A GENERALIZATION OF THE FINITE FOURIER TRANSFORMATION AND APPLICATIONS*

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Introduction. The purpose of this paper is to generalize and unify the methods used by Doetsch [8]†, Kniess [12], Koschmieder [13] and others to solve certain boundary value problems by finite Fourier integral transformations. To give an idea of the general method to be developed a formal solution of the following boundary value problem is here given by means of a particular transformation.

$$Y_{tt}(x, t) = Y_{xx}(x, t) + x, \qquad 0 < x < \pi, \qquad 0 < t,$$
$$Y(x, 0+) = 0, \qquad Y_t(x, 0+) = 0,$$
$$Y(0+, t) = 0, \qquad Y_x(\pi -, t) + hY(\pi -, t) = 0, \qquad h \neq 0.$$

Let $S\{F(x)\} = \int_0^\pi F(x) \sin k_n x dx = f_s(k_n)$, (compare section 1 below), where $\sin k_n x$ are the characteristic functions of $y''(x) + k^2 y(x) = 0$, $y(0) = 0$, $y'(\pi) + hy(\pi) = 0$, that is k_n, $n = 1, 2, \cdots$, are the roots of $\tan k\pi = -(k/h)$, $k > 0$, then, if $F(x)$, $F'(x)$ are continuous and $F''(x)$ is sectionally continuous in $(0, \pi)$

$$S\{F''(x)\} = -k_n^2 f_s(k_n) + k_n F(0) + \sin k_n\pi[F'(\pi) + hF(\pi)],$$

(see theorem 1 below).

This last formula applied to the above problem with respect to x yields the following transformed problem:

$$\frac{d^2 y_s(k_n, t)}{dt^2} = -k_n^2 y_s(k_n, t) + S\{x\},$$

$$y_s(k_n, 0+) = 0, \qquad dy_s(k_n, 0+)/dt = 0, \quad \text{where}$$
$$S\{x\} = (\sin k_n\pi) k_n^{-2} - (\pi \cos k_n\pi) k_n^{-1}.$$

The solution of the transformed problem is

$$y_s(k_n, t) = [(1 - \cos k_n t)/k_n^2] S\{x\}.$$

The inverse transformation $S^{-1}\{f_s(k_n)\}$ is given in terms of a Sturm-Liouville series (see section 2 below) as follows

$$S^{-1}\{f_s(k_n)\} = \sum_1^\infty N(k_n) f_s(k_n) \sin k_n x \quad \text{in} \quad (0, \pi),$$

where $N(k_n)$ denotes the normalization factor of the characteristic functions $\sin k_n x$, $n = 1, 2, \cdots,$.

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† The numbers in brackets refer to the bibliography.

Thus the formal solution of the above boundary value problem is given by

$$Y(x, t) = \sum_{1}^{\infty} N(k_n)\left[(\sin k_n\pi) k_n^{-4} - (\pi \cos k_n\pi) k_n^{-3}\right](1 - \cos k_n t) \sin k_n x,$$

The type of problems to which the method is applicable is described in the summary (see Sec. 11 below).

Since there are at present no readily applicable general existence theorems for solutions of the class of boundary value problems considered here, the existence and uniqueness of the solution should be established in each particular case. However, this procedure is not carried through in the problems solved in this paper, since the main purpose here is to set up a method which leads quickly to a formal solution. The problems solved are chosen merely to give an illustration of the method.

The method can be compared with that of the Laplace transformation (see remark in Sec. 3 below). As in the case of the Laplace transformation the present operational method does not claim to solve problems which cannot be solved by any other method. Its advantage lies, just as in the case of the Laplace transformation, in its direct, short and systematic approach. Problems in partial differential equations which by a suitable change of variables are brought into a form in which variables can be separated, can be solved directly. Tables of transforms, which are, except for normalization factors, tables of generalized Fourier coefficients, save time in the computation of solutions of practical problems. Furthermore this operational method has, due to certain theorems the advantage of systematically finding *closed form solutions* (see Secs. 6, 9, 10, 11 below) and thus exhibiting qualitative properties of a solution which may not readily be found by the usual methods.

1. Definitions of the transformations S, C and T. *Definition* 1. Let $K = \{k\}$ be a set of real numbers and let $F(x)$ be a sectionally continuous function in $(0, \pi)$. The transformations $S\{F(x)\}$ and $C\{F(x)\}$ are defined by the equations

$$S\{F(x)\} = \int_0^\pi F(x) \sin kx dx = f_s(k) \tag{1}$$

$$C\{F(x)\} = \int_0^\pi F(x) \cos kx dx = f_c(k) \tag{2}$$

respectively.

The transformations (1) and (2) are called the general finite Fourier sine and cosine transformation respectively, relative to the interval $(0, \pi)$ and the set K. The transformations (1) and (2) map the class of functions $F(x)$ onto a certain class of functions $f(k)$.

A Sturm-Liouville problem. The characteristic functions of the following Sturm-Liouville problem are used in the definition of the transformation T below.

$$\left.\begin{aligned} &y''(x) + k^2 y(x) = 0, \quad \text{in} \quad (0, \pi), \\ &L_1(y) = a_1 y(0) + a_2 y'(0) + a_3 y(\pi) + a_4 y'(\pi) = 0, \\ &L_2(y) = b_1 y(0) + b_2 y'(0) + b_3 y(\pi) + b_4 y'(\pi) = 0, \end{aligned}\right\} \tag{3}$$

also with specializations on part of the constant coefficients a_i, b_i, $i = 1, 2, 3, 4$. It is assumed that the L_j, $j = 1, 2$, are linearly independent and that

$$a_1b_2 - a_2b_1 = a_3b_4 - a_4b_3. \tag{4}$$

This last condition is to guarantee real characteristic values (see for instance [10] vol. 1 p. 352).

