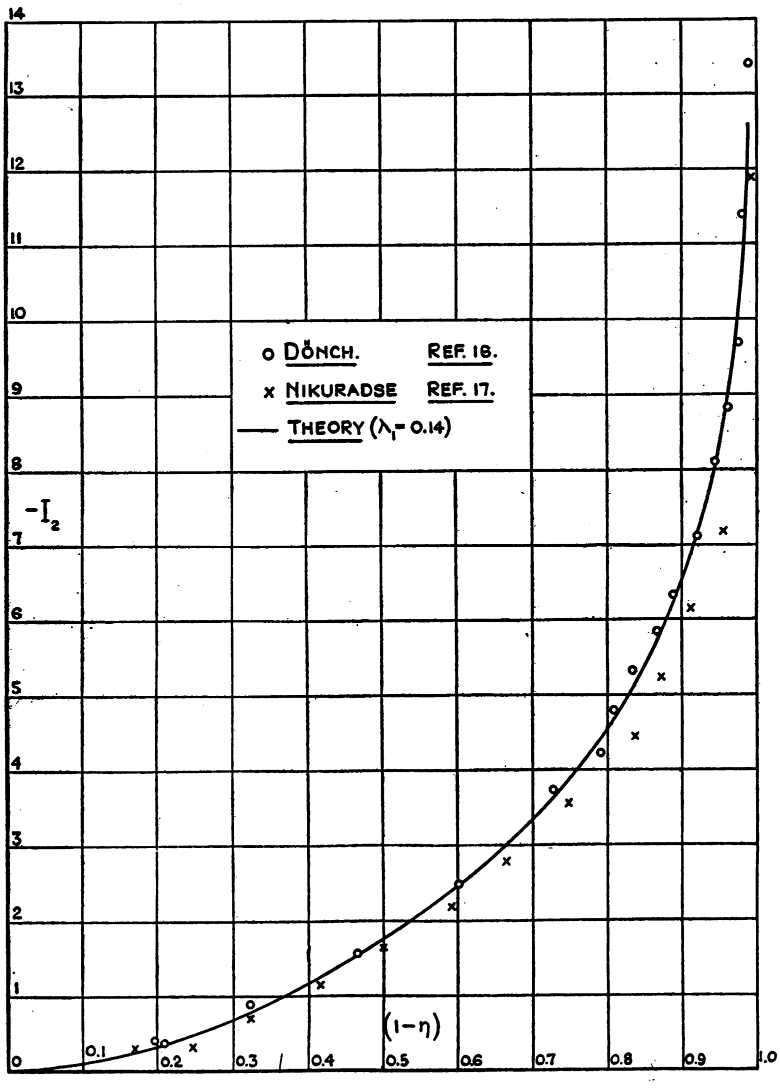

Fig. 2.

which is recognized as a form of von Kármán's equation for the length l in the similarity theory.[11] Accordingly, when $\omega_\delta=0$, Λ must be equal to the Kármán constant K whose value, from the logarithmic skin friction law for plane flow (see section 8 (c), Eq. 8.47) is 0.392. Thus, when

$$\omega_\delta = 0, \qquad \Lambda = \Lambda_0 = K = 0.392.$$

Finally, if λ_1 is to be independent of ω_δ, we may determine its value for any arbitrary condition of turbulent flow, e.g. for the case of plane flow. Like Λ_0, it must also be regarded as a constant, only ascertainable numerically by experiment. We will therefore assign to λ_1 that value which gives the best theoretical agreement with the data[16,17] of Fig. 2. This leads to $\lambda_1=0.14$, when it will be seen that the distribution of I_2 $(\omega_\delta=0)$ calculated from the numerical integration of Eqs. (8.26), (8.27), in conjunction with Eqs. (8.34), (8.35), (8.36), agrees very well with the experimental curves for all values of η outside the laminar sub-layer.

The validity of the assumption $\lambda_1=$constant, and hence of Eq. (8.37), is discussed in section 8 (c). Further evidence appears from the general comparison of the theoretical and experimental velocity profiles in section 10.

8. (c) Consideration of the general skin friction law $(\omega_\delta \geqq 0)$. The present development of the theory of turbulent flow near a surface depends essentially on the three boundary terms Λ, λ_1 and X, of which Λ and λ_1 have already been considered. As regards X, which determines the surface value of the shear stress, or skin friction intensity, previous investigators have mainly been content to ignore the effect of pressure gradients on X, and to take one of the well attested laws strictly applicable for plane flow only. We shall now show, however, that a simple, approximate relation may be derived to account for the variation of X with ω_δ, i.e. for the influence of pressure gradients on skin friction. This leads to the consideration of the flow conditions very near the surface, namely for very small values of η. Eq. (8.34) then reduces to

$$f_R = 1 + \omega_\delta \eta, \tag{8.39}$$

and, equally, Eq. (8.19) approximates to

$$g = -\Lambda. \tag{8.40}$$

Substituting, now, (8.39), (8.40) in (8.26) and integrating, we find

$$I_1 = \frac{2\Lambda}{\omega_\delta}\left(\sqrt{1+\omega_\delta\eta} - 1\right), \tag{8.41}$$

the constant of integration satisfying the condition $I_1=\eta=0$. Or, since $\omega_\delta\eta$ is small in relation to unity, (8.41) may be expanded to give simply

$$I_1 = \Lambda\eta. \tag{8.42}$$

Hence, by definition,

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[16] F. Dönsch, *Divergente und konvergente turbulente Strömung mit kleinen Öffnungswinkeln*, Forschungsber. Ver. Deutsch. Ing., p. 282 (1926).

[17] J. Nikuradse, *Untersuchungen über die Strömungen des Wassers in konvergenten und divergenten Kanälen*, Forschungsber. Ver. Deutsch. Ing., p. 289 (1929).

$$I_2 \equiv \int \frac{1}{I_1}\, d\eta = \frac{1}{\Lambda}\int \frac{1}{\eta}\, d\eta,$$
$$= \frac{1}{\Lambda}\log_e \eta + \text{const.},$$
$$= h_0 + h_1 \log_{10} \eta, \tag{8.43}$$

where h_0=constant for any given value of ω_δ, and $h_1 = 2.3026/\Lambda$. Owing to the fact that viscosity has been neglected, Eq. (8.43) leads to the well known result that $q = -\infty$ when $\eta = 0$, or, alternatively, that $q = 0$ at some small distance η_0 from the surface, η_0 being clearly of the order of thickness of the laminar sub-layer. As shown in the appendix, the original boundary layer equation (4.3) indicates that η_0 is proporportional to $1/R_\delta X$, and we therefore write

$$\eta_0 = \frac{\Omega}{R_\delta X}, \tag{8.44}$$

where Ω, the factor of proportionality, is clearly a function of ω_δ. It is also immediately apparent from (8.29) that, for the approximate condition $\eta = \eta_0$, $q = 0$,

$$(I_2)_{\eta_0} = -\frac{1}{X}\cdot \tag{8.45}$$

Hence, combining Eqs. (8.43), (8.44), (8.45),

$$\frac{1}{X} = -(h_0 + h_1 \log \Omega) + h_1 \log R_\delta X, \tag{8.46}$$

which has the form of von Kármán's skin friction law for plane flow.[11,12] Thus, when $\omega_\delta = 0$, Eq. (8.46) must be consistent with von Kármán's semi-empirical relation

$$\frac{1}{c_f^{1/2}} = 3.60 + 4.15 \log_{10} R_\delta c_f^{1/2}, \tag{8.47}$$

where $c_f^{1/2} = \sqrt{2}X$ and, by identifying (8.46) with (8.47), it follows, when h_0 and h_1 are determined from Eqs. (8.34), (8.35), (8.36) for $\omega_\delta = 0$, that Ω has the value 0.131 which compares with values given by Prandtl[18] varying between 0.089 and 0.111. Eq. (8.46) then reduces to

$$\frac{1}{X_0} = 5.97 + 5.88 \log R_\delta X_0, \tag{8.48}$$

and it follows immediately from the relation $K = \Lambda_0 = 2.3026/(h_1)_0$ that $\Lambda_0 = 0.392$.

