$$|\alpha| \leqq |1 - \tfrac{1}{2}\mu| + \tfrac{1}{2}\sum_{n=2}^{\infty} M^n/(2n-1)!.$$

Hence, by the first of the inequalities (23),

$$|\alpha| < |1 - \tfrac{1}{2}\mu| + 2 - \tfrac{1}{2}\mu.$$

On the other hand, the second of the inequalities (23) can be written in the form $\mu \geqq 2$, which means that

$$|1 - \tfrac{1}{2}\mu| = \tfrac{1}{2}\mu - 1.$$

This completes the proof, since the last two formula lines imply the inequality $|\alpha| < 1$, which is (9).

Conclusion. If μ, M are defined by (10), (11), then either (12) or (23) [and so, in particular, either (13) or (24)] is sufficient for stability.

As an illustration, let

$$f(t) = (a + b\cos 2\pi t)^{-1}, \quad \text{where} \quad 0 < b < a;$$

so that (1) becomes the equation known from the problem of frequency modulation. In this case, (10) and (11) reduce to

$$\mu = (a^2 - b^2)^{-1/2} \quad \text{and} \quad M = (a - b)^{-1},$$

and so the above inequalities supply explicit conditions for pairs (a, b) which are sure to be of stable type. Needless to say, the resulting inequalities for a and b are just sufficient for stability. Incidentally, since $f(t)$ is now positive, Liapounoff's criterion, $\mu < 4$, also is applicable.

LOWER BUCKLING LOAD IN THE NON-LINEAR BUCKLING THEORY FOR THIN SHELLS*

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For thin shells the relation between the load P and the deflection ϵ beyond the classical buckling load is very often non-linear. For instance, when a uniform thin circular cylinder is loaded in the axial direction, the load P when plotted against the end-shortening ϵ has the characteristic shown in Fig. 1. If the strain energy S and the total potential $\varphi = S - P\epsilon$ are calculated, their behavior can be represented by the curves shown in Figs. 2 and 3. It can be demonstrated that the branches OC and AB corresponds to stable equilibrium configurations and the branch BC to unstable equilibrium configurations. The point B is then the point of transition from stable to unstable equilibrium configurations.

It was proposed by the author in a previous paper[1] that the point A was the critical point for buckling of the structure under external disturbances, using the S, ϵ curve for "testing machine" loading and the φ, P curve for "deadweight" loading. The load P for the unbuckled configuration of the shell corresponding to the point A was called

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[1] H. S. Tsien, *A theory for the buckling of thin shells*, J. Aero. Sciences 9, 373–384 (1942).

the lower buckling load of the shell. The energy represented by the vertical distance from the point A to the curve BC is then the minimum external excitation required to cause the buckling at point A.

However, if the external excitation is large, there is no reason why buckling cannot occur at the point B' directly under the point B. The minimum external excitation required is then given by the energy represented by the distance $B'B$. This amount of energy is actually absorbed by the structure during buckling. Since the curve BA represents the final state of the structure after buckling, for buckling to happen between B' and A, energy is absorbed, and for buckling to happen between A and C, energy is released. But in any event, the lower limit of buckling load is definitely given by the point B', not the point A. Therefore the lower buckling load should be the load P corresponding to the point B'.

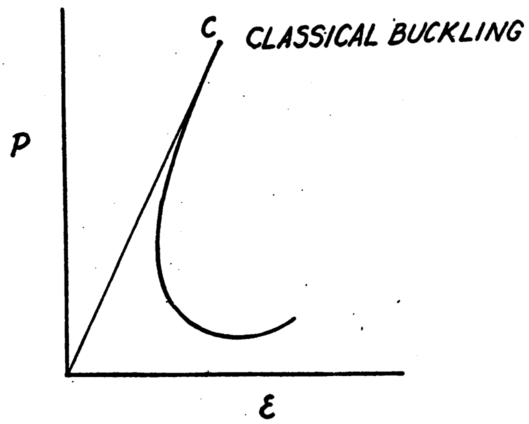

FIG. 1.

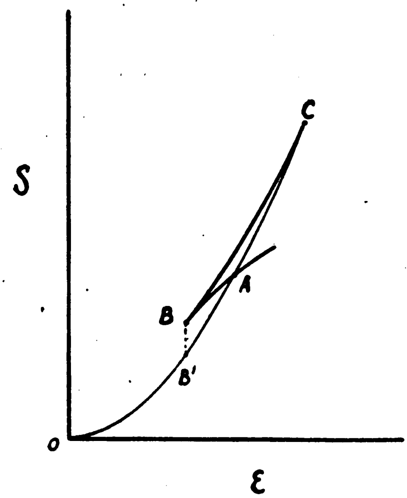

FIG. 2.

By referring to Figs. 11 and 13 of the aforementioned paper, and assuming a square wave pattern, we find the lower buckling stress σ of thin uniform cylindrical shells under axial load to be given by

$$\sigma = 0.42Et/R$$

for testing machine loading and

$$\sigma = 0.19Et/R$$

for deadweight loading. The corresponding values under the previously proposed criteria are $\sigma = 0.46Et/R$ and $\sigma = 0.298Et/R$ for the two cases.

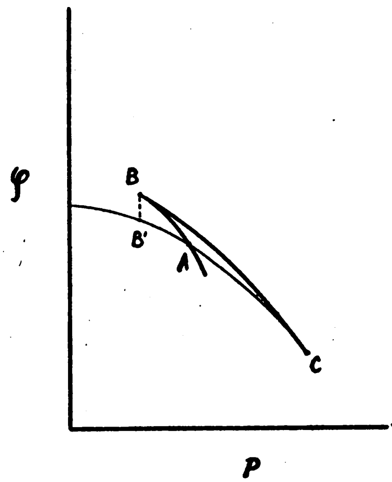

FIG. 3.