The characteristic functions $\varphi_{k_n}(x)$, $n=1, 2, \cdots$, of (3) are given by

$$\varphi_{k_n}(x) = A \sin k_n x + B \cos k_n x, \tag{5}$$

where

$$\begin{aligned}(A/B) &= -(a_1 + a_3 \cos k_n\pi - a_4k_n \sin k_n\pi)/(k_na_2 + a_3 \sin k_n\pi + a_4\, k_n \cos k_n\pi)\\ &= -(b_1 + b_3 \cos k_n\pi - b_4k_n \sin k_n\pi)/(k_nb_2 + b_3 \sin k_n\pi + b_4k_n \cos k_n\pi)\end{aligned}$$

and k_n are the roots of

$$D(k) \equiv \begin{vmatrix} (ka_2 + a_3 \sin k\pi + a_4k \cos k\pi) & (a_1 + a_3 \cos k\pi - a_4k \sin k\pi) \\ (kb_2 + b_3 \sin k\pi + b_4k \cos k\pi) & (b_1 + b_3 \cos k\pi - b_4k \sin k\pi) \end{vmatrix} = 0. \tag{6}$$

By use of (4), Eq. (6) reduces to

$$\begin{aligned}2k(a_1b_2 - a_2b_1) = {}& [(a_3b_1 - a_1b_3) + k^2(a_4b_2 - a_2b_4)] \sin k\pi\\ &+ [(a_4b_1 - a_1b_4) + (a_2b_3 - a_3b_2)]k \cos k\pi.\end{aligned} \tag{7}$$

Remarks concerning $D(k)$. (Compare [10] vol. 1, p. 362.) If $D(k)=0$ has no roots, then (3) has no solution.

If and only if $D(k)$ is of rank 0, that is, if every term in $D(k)$ is 0, then the characteristic values are called double, since in this case (3) has two linearly independent characteristic functions with the same characteristic values, e.g. $a_1=b_2=-a_3-b_4=1$ all other a_i, b_i zero. In this case the characteristic values are $2n$, $n=1, 2, \cdots$, and $\sin 2nx$ as well as $\cos 2nx$ are characteristic functions of (3).

If $D(k)$ is of rank greater than zero the characteristic values are single. This is the case in particular in the first, second and third boundary value problem.

Definition 2. Let $K=\{k_n\}$, $n=1, 2, \cdots$, be the sequence of characteristic values of (3), that is, the roots of (7). Let

$$\varphi_{k_n}(x) = A_{k_n} \sin k_nx + B_{k_n} \cos k_nx \tag{8}$$

be given by (5), where

$$\begin{aligned}A_{k_n} &= -a_1 + a_3 \cos k_n\pi - a_4k_n \sin k_n\pi,\\ B_{k_n} &= -(k_na_2 + a_3 \sin k_n\pi + a_4k_n \cos k_n\pi).\end{aligned}$$

And let $F(x)$ be a sectionally continuous function in $(0, \pi)$. The transformation $T\{F(x)\}$ is defined by the equation

$$T\{F(x)\} = \int_0^\pi F(x)\varphi_{k_n}(x)dx = f(k_n). \tag{9}$$

The transformations S, C and T are linear. The interval $(0, \pi)$ is chosen for convenience and without loss of generality. The transformation (9) maps the class of functions $F(x)$ onto a class of sequences of numbers; except for a normalization factor, each sequence is the set of Sturm-Liouville coefficients of the corresponding $F(x)$ in terms of the characteristic functions $\varphi_{k_n}(x)$. The restriction to sectionally continuous functions was made in view of applications. In general the functions $F(x)$ need only belong to $L^2(0, \pi)$: i.e. need only be Lebesgue square integrable.

2. Inverse transformations. Since the function $\varphi_{k_n}(x)$ form a complete set of orthogonal functions on the interval $(0, \pi)$, the transformation (9) has an inverse transformation in the form of a Sturm-Liouville expansion. For, let $F(x)$ be sectionally continuous and $[F'(x)]^2$ integrable in $(0, \pi)$ and at a point of discontinuity x_0, let $F(x)$ be defined as $F(x_0)=\frac{1}{2}[F(x_0+0)+F(x_0-0)]$, $0<x_0<\pi$, and let $N(k_n)$ denote the normalization factor of the functions (8), then the expansion in the characteristic functions (8) of the function $F(x)$ converges to the function $F(x)$ in $(0, \pi)$; i.e.,

$$F(x) = \sum_1^\infty N(k_n) f(k_n) \varphi_{k_n}(x), \quad \text{in} \quad (0, \pi), \tag{10}$$

and the convergence is uniform and absolute in every closed subinterval of $(0, \pi)$ which does not contain a discontinuity (see [6] vol. 1, p. 371, compare also [5] pp. 268, 272). Equation (10) gives a formula for the inverse $T^{-1}\{f(k_n)\}$, the function whose T-transform is $f(k_n)$. The inverse is unique. Thus

$$T^{-1}\{f(k_n)\} = \sum_1^\infty N(k_n) f(k_n) \varphi_{k_n}(x), \quad \text{in} \quad (0, \pi), \tag{11}$$

where $N(k_n)$ is given by

$$\begin{aligned} 1/N(k_n) = {} & \tfrac{1}{2}[\pi(a_1^2 + a_3^2) + k_n^2\pi(a_2^2 + a_4^2) - a_1a_2 + a_3a_4] \\ & + \sin k_n\pi[k_n\pi(a_2a_3 - a_1a_4) + (1/k_n)(k_n^2a_2a_4 - a_1a_3)] \\ & + (\sin 2k_n\pi/4k_n)[k_n^2(a_4^2 + a_2^2) - (a_1^2 + a_3^2)] \\ & + \pi(k_n^2a_2a_4 + a_1a_3)\cos k_n\pi + (\cos 2k_n\pi/2)(a_1a_2 - a_3a_4). \end{aligned}$$