For the case when $\omega_\delta \neq 0$, it is necessary to know Ω as a function of ω_δ in order to obtain X from Eq. (8.46). The difficulty of establishing this relation theoretically may be avoided, however, in the following approximate treatment, which appears to be reasonably valid up to values of ω_δ at which separation is imminent.

On splitting the product of Reynolds number and skin friction coefficient in (8.46), we have the alternative arrangement

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[18] L. Prandtl in W. F. Durand, *Aerodynamic theory*, vol. 3, J. Springer Berlin, p. 140 (1935).

$$\frac{1}{X} = h_1 \log R_\delta + h_1 \log X - (h_0 + h_1 \log \Omega), \tag{8.49}$$

or by writing

$$h_1 \log X - (h_0 + h_1 \log \Omega) = \sigma h_1,$$

where σ is also a function of ω_δ,

$$\frac{1}{X} = h_1(\log R_\delta + \sigma). \tag{8.50}$$

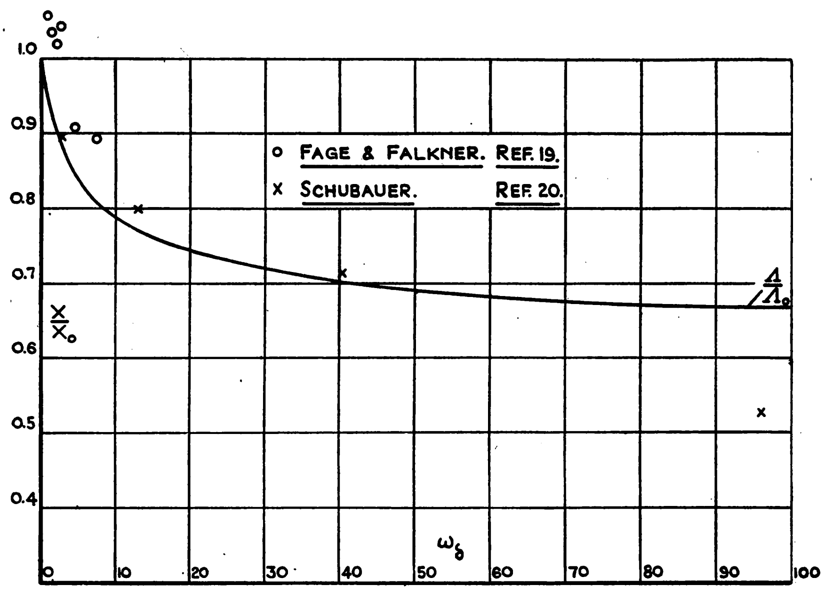

FIG. 3. Comparison of X/X_0 and Λ/Λ_0.

Consider, now, the variation of X with ω_δ when R_δ remains constant. Then

$$\frac{X}{X'} = \frac{h_1'(\log R_\delta + \sigma')}{h_1(\log R_\delta + \sigma)}, \tag{8.51}$$

where h_1, X, σ correspond with ω_δ, and h_1', X', σ' with ω_δ'.

It further appears from experiment that σ varies only slightly, and is small in relation to $\log R_\delta$, provided R_δ is moderate or large. Hence, for the ratio (8.51) no serious error is introduced in neglecting σ, and we then have the simple relation

$$\frac{X}{X'} = \frac{h_1'}{h_1} = \frac{\Lambda}{\Lambda'}, \tag{8.52}$$

which, when $\omega_\delta' = 0$, becomes

$$\frac{X}{X_0} = \frac{\Lambda}{\Lambda_0}, \tag{8.53}$$

Λ/Λ_0 being given by (8.37) and X_0 by (8.48).

Evidence[19,20] as to the validity of (8.53) is shown in Fig. 3.

From the very limited data at present available, it would appear that (8.53) is a reasonably good approximation in the range $0 < \omega_\delta < 50$, but clearly a good deal more experimental information is required before the precise significance of Eq. (8.53) can be ascertained.

9. Laminar sub-layer. The solution of Eqs. (5.9), (5.10) depends on a knowledge of the momentum thickness, and the parameters H and θ_1, each of which is a function of the velocity profile. If, however, the laminar sub-layer is neglected in evaluating these quantities, appreciable errors will arise, and it is necessary, therefore, to consider the flow in this region as well as in the fully developed turbulent part of the boundary layer. Before doing so, it will be convenient to express the displacement and momentum lengths in the non-dimensional forms

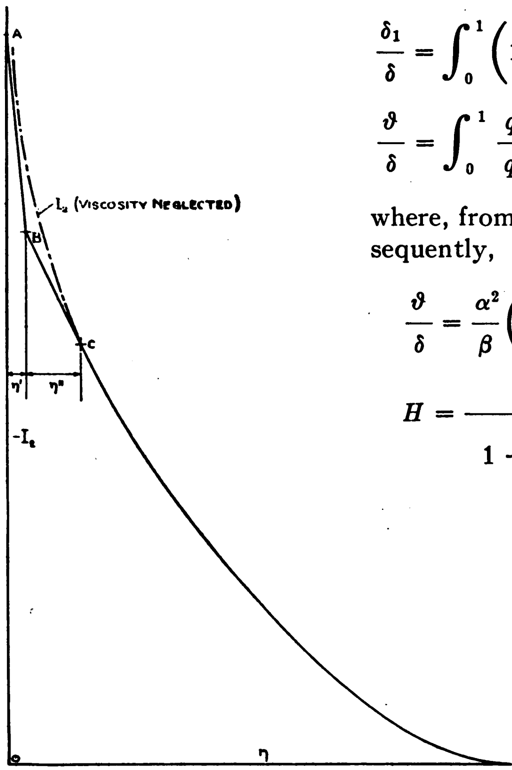

FIG. 4. Distribution of I_2 in laminar sub-layer.