Similarly the inverse transformations $S^{-1}\{f_s(k)\}$ and $C^{-1}\{f_c(k)\}$ of the transformations $S\{F(x)\}$ and $C\{F(x)\}$ respectively are given in terms of a sine and cosine series respectively if K is the sequence of characteristic values of certain special cases of (3). e.g. $a_1\neq 0$, $b_3\neq 0$ all other a_i, b_i, zero, then $K=\{n\}$, $n=1, 2, \cdots$, and $S\{F(x)\}$ is the finite Fourier sine transformation as defined by Kniess (see [12]) and $S^{-1}\{f_s(n)\}$ is given by a Fourier sine series.

3. Transformation of derivatives. The following two lemmas can be proved by integration by parts.

Lemma 1. If $F(x)$, $F'(x)$, $\cdots$ $F^{(2r-1)}(x)$, are continuous and $F^{(2r)}(x)$ is sectionally continuous in $(0, \pi)$, $r=1, 2, \cdots$, then

$$\begin{aligned} S\{F^{(2r)}(x)\} = (-1)^r k^{2r} S\{F(x)\} + \sum_{s=0}^{s=r-1} (-1)^s k^{2s}[k\{F^{(2r-2s-2)}(0) \\ - F^{(2r-2s-2)}(\pi)\cos k\pi\} + F^{(2r-2s-1)}(\pi)\sin k\pi] \end{aligned} \tag{12}$$

and

$$\begin{aligned} C\{F^{(2r)}(x)\} = (-1)^r k^{2r} C\{F(x)\} + \sum_{s=0}^{s=r-1} (-1)^s k^{2s}[kF^{(2r-2s-2)}(\pi)\sin k\pi \\ + F^{(2r-2s-1)}(\pi)\cos k\pi - F^{(2r-2s-1)}(0)]. \end{aligned} \tag{13}$$

Lemma 2. If $F(x)$, $F'(x)$, $\cdots$ $F^{(2r-2)}(x)$, are continuous and $F^{(2r-1)}(x)$ is sectionally continuous in $(0, \pi)$, $r=1, 2, \cdots$, then

$$S\{F^{(2r-1)}(x)\} = (-1)^r k^{2r-1} C\{F(x)\} + \sin k\pi \sum_{s=0}^{s=r-1} (-1)^s k^{2s} F^{(2r-2s-2)}(\pi)$$
$$+ \sum_{s=1}^{s=r-1} (-1)^s k^{2s-1} [F^{(2r-2s-1)}(\pi) \cos k\pi - F^{(2r-2s-1)}(0)] \qquad (14)$$

and

$$C\{F^{(2r-1)}(x)\} = (-1)^{r+1} k^{2r-1} S\{F(x)\} + \sin k\pi \sum_{s=1}^{s=r-1} (-1)^{s+1} k^{2s-1} F^{(2r-2s-1)}(\pi)$$
$$+ \sum_{s=0}^{s=r-1} (-1)^s k^{2s} [F^{(2r-2s-2)}(\pi) \cos k\pi - F^{(2r-2s-2)}(0)]. \qquad (15)$$

Theorem 1. If $F(x)$, $F'(x)$, $\cdots$ $F^{(2r-1)}(x)$, are continuous and $F^{(2r)}(x)$ is sectionally continuous in $(0, \pi)$, $r=1, 2, \cdots$, and if $u_1 = \varphi_{k_n}(0)$, $u_2 = \varphi'_{k_n}(0)$, $u_3 = \varphi_{k_n}(\pi)$, $u_4 = \varphi'_{k_n}(\pi)$, ($\varphi_{k_n}(x)$ as given by (8)), then there exist numbers λ and μ such that

$$T\{F^{(2r)}(x)\} = (-1)^r k_n^{2r} T\{F(x)\}$$
$$+ \sum_{s=0}^{s=r-1} (-1)^s k_n^{2s} [\lambda L_1(F^{(2r-2s-2)}) + \mu L_2(F^{(2r-2s-2)})], \qquad (16)$$

where

$$\lambda = (u_2 b_2 + u_1 b_1)/(a_1 b_2 - a_2 b_1)$$
$$= k_n + (a_1 b_2 - a_2 b_1)^{-1} [(a_3 b_2 - a_4 b_1) k_n \cos k_n \pi - (a_3 b_1 + k_n^2 a_4 b_2) \sin k_n \pi]$$

and

$$\mu = (u_3 a_3 + u_4 a_4)/(a_1 b_2 - a_2 b_1)$$
$$= (a_1 b_2 - a_2 b_1)^{-1} [(a_1 a_4 - a_2 a_3) k_n \cos k_n \pi + (a_1 a_3 + k_n^2 a_2 a_4) \sin k_n \pi].$$