$$\frac{\delta_1}{\delta} = \int_0^1 \left(1 - \frac{q}{q_1}\right) d\eta = \alpha X; \tag{9.1}$$

$$\frac{\vartheta}{\delta} = \int_0^1 \frac{q}{q_1}\left(1 - \frac{q}{q_1}\right) d\eta = \alpha X - \beta X^2, \tag{9.2}$$

where, from Eq. (8.29), $\alpha \equiv -\int_0^1 I_2 d\eta$, $\beta \equiv \int_0^1 I_2^2 d\eta$. Consequently,

$$\frac{\vartheta}{\delta} = \frac{\alpha^2}{\beta}\left(\frac{H-1}{H^2}\right), \tag{9.3}$$

$$H = \frac{1}{1 - \dfrac{\beta}{\alpha} X}. \tag{9.4}$$

Turning our attention, now, to the laminar sub-layer, we note first that, for moderate or large Reynolds numbers, it is quite thin. Since, also, the velocity distribution in this region is mainly linear, a good approximation to the corresponding distribution of I_2 may be represented by the discontinuous curve ABC (Fig. 4). The characteristics of his curve will be determined by satisfying the essential

[19] A. Fage and V. M. Falkner, *An experimental determination of the intensity of friction on the surface of an aerofoil,* Proc. Roy. Soc. (A) **129,** 378–410 (1930); also Tech. Rep. of the Aeron. Res. Comm., R & M. No. 1315, **1,** 117–140 (1930).

[20] G. B. Schubauer, *Airflow in the boundary layer of an elliptic cylinder,* Twenty-fifth Ann. Rep. Nat. Ad. Comm. for Aeron. Rep. No. 652, 207–226 (1939).

wall conditions, and by preserving continuity with the solution (ν neglected) where the laminar flow merges with the turbulent flow. For

$$0 < \eta < \eta',$$

let

$$I_2 = j_0 + j_1\eta, \tag{9.5}$$

and for

$$\eta' < \eta < (\eta' + \eta''),$$

$$I_2 = j_0 + (j_1 - j_2)\eta' + j_2\eta, \tag{9.6}$$

where j_0, j_1, j_2 are coefficients satisfying the necessary boundary conditions, and η', η'' are defined in Fig. 4. Then, if η_l is the nominal thickness of the laminar sub-layer,

$$\eta_l = \eta' + \eta'' = \frac{\sqrt{m}}{\Lambda R_\delta X} \tag{9.7}$$

according to the appendix.

Also, from Eqs. (8.45), (9.5), we have, when the approximate condition $\eta = \eta_0$ is replaced by the correct condition $\eta = 0$,

$$j_0 = -\frac{1}{X}, \tag{9.8}$$

and, from the differentiation of (9.5)

$$j_1 = \left(\frac{\partial I_2}{\partial \eta}\right)_{\eta=0} = \left(\frac{1}{I_1}\right)_{\eta=0}.$$

But, according to Eq. (8.28)

$$\left(\frac{1}{I_1}\right)_{\eta=0} = \frac{1}{X}\left[\frac{\partial}{\partial \eta}\left(\frac{q}{q_1}\right)\right]_{\eta=0},$$

which, from the boundary condition

$$\left[\frac{\partial}{\partial \eta}\left(\frac{q}{q_1}\right)\right]_{\eta=0} = R_\delta X^2,$$

gives

$$j_1 = \frac{1}{X}\left[\frac{\partial}{\partial \eta}\left(\frac{q}{q_1}\right)\right]_{\eta=0} = R_\delta X. \tag{9.9}$$

Finally, if the distribution of I_2 in the laminar layer is to be continuous with that in the turbulent region, we have from (9.6)

$$j_2 = \left(\frac{\partial I_2}{\partial \eta}\right)_{\eta_l} = \left(\frac{1}{I_1}\right)_{\eta_l}, \tag{9.10}$$

where $(I_1)_{\eta_l}$, $(I_2)_{\eta_l}$ are the values of I_1 and I_2 at $\eta=\eta_l$ given by the solution of sections 8 (a), 8 (b) when ν is neglected.

It follows from (9.7), (9.9) that

$$\eta_l = \eta' + \eta'' = \frac{\sqrt{m}}{\Lambda j_1}, \tag{9.11}$$

and hence, from the relation

$$j_0 + j_1\eta' = (I_2)_{\eta_l} - j_2\eta'',$$

that

$$\eta' = \frac{(I_2)_{\eta_l} - j_0 - j_2\sqrt{m}/j_1\Lambda}{(j_1 - j_2)}. \tag{9.12}$$

Hence, by definition,

$$\begin{aligned}\alpha &= -\int_0^{\eta'} (j_0 + j_1\eta)d\eta - \int_{\eta'}^{\eta_l} \{j_0 + (j_1 - j_2)\eta' + j_2\eta\}d\eta - \int_{\eta_l}^1 I_2 d\eta,\\ &= -j_0\eta_l - \tfrac{1}{2}(j_1\eta'^2 + 2j_1\eta'\eta'' + j_2\eta''^2) - \int_{\eta_l}^1 I_2 d\eta,\end{aligned} \tag{9.13}$$

and

$$\begin{aligned}\beta &= \int_0^{\eta'} (j_0 + j_1\eta)^2 d\eta + \int_{\eta'}^{\eta_l} \{j_0 + (j_1 - j_2)\eta' + j_2\eta\}^2 d\eta + \int_{\eta_l}^1 I_2^2 d\eta,\\ &= j_0^2\eta_l + j_1(j_0 + j_1\eta'')\eta'^2 + j_2(j_0 + j_1\eta')\eta''^2\\ &\quad + 2j_0j_1\eta'\eta'' + \tfrac{1}{3}(j_1^2\eta'^3 + j_2^2\eta''^3) + \int_{\eta_l}^1 I_2^2 d\eta.\end{aligned} \tag{9.14}$$

10. Velocity profiles. Comparison of theory and experiment. Von Doenhoff and Tetervin[1] have analyzed a large amount of data from velocity measurements in the turbulent boundary layer, and it is of interest, therefore, to compare their empirical curves with the theoretical conclusions of the present paper. For this purpose it has been most convenient to take the actual velocity distribution in the form $q/q_1=f(n/\vartheta)$ as a basis of comparison, the experimental data being obtained from cross-plots of Fig. 9, of the paper quoted in Footnote 1, where q/q_1 is plotted with respect to H for a series of values of $n/\vartheta=$constant.

Fig. 5 gives typical results at $\omega_\delta=30$ and for log $R_\delta=3.5$ and log $R_\delta=6.0$. In making the theoretical calculations, I_2 for the turbulent layer (ν neglected) is first determined from the numerical integration of the reciprocals of Eqs. (8.26), (8.27) in conjunction with Eqs. (8.34), (8.35), (8.36) and (8.37); X then follows from (8.37), (8.48), (8.53), and hence the velocity distribution in the turbulent part of the boundary layer from (8.29). The integrals α, β are next evaluated in the turbulent region by numerical integration of I_2 and I_2^2 respectively, and thence for the laminar layer by direct calculation from Eqs. (9.11), (9.12), (9.13), (9.14). Finally, H follows from Eq. (9.4), ϑ/δ from (9.3) and hence n/ϑ from

$$\frac{n}{\vartheta} = \eta \Big/ \left(\frac{\vartheta}{\delta}\right).$$

The results of Fig. 5, which are representative of a number of such calculations, indicate satisfactory agreement between theory and experiment up to values of H in the region of separation.

11. Approximate treatment of Eq. (4.6) in the solution of Eq. (5.9). We are now in a position to consider the solution of Eq. (5.9). This equation contains the important term θ_1, which, as will be seen from Eq. (4.6), depends on the ratio of $(\partial f/\partial\eta - \omega_\delta)$

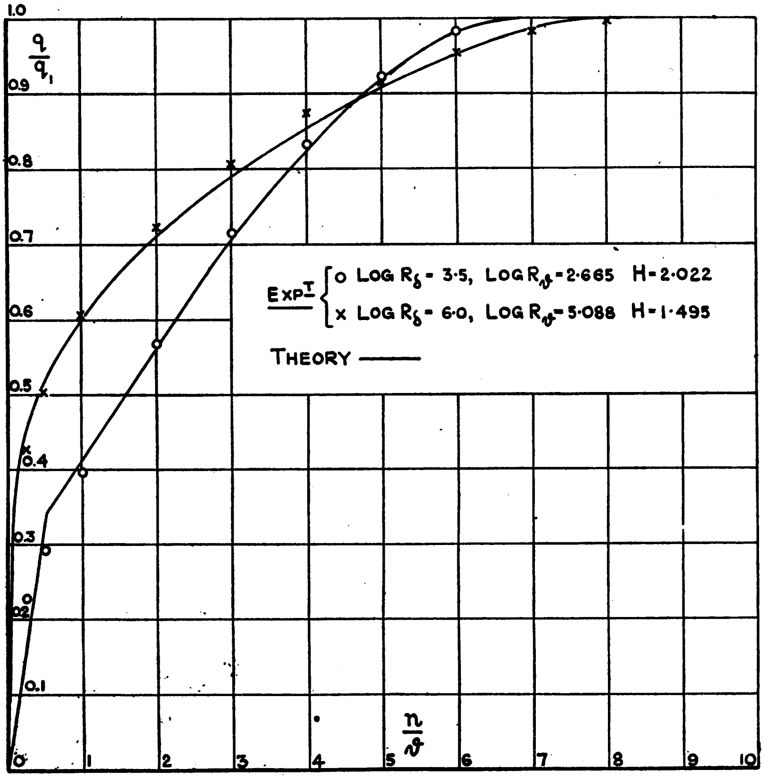

FIG. 5. Theoretical and experimental velocity profiles.

to $(q/q_1)^2$. Near the surface, within the laminar sub-layer, $\partial\theta/\partial\eta$ becomes large, and it appears that the value of θ_1 is critically dependent on the distribution of $\partial\theta/\partial\eta$ in this region. Consequently, viscosity is by no means a negligible factor, and it must be taken into account, not only with regard to the local velocity distribution, already considered in section (9), but also in so far as it affects the stress function f. In this respect, the solution is in marked contrast to that of the turbulent, velocity distribution, for which f may be equated to f_R. Hence, Eq. (8.34) is no longer tenable, and we must consider a more accurate form, at least for small values of η, when the solution will depend predominantly on the viscous terms previously ignored. In dealing first with the term θ_1, we shall accordingly divide the analysis into two parts; (a) that in which the effects of viscosity are all important, (b) that region outside the laminar sub-layer where, again, viscosity may be neglected. As before, let us represent the stress function near the surface by a power series, namely,

$$f = B_0 + B_1\eta + B_2\eta^2 + \cdots, \tag{11.1}$$

and solve for the B coefficients from the boundary conditions at $\eta = 0$. Then, as before (see Sec. 8 (b))

$$B_0 = A_0 = 1, \qquad B_1 = A_1 = \omega_\delta, \qquad B_2 = A_2 = 0, \tag{11.2}$$

$$\begin{aligned} B_3 = \frac{1}{6}\left(\frac{\partial^3 f}{\partial\eta^3}\right)_{\eta=0} &= \frac{2}{3} X R_\delta^2 \delta \left[\frac{dX}{ds} + \frac{X}{q_1}\frac{dq_1}{ds}\right], \\ &= \frac{2}{3} X R_\delta^2 \delta \left[\frac{dX}{dR_\vartheta}\frac{dR_\vartheta}{ds} + \frac{X}{q_1}\frac{dq_1}{ds}\right], \end{aligned} \tag{11.3}$$

where $R_\vartheta = q_1\vartheta/\nu$.

To a first approximation we now take the relation between X and R_ϑ as that for a flat plate, and determine dX/dR_ϑ from Falkner's power law form.[8] This in terms of R_ϑ may be written

$$X = 0.0808 R_\vartheta^{-1/12}, \tag{11.4}$$

so that

$$\frac{dX}{dR_\vartheta} = -0.00673 R_\vartheta^{-13/12}. \tag{11.5}$$

Also,

$$\frac{dR_\vartheta}{ds} = \frac{q_1}{\nu}\left[\frac{d\vartheta}{ds} + \frac{\vartheta}{q_1}\frac{dq_1}{ds}\right]. \tag{11.6}$$

Hence, substituting (11.5), (11.6) in (11.3)

$$\frac{3}{2} B_3 = -0.00673 X R_\vartheta^{-13/12} R_\delta^3 \left[\frac{d\vartheta}{ds} + \frac{\vartheta}{q_1}\frac{dq_1}{ds}\right] + \frac{\delta}{q_1}\frac{dq_1}{ds} X^2 R_\delta^2,$$

or, from Eq. (5.10) and the Bernoulli relation

$$\frac{1}{q_1}\frac{dq_1}{ds} = -\frac{1}{\rho q_1^2}\frac{dp}{ds},$$

$$\begin{aligned} \frac{3}{2} B_3 = &-0.00673 X R_\vartheta^{-13/12} R_\delta^3 \left[X^2\left\{1 + \omega_\delta(H+2) - \frac{\vartheta}{\rho q_1^2 X^2}\frac{dp}{ds}\right\}\right] \\ &- \frac{\delta}{\rho q_1^2 X^2}\frac{dp}{ds} X^4 R_\delta^2, \end{aligned}$$

so that

$$3B_3 = -2X^3R_\delta^2[0.00673R_\delta R_\vartheta^{-13/12}\{1 + \omega_\vartheta(H+1)\} + \omega_\delta X],$$

$$= -2X^3R_\delta^2\left[0.00673\left(\frac{\vartheta}{\delta}\right)^{-13/12}R_\delta^{-1/12}\{1 + \omega_\vartheta(H+1)\} + \omega_\delta X\right]. \quad (11.7)$$