Remark. Property (16) exhibits the usefulness of the transformation T in obtaining quickly a formal solution of the following type of boundary value problem:

$$\sum_{r=0}^{m} A_{2r} \frac{d^{2r} F(x)}{dx^{2r}} = G(x), \quad \text{in } (0, \pi), \quad \text{where}$$

$G(x)$, the constants A_{2r} and the quantities $L_j(F^{(2r-2s-2)})$, $j=1, 2$, $s=1, 2, \cdots r-1$, are assigned. The constants a_i, b_i, $i=1, 2, 3, 4$, are allowed to be different for each s. The transformation is also useful in boundary value problems in partial differential equations, where one or more variables behave like x in the above problem. Property (16) is analogous to the property of the Laplace transformation which expresses the transform of the nth derivative of a function in terms of the transform of the function and the value of the 0th, $\cdots$ $(n-1)$th derivative of the function at 0 (see for instance [5] p. 8).

Proof of theorem 1. By combining (12) and (13) $T\{F^{(2r)}(x)\}$ can be expressed as follows:

$$T\{F^{(2r)}(x)\} = (-1)^r k_n^{2r} T\{F(x)\} + \sum_{s=0}^{s=r-1} (-1)^s k_n^{2s} [u_2 F^{(2r-2s-2)}(0) - u_1 F^{(2r-2s-1)}(0) - u_4 F^{(2r-2s-2)}(\pi) + u_3 F^{(2r-2s-1)}(\pi)].$$

If

$$F^{(2r-2s-2)}(0) = w_1, \qquad F^{(2r-2s-1)}(0) = w_2,$$
$$F^{(2r-2s-2)}(\pi) = w_3, \qquad F^{(2r-2s-1)}(\pi) = w_4,$$

it remains to be shown that

$$u_2 w_1 - u_1 w_2 + u_3 w_4 - u_4 w_3 = \lambda \sum_1^4 a_i w_i + \mu \sum_1^4 b_i w_i,$$

which means it has to be shown that the following four equations are consistent:

$$u_2 = \lambda a_1 + \mu b_1$$
$$-u_1 = \lambda a_2 + \mu b_2$$
$$-u_4 = \lambda a_3 + \mu b_3$$
$$u_3 = \lambda a_4 + \mu b_4.$$

It can be left to the reader to obtain for instance λ from the first two equations and μ from the last two equations and to show that with these values of λ and μ the above equations are consistent.

4. Illustration of the operational method. Particular cases of transformations. As an illustration of the use of the operator T a formal solution of the following boundary value problem is to be established.

$$\Psi_{xx} + \Psi_{yy} = Q(x), \qquad 0 < x < \pi, \qquad 0 < y,$$

$\Psi(x, y)$ is to remain bounded as y approaches infinity and $\Psi(x, 0+)=0$, furthermore, $a\Psi(0+, y)+b\Psi_x(0+, y)=E$, where E is a constant, and $c\Psi(\pi-, y)+d\Psi_x(\pi-, y)=0$, where a, b, c, d are given non-zero constants such that $ad-bc\neq 0$.

This problem calls for the operator (9), where $\varphi_{k_n}(x)$ are the characteristic functions of $y''(x)+k^2y(x)=0$, $ay(0)+by'(0)=0$, $cy(\pi)+dy'(\pi)=0$. From (7) follows that the characteristic values k_n are the roots of $\tan k\pi = k(bc-ad)/(ac+k^2bd)$, $k>0$. From (9) it follows that

$$T\{F(x)\} = \int_0^\pi F(x)(a \sin k_n x - bk_n \cos k_n x)dx = f(k_n),$$

$n=1, 2, \cdots$, and (11) yields

$$T^{-1}\{f(k_n)\} = \sum_1^\infty N(k_n)f(k_n)(a \sin k_n x - bk_n \cos k_n x), \quad \text{in} \quad (0, \pi), \tag{17}$$

where $N(k_n)$ is given by

$$1/N(k_n) = \tfrac{1}{2}(\pi a^2 + \pi k_n^2 b^2 - ab) + [(k_n^2 b^2 - a^2)/4k_n] \sin 2k_n\pi + (ab/2) \cos 2k_n\pi.$$

Equation (16) with $r=1$ yields

$$T\{F''(x)\} = -k_n^2 f(k_n) + k_n[aF(0) + bF'(0)] - [k_n^2 b/(c \sin k_n\pi + k_n d \cos k_n\pi)][cF(\pi) + dF'(\pi)]. \tag{18}$$

Formula (18) applied to the above problem with respect to x leads to the following transformed problem:

$$\psi_{yy}(k_n, y) - k_n^2\psi(k_n, y) + k_n E - q(k_n) = 0,$$

$\psi(k_n, y)$ remains bounded as $y\to\infty$ and $\psi(k_n, 0+)=0$, where ψ and q stand for the T-transforms of Ψ and Q respectively.