We also obtain a relation for B_4 as follows. Equation (4.3) may be written

$$1 + \omega_\delta\eta + \frac{1}{X^2}\int_0^\eta \left(\frac{q}{q_1}\right)^2 \frac{\partial\theta}{\partial\eta}\,d\eta = \frac{1}{X^2R_\delta}\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right)$$

$$+ \frac{\lambda^2}{X^2}\left|\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right)\right|\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right), \quad (11.8)$$

which on differentiation with respect to η gives

$$\left(\frac{q}{q_1}\right)^2\frac{\partial\theta}{\partial\eta} = \frac{1}{R_\delta}\frac{\partial^2}{\partial\eta^2}\left(\frac{q}{q_1}\right) + 2\lambda\frac{\partial\lambda}{\partial\eta}\left|\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right)\right|\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right)$$

$$+ 2\lambda^2\left|\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right)\right|\frac{\partial^2}{\partial\eta^2}\left(\frac{q}{q_1}\right) - \omega_\delta X^2, \quad (11.9)$$

and when $\eta=0$, $q=\lambda=0$; therefore, from (11.9),

$$\frac{1}{R_\delta}\left[\frac{\partial^2}{\partial\eta^2}\left(\frac{q}{q_1}\right)\right]_{\eta=0} = \omega_\delta X^2,$$

or

$$\left[\frac{\partial^2}{\partial\eta^2}\left(\frac{q}{q_1}\right)\right]_{\eta=0} = X^2R_\delta\omega_\delta. \quad (11.10)$$

We also have from Sec. 8 (b)

$$\frac{\partial^4 f}{\partial\eta^4} = \frac{\delta}{X^2}\left[\frac{\partial^3}{\partial\eta^3}\left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\left(\frac{q}{q_1}\right) + 3\frac{\partial^2}{\partial\eta^2}\left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right)\right.$$

$$\left. + 3\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\frac{\partial^2}{\partial\eta^2}\left(\frac{q}{q_1}\right) + \left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\frac{\partial^3}{\partial\eta^3}\left(\frac{q}{q_1}\right)\right],$$

and when $\eta=0$

$$\left(\frac{\partial^4 f}{\partial\eta^4}\right)_{\eta=0} = \frac{3\delta}{X^2}\left[\frac{\partial^2}{\partial\eta^2}\left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right) + \frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\frac{\partial^2}{\partial\eta^2}\left(\frac{q}{q_1}\right)\right]. \quad (11.11)$$

From (11.1) there is also the relation

$$B_4 = \frac{1}{24}\left(\frac{\partial^4 f}{\partial\eta^4}\right)_{\eta=0}. \quad (11.12)$$

Then, from (11.10), (11.11), the first term of (11.12) yields

$$\frac{1}{8}\frac{\delta}{X^2}\frac{\partial^2}{\partial\eta^2}\left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right)=\frac{1}{4}XR_\delta^2\omega_\delta\delta\left[\frac{dX}{ds}+\frac{X}{q_1}\frac{dq_1}{ds}\right],$$

$$=\frac{3}{8}B_3\omega_\delta, \tag{11.13}$$

according to Eq. (11.3).

Again, for the second term of (11.12), we note first that (11.10) may be written alternatively as

$$\left[\frac{\partial^2}{\partial\eta^2}\left(\frac{q}{q_1}\right)\right]_{\eta=0}=\frac{R_\delta\delta}{\rho q_1^2}\frac{dp}{ds},$$

and therefore

$$\frac{1}{8}\frac{\delta}{X^2}\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\frac{\partial^2}{\partial\eta^2}\left(\frac{q}{q_1}\right)=\frac{1}{8}R_\delta^2\frac{\delta^2}{\rho q_1^2}\frac{d^2p}{ds^2},$$

or, since $\omega_\delta=\delta dp/\tau_0 ds=(\delta/X^2\rho q_1^2)(dp/ds)$, we have, substituting for δ in terms of ω_δ,

$$\frac{1}{8}\frac{\delta}{X^2}\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\frac{\partial^2}{\partial\eta^2}\left(\frac{q}{q_1}\right)=\frac{1}{8}R_\delta^2\omega_\delta^2X^4\rho q_1^2\frac{d^2p}{ds^2}\bigg/\left(\frac{dp}{ds}\right)^2.$$

Further,

$$\frac{dp}{ds}=-\rho q_1\frac{dq_1}{ds},$$

$$\frac{d^2p}{ds^2}=-\rho\left[\left(\frac{dq_1}{ds}\right)^2+q_1\frac{d^2q_1}{ds^2}\right],$$

and if the Bernoulli velocity distribution is assumed to be linear, which in general is a reasonable approximation,

$$\frac{d^2p}{ds^2}\bigg/\left(\frac{dp}{ds}\right)^2=-\frac{1}{\rho q_1^2}.$$

Hence,

$$\frac{1}{8}\frac{\delta}{X^2}\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\frac{\partial^2}{\partial\eta^2}\left(\frac{q}{q_1}\right)=-\frac{1}{8}X^3R_\delta^2\omega_\delta^2X.$$

But from (11.7)

$$X^3R_\delta^2=\frac{3B_3}{2\left[0.00673\left(\frac{\vartheta}{\delta}\right)^{-13/12}R_\delta^{-1/12}\{1+\omega_\vartheta(H+1)+\omega_\delta X\}\right]}.$$

For brevity write now

$$a \equiv 2\left[0.00673\left(\frac{\vartheta}{\delta}\right)^{-13/12} R_\delta^{-1/12}\{1 + \omega_\vartheta(H + 1) + \omega_\delta X\}\right],$$

$$b \equiv 3B_3 = -aX^3R_\delta^2. \tag{11.14}$$

Then the second term of (11.12) reduces to

$$\frac{1}{8}\frac{\delta}{X^2}\frac{\partial}{\partial\eta}\left(\frac{q}{q_1}\right)\frac{\partial}{\partial s}\frac{\partial^2}{\partial\eta^2}\left(\frac{q}{q_1}\right) = \frac{1}{8}\frac{b}{a}\cdot\omega_\delta^2 X, \tag{11.15}$$

and adding (11.13), (11.15),

$$B_4 = \frac{1}{8}b\omega_\delta\left(1 + \frac{\omega_\delta X}{a}\right). \tag{11.16}$$

Higher coefficients become progressively more complex, but in view of the fact that the values of η concerned are very small, we may, to a sufficient approximation, restrict the series (11.1) to a quartic, in which case we have

$$\frac{\partial f}{\partial\eta} - \omega_\delta = 3B_3\eta^2 + 4B_4\eta^3,$$

$$= b\eta^2\left[1 + \frac{1}{2}\omega_\delta\left(1 + \frac{\omega_\delta X}{a}\right)\eta\right]. \tag{11.17}$$

This equation, substituted in (4.6), then enables the distribution of θ near the surface to be calculated.