The solution of the transformed problem, that is the coefficient in the Sturm-Liouville expansion of the solution is

$$\psi(k_n, y) = [1 - \exp(-k_n y)][k_n E - q(k_n)]/k_n^2$$

and according to (17) the formal solution of the given problem is

$$\Psi(x, y) = \sum_1^\infty N(k_n)\{[1 - \exp(-k_n y)][k_n E - q(k_n)]/k_n^2\}(a \sin k_n x - bk_n \cos k_n x).$$

In the following sections a special case of the transformations S and C will be needed (for additional special cases see section 12). The kernels sin kx and cos kx respectively of the transformations are the characteristic functions of $y''(x)+k^2y(x)=0$ $y(0)=0$, $y'(\pi)=0$; and $y'(0)=0$, $y(\pi)=0$ respectively.
In these two cases $K=\{n-\frac{1}{2}\}$, $n=1, 2, \cdots$. The transformations are

$$S\{F(x)\} = \int_0^\pi F(x) \sin(n - \tfrac{1}{2})x dx = f_s(n - \tfrac{1}{2}), \qquad n = 1, 2, \cdots \tag{19}$$

$$C\{F(x)\} = \int_0^\pi F(x) \cos(n - \tfrac{1}{2})x dx = f_c(n - \tfrac{1}{2}), \qquad n = 1, 2, \cdots. \tag{20}$$

The inverse transformations are given by

$$S^{-1}\{f_s(n - \tfrac{1}{2})\} = \frac{2}{\pi}\sum_1^\infty f_s(n - \tfrac{1}{2}) \sin(n - \tfrac{1}{2})x \quad \text{in} \quad (0, \pi) \tag{21}$$

$$C^{-1}\{f_c(n - \tfrac{1}{2})\} = \frac{2}{\pi}\sum_1^\infty f_c(n - \tfrac{1}{2}) \cos(n - \tfrac{1}{2})x \quad \text{in} \quad (0, \pi). \tag{22}$$

And for the transform of an even derivative (12) and (13) yield

$$S\{F^{(2r)}(x)\} = -(n - \tfrac{1}{2})^{2r}f_s(n - \tfrac{1}{2}) + \sum_{s=0}^{s=r-1}(-1)^s(n - \tfrac{1}{2})^{2s}[(-1)^{n+1}F^{(2r-2s-1)}(\pi) + (n - \tfrac{1}{2})F^{(2r-2s-1)}(0)] \tag{23}$$

$$C\{F^{(2r)}(x)\} = -(n - \tfrac{1}{2})^{2r}f_c(n - \tfrac{1}{2}) + \sum_{s=0}^{s=r-1}(-1)^{s+1}(n - \tfrac{1}{2})^{2s}[F^{(2r-2s-1)}(0) + (n - \tfrac{1}{2})(-1)^nF^{(2r-2s-2)}(\pi)]. \tag{24}$$

Remark. In the above listed cases $D(k)$ is of rank greater than zero. If the rank of $D(k)$ is zero it merely has to be kept in mind that the sine as well as the cosine coefficients in the Sturm-Liouville expansion of the solution have to be found. An illustration of the operational method in this case is given in section 10.

5. Some properties of the transformations S **and** C **when** $K=\{n-\frac{1}{2}\}$, $n=1, 2, \cdots$. Theorems 2 to 5 below are analogous to theorems which holds for the finite Fourier sine and cosine transformations (compare [5] sections 95 and 96 also [19] chap. II sect. 2.1). and can be proved in a similar way.

Definition 3. Let the function $F(x)$ be defined in $(0, p)$. By the odd antiperiodic extension $F_1(x)$ of $F(x)$ with antiperiod $2p$ is meant $F_1(x)=F(x)$, in $(0, p)$; $F_1(-x) = -F_1(x)$, $F_1(x+2p)=-F_1(x)$ in $(-\infty, \infty)$.

Definition 4. Let the function $F(x)$ be defined in $(0, p)$. By the even antiperiodic extension $F_2(x)$ of $F(x)$ with antiperiod $2p$ is meant $F_2(x)=F(x)$, in $(0, p)$; $F_2(-x) = F_2(x)$, $F_2(x+2p)=-F_2(x)$, in $(-\infty, \infty)$.

Theorem 2. If $F(x)$ is sectionally continuous in $(0, \pi)$ and if $F_1(x)$ is the odd antiperiodic extension of $F(x)$ with antiperiod 2π and if a is any constant, then

$$\sin (n - \tfrac{1}{2})aS\{F(x)\} = \tfrac{1}{2}C\{F_1(x + a) - F_1(x - \mathrm{a})\}.$$

Theorem 3. Under the same assumption as in theorem 2

$$\cos (n - \tfrac{1}{2})aS\{F(x)\} = \tfrac{1}{2}S\{F_1(x + a) + F_1(x - a)\}.$$

Theorem 4. If $F(x)$ is sectionally continuous in $(0, \pi)$ and if $F_2(x)$ is the even antiperiodic extension of $F(x)$ with antiperiod 2π and if a is any constant, then

$$\sin (n - \tfrac{1}{2})aC\{F(x)\} = \tfrac{1}{2}S\{F_2(x - a) - F_2(x + a)\}.$$

Theorem 5. Under the same assumption as in theorem 4

$$\cos (n - \tfrac{1}{2})aC\{F(x)\} = \tfrac{1}{2}C\{F_2(x - a) + F_2(x + a)\}.$$

Similar theorems in the case when K is the sequence of characteristic values of (3) and for more general K can be proved by the use of almost periodic functions (see [15]).

6. Vibration of a horizontal string with one end fixed and one end sliding. Compare [5] sections 98, 99.) Let the end $x=0$ of a string be fixed and the end $x=\pi$ be looped about a vertical support along the line $x=\pi$. If a constant upward force acts on the loop, the displacements $Y(x, t)$ as the string is released from the position $Y=0$ satisfy the conditions:

$$\begin{aligned} Y_{tt}(x, t) &= a^2Y_{xx}(x, t) + g, \qquad 0 < x < \pi, \qquad 0 < t,\\ Y(x, 0+) &= 0 \qquad Y_t(x, 0+) = 0,\\ Y(0+, t) &= 0 \qquad Y_x(\pi -, t) = -b, \end{aligned}$$

where g is the acceleration of gravity and b is the magnitude of the vertical force divided by the tension.

The S-transformation (19) applied with respect to x gives, according to (23), the following transformed problem:

$$\frac{d^2y(n-\frac{1}{2},t)}{dt^2}+(n-\tfrac{1}{2})^2a^2y(n-\tfrac{1}{2},t)+(-1)^{n+1}a^2b-gS\{1\}=0$$

$$y(n-\tfrac{1}{2},0+)=0,\qquad y_t(n-\tfrac{1}{2},0+)=0.$$

The solution of this transformed problem is

$$y(n-\tfrac{1}{2},t)=[1-\cos(n-\tfrac{1}{2})at]\{[g/a^2(n-\tfrac{1}{2})^3]+[b(-1)^n/(n-\tfrac{1}{2})^2]\}.$$

From the table in section 12 below it follows that

$$S^{-1}\{[b(-1)^n/(n-\tfrac{1}{2})^2]+[g/a^2(n-\tfrac{1}{2})^3]\}=-bx+[gx(2\pi-x)/2a^2]=F(x).$$

Using theorem 3 the solution of the given problem can be written in the form, $Y(x,t)=F(x)-\frac{1}{2}[F_1(x+at)+F_1(x-at)]$, where F_1 denotes the odd antiperiodic extension with antiperiod 2π of $F(x)$.

Similarly the problem with the end $x=0$ sliding and the end $x=\pi$ fixed can be solved by use of the C-transformation (20).

7. The convolution in the case $K=\{n-\frac{1}{2}\}$. The purpose of this section is to give formulas for the product of two transforms in terms of one transform. Kniess (see [12], compare also [19] 2.1) and Doetsch (see [8]) give such results for the finite Fourier sine and cosine transformations, i.e. when $K=\{n\}$, $n=1,2,\cdots$. Here the analogous results are given for the S- and C-transformations when $K=\{n-\frac{1}{2}\}$ (compare also Koschmieder [13]). Analogous results when K is the set of characteristic values of certain Sturm-Liouville problems can be obtained by the use of almost period functions (see [15]). The proofs of theorems 6–9 are analogous to those of Kniess.

Definition 5. If $F(x)$ in $(-2\pi,2\pi)$ and $G(x)$ in $(-\pi,\pi)$, are sectionally continuous, then the function

$$F(x)*G(x)=\int_{-\pi}^{\pi}F(x-y)G(y)dy$$

is called the convolution of F and G on the interval $(-\pi,\pi)$ (compare [12] p. 270 and [5] p. 274.)

Lemma 3. If $F(x)$ and $G(x)$ are sectionally continuous and $F(x+2\pi)=-F(x)$ and $G(x+2\pi)=-G(x)$, then

$$F(x)*G(x)=G(x)*F(x).$$

Theorem 6. If $F(x)$ and $G(x)$ are sectionally continuous and even functions and if $F(x+2\pi)=-F(x)$, then

$$C\{F(x)\}C\{G(x)\}=\tfrac{1}{2}C\{F(x)*G(x)\}.$$

Theorem 7. If $F(x)$ and $G(x)$ are sectionally continuous and odd functions and if $F(x+2\pi)=-F(x)$, then

$$S\{F(x)\}S\{G(x)\}=-\tfrac{1}{2}C\{F(x)*G(x)\}.$$

Theorem 8. If $F(x)$ is an even and $G(x)$ is an odd sectionally continuous function and if $F(x+2\pi)=-F(x)$, then

$$C\{F(x)\}S\{G(x)\}=\tfrac{1}{2}S\{F(x)*G(x)\}.$$

Theorem 9. If $F(x)$ is an odd and $G(x)$ is an even sectionally continuous function and if $F(x+2\pi)=-F(x)$, then

$$S\{F(x)\}C\{G(x)\} = \tfrac{1}{2}S\{F(x)*G(x)\}.$$

Remark. If in the above four theorems $G(x)$ satisfies the condition $G(x+2\pi)=-G(x)$, then according to lemma 3 the convolution is commutative. That being the case theorems 8 and 9 say the same thing.

Example. Given $q_c(n-\frac{1}{2})=(n-\frac{1}{2})^{-2}[1-(n-\frac{1}{2})^2]^{-1}$, find $Q(x)$. $q_c(n-\frac{1}{2})=S\{x\}\times S\{\sin x\}$ (see tables section 12). According to theorem 7 if $F(x)=x$ in $(0,\pi)$ and

$$\left.\begin{aligned} F(x+2\pi) &= -F(x) \\ G(x) &= \sin x \end{aligned}\right\} \quad \text{in} \quad (-\infty, \infty)$$

$$\begin{aligned} Q(x) &= -\frac{1}{2}\left[\int_{-\pi}^{x-\pi}(2\pi - x + y)\sin y dy + \int_{x-\pi}^{\pi}(x-y)\sin y dy\right] \\ &= \pi - x + \sin x. \end{aligned}$$