For the corresponding distribution in the turbulent part of the boundary layer, we should, to be consistent, correlate Eq. (11.1) with Eq. (8.34), but, as already pointed out, the value of θ_1 depends primarily on the conditions near the surface, and the precise form of $(\partial f/\partial\eta) - \omega_\delta$ at larger values of η appears to be less important, provided the essential conditions relating to f in the turbulent region are satisfied. We will therefore write a second approximation for f outside the laminar layer as

$$f = B_0' + B_1'(1 - \eta) + B_2'(1 - \eta)^2, \tag{11.18}$$

and satisfy the conditions at $\eta = 1$ and $\eta = \eta^*$, where η^* (see Fig. 6) represents the effective thickness of the laminar sub-layer. Then, from (11.1), (11.2), (11.14), (11.16), (11.17), we have, when $\eta = \eta^*$,

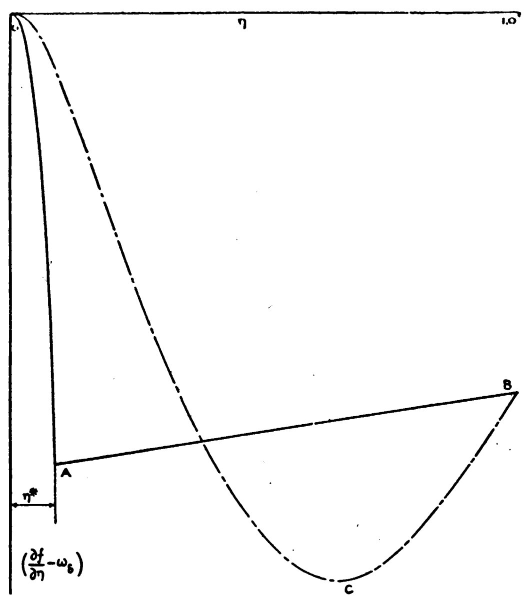

FIG. 6.

$$f = 1 + \omega_\delta \eta^* + \frac{1}{3} b\eta^{*3}\left[1 + \frac{3}{8}\omega_\delta\left(1 + \frac{\omega_\delta X}{a}\right)\eta^*\right],$$

$$\frac{\partial f}{\partial \eta} = \omega_\delta + b\eta^{*2}\left[1 + \frac{1}{2}\omega_\delta\left(1 + \frac{\omega_\delta X}{a}\right)\eta^*\right],$$

and when $\eta = 1, f = \partial f/\partial\eta = 0$.

Hence, equating f and $\partial f/\partial \eta$, as given by (11.18), to the above boundary conditions at $\eta = \eta^*$ and $\eta = 1$, we obtain

$$B_0' = B_1' = 0,$$

$$B_2' = -\frac{1}{2}\left[\frac{\omega_\delta + b\eta^{*2}\left\{1 + \frac{1}{2}\omega_\delta\left(1 + \frac{\omega_\delta X}{a}\right)\eta^*\right\}}{(1 - \eta^*)}\right], \tag{11.19}$$

and

$$1 + \omega_\delta + \frac{1}{3} b\eta^{*3}\left[1 + \frac{3}{8}\omega_\delta\left(1 + \frac{\omega_\delta X}{a}\right)\eta^*\right]$$
$$+ \frac{1}{2}\left[b\eta^{*2}\left\{1 + \frac{1}{2}\omega_\delta\left(1 + \frac{\omega_\delta X}{a}\right)\eta^*\right\} - \omega_\delta\right](1 - \eta^*) = 0, \tag{11.20}$$

which are the two equations from which to solve for B_2' and η^*.

The distribution of the quantity $(\partial f/\partial\eta) - \omega_\delta$ in the turbulent layer is then given by the linear relationship

$$\left(\frac{\partial f}{\partial \eta} - \omega_\delta\right) = -\omega_\delta - 2B_2'(1 - \eta). \tag{11.21}$$

This is illustrated in Fig. 6, where the line AB represents the solution in the turbulent region (Eq. 11.21), and the curve OA the corresponding solution in the laminar layer, as given by Eq. (11.17). The chain curve OCB has been included for comparison, and is the solution developed in section 8 (b). The curves are approximately to scale, and refer to the same flow conditions. The effect of viscosity near the surface evidently has a large influence on the initial distribution of the shear stress, and hence upon θ_1. For small values of η there is no reason to doubt the accuracy of the curve OA, but further from the surface the treatment of section 8 (b) is probably a closer approximation than that afforded by the series (11.18). The true solution would therefore appear to be represented by a transition curve linking OA as $\eta \to 0$ with OCB as $\eta \to 1$.

One final point needs consideration. When $\eta = 0$, so also $(\partial f/\partial\eta) - \omega_\delta = q = 0$, and therefore the initial value of $\partial\theta/\partial\eta$ becomes indeterminate. This may be circumvented by the following argument. From (11.16)

$$\mathop{\mathcal{L}}_{\eta\to 0}\left(\frac{\partial f}{\partial \eta} - \omega_\delta\right) = b\eta^2.$$

Likewise,

$$\mathcal{L}_{\eta\to 0}\left(\frac{q}{q_1}\right)^2 = R_\delta^2 X^4 \eta^2.$$

Hence, from Eq. (4.5),

$$\mathcal{L}_{\eta\to 0}\left(\frac{\partial\theta}{\partial\eta}\right) = \frac{b}{R_\delta^2 X^2} = -\, aX.$$

12. Solution of Eq. (5.9) for $\vartheta dH/ds$. Since, by definition,

$$\omega_\vartheta = \omega_\delta\left(\frac{\vartheta}{\delta}\right),$$

we have, according to Eq. (9.3),

$$\omega_\vartheta = \omega_\delta \frac{\alpha^2}{\beta}\left(\frac{H-1}{H^2}\right). \tag{12.1}$$

Combining, now, Eqs. (4.6), (8.29), (8.53), (12.1) with Eq. (5.9),

$$\vartheta\frac{dH}{ds} = -\, X_0^2\left(\frac{\Lambda}{\Lambda_0}\right)^2\left[\omega_\delta + H\left\{1 + \omega_\delta\frac{\alpha^2}{\beta}\left(\frac{H^2-1}{H^2}\right)\right\} + \int_0^1 \frac{\left(\dfrac{\partial f}{\partial\eta} - \omega_\delta\right)}{\left(1 + I_2 X_0\dfrac{\Lambda}{\Lambda_0}\right)^2}\, d\eta\right], \tag{12.2}$$

where I_2, α, β are found as described in section (10), and $(\partial f/\partial\eta)-\omega_\delta$ is given by Eq. (11.17) in the range $0<\eta<\eta^*$, and by Eq. (11.21) in the range $\eta^*<\eta<1$.

Considering the functional nature of the terms, excluding H for the moment, on the right hand side of Eq. (12.2), it is evident that they are either functions of ω_δ and R_δ, or of ω_δ or R_δ separately. Hence we may write

$$\vartheta\frac{dH}{ds} = \phi_1(\omega_\delta, R_\delta, H). \tag{12.3}$$

Actually, from Eq. (9.4), H is also a function of ω_δ and R_δ, so that (12.3) reduces to

$$\vartheta\frac{dH}{ds} = \phi_2(\omega_\delta, R_\delta). \tag{12.4}$$

For the numerical integration of Eqs. (5.9), (5.10) along the lines of refs. (1) and (2), however, it is desirable to retain the form (12.3).

We now show how (12.2) may be evaluated in terms of the data discussed in section (6), namely, when the static pressure and the initial values of ϑ and H at each step are known.

From Eqs. (9.3), (9.4) we have

$$\delta = \frac{\vartheta H}{\alpha X} = \frac{\vartheta H}{\alpha X_0 \dfrac{\Lambda}{\Lambda_0}}, \tag{12.5}$$

or,

$$\omega_\vartheta = (\omega_\vartheta)_0 \frac{H}{\alpha X_0 \left(\dfrac{\Lambda}{\Lambda_0}\right)^3}, \tag{12.6}$$

where

$$(\omega_\vartheta)_0 = \frac{\vartheta}{\rho q_1^2 X_0^2} \frac{dp}{ds} = -\frac{\vartheta}{X_0^2} \frac{1}{q_1} \frac{dq_1}{ds}.$$

Eq. (12.5), or (12.6), can be solved as follows. Given ϑ, X_0 is determined from the law relating X_0 and R_ϑ, i.e. Falkner's power law,[8] or the logarithmic law of Squire and Young.[6] Now assume a value of ω_δ; hence we obtain Λ/Λ_0 (which is purely a function of ω_δ) by calculation according to section 8 (b), or, alternatively, from Fig. 3. X then follows. This enables a value of δ to be found consistent with the assumed value of ω_δ, and the corresponding value of R_δ is calculated. Finally, knowing both ω_δ and R_δ, we obtain α from section (9), and, since H is given, a second approximation for δ (or ω_δ) follows from Eqs. (12.5), (12.6). Hence, by trial and error, or by pre-established families of curves, we may solve for the relation between δ and ϑ. By virtue of Eqs. (12.5), (12.6), therefore, (12.2) is expressible in terms of ϑ, H and the pressure distribution, so that formally we have

$$\vartheta \frac{dH}{ds} = \phi_3[(\omega_\vartheta)_0, R_\vartheta, H]. \tag{12.7}$$

In conclusion it is worth noting that $(\omega_\vartheta)_0$ is a multiple of Garner's variable Γ, as it is of von Doenhoff's corresponding term. Thus, Garner takes

$$\Gamma \equiv \vartheta R_\vartheta^{1/6} \frac{1}{q_1} \frac{dq_1}{ds}, \tag{12.8}$$

and assumes Falkner's power law for the skin friction, viz.

$$X_0^2 = 0.006534 R_\vartheta^{-1/6}, \tag{12.9}$$

so that combining (12.8), (12.9)

$$\Gamma = 0.006534 \frac{\vartheta}{X_0^2} \frac{1}{q_1} \frac{dq_1}{ds}, \\ = -0.006534(\omega_\vartheta)_0.$$

Similarly, von Doenhoff's term, which we will denote by Γ', is

$$\Gamma' = 2 \frac{\vartheta}{X_0^2} \frac{1}{q_1} \frac{dq_1}{ds} = -2(\omega_\vartheta)_0.$$

Consequently, as discussed in section (6), both the theoretical Eq. (12.2) and the empirical equations for $\vartheta dH/ds$ of refs. (1) and (2) contain the same basic parameters.

13. Comparison of Eq. (12.2) with the corresponding relations of von Doenhoff and Garner. For comparison we have taken, as examples, the case (a) when $\omega_\delta = 0$, (b) when $\omega_\delta = 30$, the range of log R_δ considered being in both cases $3.5 < \log R_\delta < 6.0$. For (b) this range of Reynolds number allows a wide variation of flow conditions to be studied, from virtually the plane flow state up to separation, which is imminent when H exceeds 1.8.

TABLE 1.

$\omega_\delta = 0$

Log $q_1\delta/\nu$	α	β	X	H	ϑ/δ	Log $q_1\vartheta/\nu$	$\vartheta dH/ds \times 10^3$		
							Ref. 1	Ref. 2	Theory
3.5	2.845	20.21	0.0526	1.596	0.0937	2.471	−0.996	−2.731	−0.047
4.0	2.724	17.29	0.0464	1.421	0.0896	2.952	−0.190	−0.102	+0.306
4.5	2.682	16.10	0.0413	1.330	0.0835	3.422	−0.041	+0.179	0.298
5.0	2.667	15.57	0.0371	1.277	0.0776	3.890	+0.007	0.202	0.271
5.5	2.662	15.31	0.0338	1.242	0.0727	4.362	0.027	0.182	0.190
6.0	2.660	15.23	0.0309	1.215	0.0677	4.831	0.038	0.155	0.157

TABLE 2.

$\omega_\delta = 30$

Log $q_1\delta/\nu$	α	β	X	H	ϑ/δ	Log $q_1\vartheta/\nu$	$\vartheta dH/ds \times 10^3$		
							Ref. 1	Ref. 2	Theory
3.5	7.770	103.49	0.0379	2.020	0.1460	2.665	35.6	72.7	62.0
4.0	8.086	114.81	0.0334	1.902	0.1421	3.153	20.7	35.1	40.9
4.5	8.187	119.66	0.0297	1.770	0.1376	3.639	12.65	15.4	24.6
5.0	8.223	121.68	0.0267	1.654	0.1328	4.123	7.58	7.58	14.9
5.5	8.233	122.36	0.0243	1.566	0.1279	4.607	5.06	4.34	9.23
6.0	8.237	122.67	0.0222	1.495	0.1224	5.088	3.60	2.62	5.68

Tables 1 and 2 and Fig. 7 summarize the results, those of refs. (1) and (2) having been calculated from the formulae there given, which, for the present purpose, may be most conveniently expressed in the notation of the present paper as follows:

Von Doenhoff and Tetervin[1]

$$\vartheta \frac{dH}{ds} = e^{4.68(H-2.975)}\left[2(\omega_\vartheta)_0 - 2.035(H - 1.286)\right], \tag{13.1}$$

with X_0 given by Squire's and Young's formula

$$X_0^2 = \left[5.890 \log_{10} (4.075 R_\vartheta)\right]^{-2};$$

Garner[2]

$$\vartheta \frac{dH}{ds} = X_0^2 e^{5(H-1.4)} [(\omega_\vartheta)_0 - 2.068(H - 1.4)], \tag{13.2}$$