8. Basic problem. In section 9 an application of the solution of the following problem will be made.

$$\frac{\partial Y}{\partial t} + \sum_{r=1}^{m}(-1)^r\frac{\partial^{(2r)}Y}{\partial x^{2r}} = 0, \qquad 0 < x < \pi, \qquad 0 < t,$$

$$\frac{\partial^{(2m-2s-2)}Y(0+,t)}{\partial x^{2m-2s-2}} = 0, \qquad \frac{\partial^{(2m-2s-1)}Y(\pi-,t)}{\partial x^{2m-2s-1}} = 0,$$

$$s = 0, 1, 2, \cdots, m-1, \qquad Y(x, 0+) = 1.$$

The S-transformation (19) with respect to x gives according to (23) the following transformed problem:

$$\frac{dy(n-\frac{1}{2},t)}{dt} + y(n-\tfrac{1}{2},t)\sum_{r=1}^{m}(n-\tfrac{1}{2})^{2r} = 0$$

$$y(n-\tfrac{1}{2}, 0+) = S\{1\} = 1/(n-\tfrac{1}{2}).$$

The solution of this transformed problem is

$$y(n-\tfrac{1}{2},t) = (n-\tfrac{1}{2})^{-1}\exp\left[-\sum_{r=1}^{m}t(n-\tfrac{1}{2})^{2r}\right].$$

According to (21) the solution of the basic problem is thus

$$Y(x,t) = (2/\pi)\sum_{n=1}^{\infty}(n-\tfrac{1}{2})^{-1}\exp\left[-\sum_{r=1}^{m}t(n-\tfrac{1}{2})^{2r}\right]\sin(n-\tfrac{1}{2})x.$$

9. A generalized heat conduction problem. A formal solution of the following generalized heat conduction problem is obtained by use of the operator C when $K=\{n-\frac{1}{2}\}$ (see 20)).

$$L(U) = \frac{\partial U}{\partial t} + A_0(t)U(x, t) + \sum_{r=1}^{m} A_{2r}(t)(-1)^r \frac{\partial^{(2r)}U}{\partial x^{2r}} = Q(x, t),$$

$$0 < x < \pi, \qquad t > 0, \qquad A_{2r}(t) > 0,$$

$$\left.\begin{aligned} \frac{\partial^{(2m-2s-1)}U(0+, t)}{\partial x^{2m-2s-1}} &= B_{2m-2s-1}(t), \\ \frac{\partial^{(2m-2s-2)}U(\pi-, t)}{\partial x^{2m-2s-2}} &= D_{2m-2s-2}(t), \end{aligned}\right\} \qquad s = 0, 1, 2, \cdots, m-1,$$

$$U(x, 0+) = F(x).$$

This problem can be resolved into $2m+2$ problems each of which has $2m+1$ homogeneous conditions and one non-homogeneous condition. The sum of the solutions of these problems is the solution of the given problem. There are four essentially different types of problems, a formal solution of which is given in the following.

Problem I. $L(U)=Q(x, t)$, $0<x<\pi$, $0<t$, $0<A_{2r}(t)$,

$$\frac{\partial^{(2m-2s-1)}U(0+, t)}{\partial x^{2m-2s-1}} = 0, \qquad \frac{\partial^{(2m-2s-2)}U(\pi-, t)}{\partial x^{2m-2s-2}} = 0,$$

$$s = 0, 1, 2, \cdots, m-1, \qquad U(x, 0+) = 0.$$

The C-transformation (20) applied with respect to x leads according to (30) to the following transformed problem:

$$\frac{du_c(n-\frac{1}{2}, t)}{dt} + \left[A_0(t) + \sum_{r=1}^{m} A_{2r}(t)(n-\tfrac{1}{2})^{2r}\right]u_c(n-\tfrac{1}{2}, t) - q_c(n-\tfrac{1}{2}, t) = 0,$$

$$u_c(n-\tfrac{1}{2}, 0+) = 0.$$

This equation is of the type $y'(x)+P(x)y(x)-Q(x)=0$ with initial condition $y(0)=0$. The solution thereof can be written in the form

$$y(x) = \int_0^x \exp\,[-p(x, w)]Q(w)dw, \quad \text{where} \quad p(x, w) = \int_w^x P(v)dv.$$

Hence the solution of the transformed problem can be written in the form

$$u_c(n-\tfrac{1}{2}, t) = \int_0^t \exp\,[-\overline{A}_0(t, v)]\exp\left[-\sum_{r=1}^{m}\overline{A}_{2r}(t, v)(n-\tfrac{1}{2})^{2r}\right]q_c(n-\tfrac{1}{2}, v)dv,$$

where

$$\overline{A}_{2r}(t, v) = \int_v^t A_{2r}(y)dy.$$

Now

$$\exp\left[-\sum_{r=1}^{m}\overline{A}_{2r}(t, v)(n-\tfrac{1}{2})^{2r}\right] = (n-\tfrac{1}{2})S\{Y(x, \overline{A}_{2r}(t, v))\},$$

where Y is the solution of the basic problem of section 8. And according to (15) with $k=n-\frac{1}{2}$ and $r=1$.

$$(n-\tfrac{1}{2})S\{Y(x,\overline{A}_{2r}(t,v))\} = C\{Y_x(x,\overline{A}_{2r}(t,v))\},$$

so that

$$u_c(n-\tfrac{1}{2},t) = \int_0^t \exp\,[-\overline{A}_0(t,v)]C\{Y_x(x,\overline{A}_{2r}(t,v))\}C\{Q(x,v)\}dv.$$