where X_0 is based on Falkner's formula

$$X_0^2 = 0.006534 R_\vartheta^{-1/6}.$$

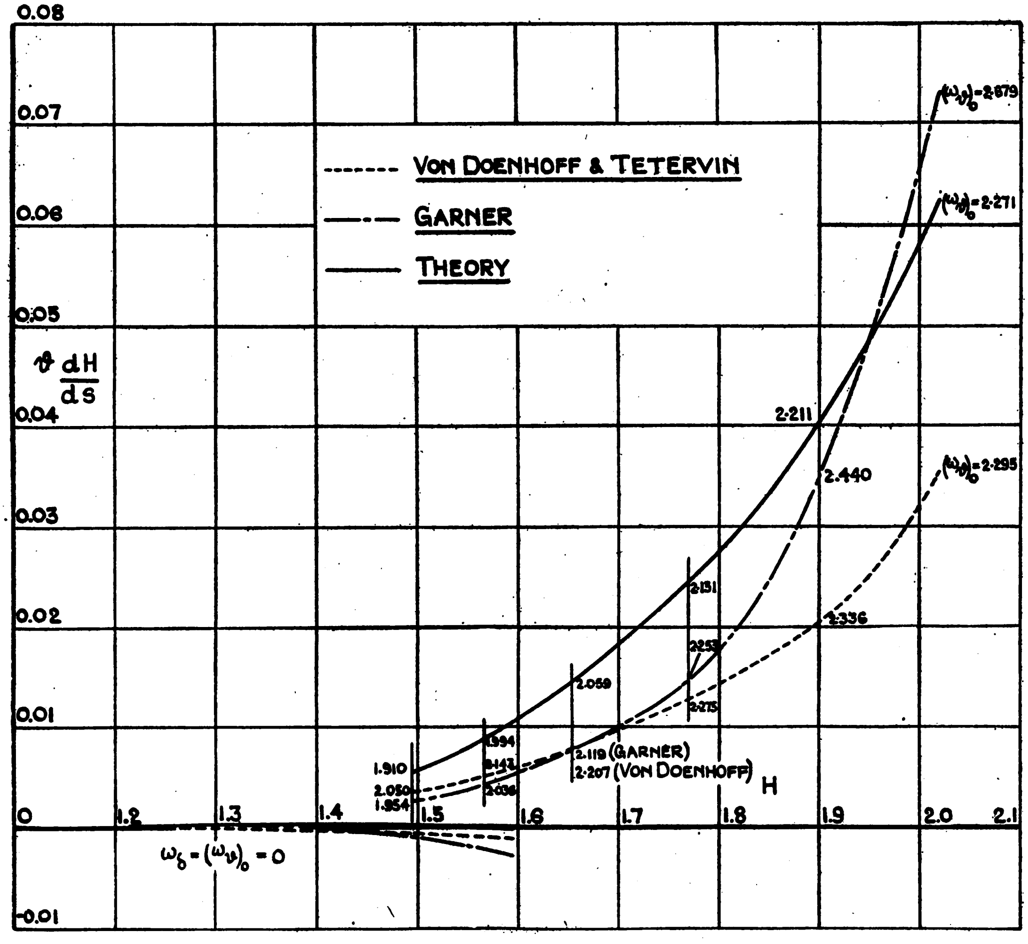

FIG. 7. Comparison of theoretical and empirical solutions for $\vartheta dH/ds$.

In estimating $(\omega_\vartheta)_0$, when $\omega_\delta = 30$, the skin friction law adopted in each case has been adhered to. Thus, for theoretical values of $(\omega_\vartheta)_0$ Eq. (8.48) applies, whilst for the empirical formulae of von Doenhoff and Garner the particular skin friction equations given above have been used. This leads to slight variations of $(\omega_\vartheta)_0$ under otherwise similar conditions, as will be seen from Fig. 7, where the appropriate value of $(\omega_\vartheta)_0$ is indicated against each point to facilitate comparison.

For plane flow ($\omega_\delta = 0$), both the theoretical and empirical results are in good agreement at large Reynolds numbers, but there is some discrepancy at lower values when H becomes abnormally large due to the very thick laminar sub-layer. Under these circumstances the theory is probably unreliable, so that some deviation is to be expected. The condition that $dH/ds \doteqdot 0$, or, as is generally assumed, H = constant when the pressure gradient is zero or small, is, however, well substantiated by theory.

Under conditions of a pronounced pressure gradient, tending to separation, agreement is less satisfactory, not only between the theoretical and semi-empirical solutions, but also between von Doenhoff's and Garner's results.

Some comment on these discrepancies was made in the introduction. It was there pointed out that the theory is consistent with Garner's relation in that both indicate $\vartheta dH/ds$ to be a function of ω_ϑ, R_ϑ and H, whereas the von Doenhoff-Tetervin equation does not contain R_ϑ as an independent parameter. Consequently, in this respect, theoretical calculations might be expected to agree (as is the case) more nearly with Eq. (13.2) than with Eq. (13.1), the latter probably being less generally representative on account of the above restriction.

A further point of interest in regard to the empirical formulae is the value of H when $dH/ds = \omega_\delta = \omega_\vartheta = 0$. From (13.1), (13.2) it will be noted that, according to von Doenhoff and Tetervin, H then has the value 1.286, whereas Garner, after investigating the variation of H at transition in some detail, concludes that the value 1.4 is a better approximation. As already observed, $\vartheta dH/ds$ is very small within the normal range of H when ω_δ is zero, so that any slight discrepancy to which the above quantity may be subject will introduce a large change in the value of H at which dH/ds is precisely zero. In the empirical approach dH/ds was obtained graphically, and therefore the difference between the values of H under the afore-mentioned conditions, as given in refs. (1) and (2), is probably due to errors in the graphical method. By interpolation of the theoretical results in Table 1, the corresponding value of H is found to be about 1.56. Allowing for the approximate nature of the theory, which may accordingly imply errors in the above H value of the same order as for the empirical formulae, it follows, nevertheless, that Garner's figure of 1.4 is perhaps a better estimate than that of von Doenhoff and Tetervin.

In conclusion, the general form of the equation for $\vartheta dH/ds$ appears to be established, and a method of analyzing the growth of the turbulent boundary layer has been developed. In practice, however, it is laborious to use, though by extensive graphical treatment it is considered that the work could be reduced to reasonable proportions. Alternatively, if some degree of empiricism is acceptable, a reasonably reliable, simple and rapid means of reaching a solution is possible on the lines of refs. (1) and (2).

APPENDIX

According to Eq. (11.8)

$$\frac{\tau}{\rho q_1^2} = \frac{1}{R_\delta}\frac{\partial}{\partial \eta}\left(\frac{q}{q_1}\right) + \lambda^2 \left|\frac{\partial}{\partial \eta}\left(\frac{q}{q_1}\right)\right| \frac{\partial}{\partial \eta}\left(\frac{q}{q_1}\right). \tag{A}$$

The first term on the right hand side of (A) represents the true viscous shear stress, whilst the second term represents the Reynolds stress. Where the laminar sub-layer merges with the fully turbulent layer, the former stress becomes vanishingly small in relation to the latter. Let η_l denote the value of η at which the viscous stress is virtually negligible, and let the ratio of Reynolds stress to viscous stress at this point be m. Then

$$\lambda^2 \left| \frac{\partial}{\partial \eta}\left(\frac{q}{q_1}\right) \right| \frac{\partial}{\partial \eta}\left(\frac{q}{q_1}\right) = \frac{m}{R_\delta} \frac{\partial}{\partial \eta}\left(\frac{q}{q_1}\right). \quad \text{(B)}$$

Also near the surface

$$\lambda = \Lambda \eta, \quad \text{(C)}$$

and approximately we may write

$$\frac{\partial}{\partial \eta}\left(\frac{q}{q_1}\right) = R_\delta X^2. \quad \text{(D)}$$

Hence, by combining (B), (C), (D) and re-arranging, we have

$$\eta_l = \frac{\sqrt{m}}{\Lambda R_\delta X}, \quad \text{(E)}$$

so that if η_0 of section 8 (c) is regarded as proportional to η_l,

$$\eta_0 \sim \eta_l \sim \frac{1}{R_\delta X}. \quad \text{(F)}$$

By writing $\sqrt{m}/\Lambda =$ constant, Eq. (E) becomes identical with von Kármán's equation[12] for the non-dimensional thickness of the laminar sub-layer. For the constant von Kármán finds the approximate value 30. Then if we take the condition of plane flow, when

$$\Lambda = \Lambda_0 = 0.392,$$

we see that

$$\sqrt{m} = 30 \times 0.392 = 11.76.$$

or

$$m = 138.3.$$

Thus the viscous stress is locally 1/138.3 of the Reynolds stress, i.e. it amounts to slightly less than 1% of it.