Using theorem 6 the solution of problem I can be written in the following form:

$$U(x,t) = \int_0^t \exp\,[-\overline{A}_0(t,v)][\tfrac{1}{2}Y_x(x,\overline{A}_{2r}(t,v)) * Q_0(x,v)]dv,$$

where

$$\begin{aligned} Q_0(x,v) &= Q(x,v) & 0 \leqq x \leqq \pi \\ Q_0(-x,v) &= Q_0(x,v), & -\pi \leqq x \leqq \pi. \end{aligned}$$

Problem II. $L(U)=0,\ 0<x<\pi,\ 0<t,\ A_{2r}(t)>0,$

$$\begin{aligned} \frac{\partial^{(2m-2s-1)}U(0+,t)}{\partial x^{2m-2s-1}} &= B_{2m-2s-1}(t) \quad \text{for} \quad s=i, \\ &= 0, \qquad\qquad \text{for} \quad s\neq i, \\ \frac{\partial^{(2m-2s-2)}U(\pi-,t)}{\partial x^{2m-2s-2}} &= 0, \qquad U(x,0+)=0, \end{aligned}$$

$$s=0,1,2,\cdots,m-1.$$

According to (24) the C-transformation applied with respect to x leads to the following transformed problem.

$$\frac{du_c(n-\frac{1}{2},t)}{dt} + \left[A_0(t) + \sum_{r=1}^{m} A_{2r}(t)(n-\tfrac{1}{2})^{2r}\right]u_c(n-\tfrac{1}{2},t)$$

$$+ \sum_{r=0}^{m} A_{2r}(t)(-1)^r(-1)^{i+1}(n-\tfrac{1}{2})^{2i}B_{2r-2i-1}(t) = 0,$$

$$u_c(n-\tfrac{1}{2},0+) = 0.$$

With the notation of problem I the solution of this transformed problem reads:

$$u_c(n-\tfrac{1}{2},t) = \int_0^t \exp\,[-\overline{A}_0(t,v)]\exp\left[-\sum_{r=1}^{m}\overline{A}_{2r}(t,v)(n-\tfrac{1}{2})^{2r}\right](n-\tfrac{1}{2})^{2i}(-1)^i$$

$$\cdot \sum_{r=0}^{m}(-1)^r B_{2r-2i-1}(v)A_{2r}(v)dv.$$

By use of the solution $Y(x,t)$ of the basic problem and (15) with $r=i+1$ and $k=n-\frac{1}{2}$

$$\begin{aligned} (n-\tfrac{1}{2})^{2i}\exp\left[-\sum_{r=1}^{m}\overline{A}_{2r}(t,v)(n-\tfrac{1}{2})^{2r}\right] &= (n-\tfrac{1}{2})^{2i+1}S\{Y(x,\overline{A}_{2r}(t,v))\} \\ &= (-1)^{i+2}C\left\{\frac{\partial^{(2i+1)}Y(x,\overline{A}_{2r}(t,v))}{\partial x^{2i+1}}\right\}. \end{aligned}$$

Hence the solution of problem II reads

$$U(x, t) = \int_0^t \exp\left[-\overline{A}_0(t, v)\right] \frac{\partial^{(2i+1)} Y(x, \overline{A}_{2r}(t, v))}{x^{2i+1}} \left(\sum_{r=0}^{m} (-1)^r A_{2r}(v) B_{2r-2i-1}(v) \right) dv.$$

Problem III. $L(U)=0,\ 0<x<\pi,\ 0<t,\ A_{2r}(t)>0,$

$$\frac{\partial^{(2m-2s-1)} U(0+, t)}{\partial x^{2m-2s-1}} = 0, \qquad s = 0, 1, 2, \cdots, m-1,$$

$$\frac{\partial^{(2m-2s-2)} U(\pi-, t)}{\partial x^{2m-2s-2}} = D_{2m-2s-2}(t), \quad s = i,$$

$$= 0, \qquad s \neq i,$$

$$U(x, 0+) = 0,$$

According to (24) the transformed problem reads:

$$\frac{du_c(n-\frac{1}{2}, t)}{dt} + \left[A_0(t) + \sum_{r=1}^{m} A_{2r}(t)(n-\tfrac{1}{2})^{2r} \right] u_c(n-\tfrac{1}{2}, t)$$

$$+ \sum_{r=0}^{m} (-1)^r A_{2r}(t)(-1)^{i+1}(-1)^n (n-\tfrac{1}{2})^{2i+1} D_{2r-2i-2}(t) = 0$$

$$u_c(n-\tfrac{1}{2}, 0+) = 0.$$

The solution thereof in the notation of problem I is

$$u_c(n-\tfrac{1}{2}, t) = \int_0^t \exp\left[-\overline{A}_0(t, v)\right] \exp\left[-\sum_{r=1}^{m} \overline{A}_{2r}(t, v)(n-\tfrac{1}{2})^{2r} \right] (-1)^{n+i} (n-\tfrac{1}{2})^{2i+1}$$

$$\cdot \sum_{r=0}^{m} (-1)^r A_{2r}(v) D_{2r-2i-2}(v) dv.$$

By use of the solution $Y(x, t)$ of the basic problem and (23) with $r=i+1$

$$(n-\tfrac{1}{2})^{2i+1} \exp\left[-\sum_{r=1}^{m} \overline{A}_{2r}(t, v)(n-\tfrac{1}{2})^{2r} \right] = (n-\tfrac{1}{2})^{2i+2} S\left\{ Y(x, \overline{A}_{2r}(t, v)) \right\}$$

$$= (-1)^{i+1} S \left\{ \frac{\partial^{(2i+2)} Y(x, \overline{A}_{2r}(t, v))}{\partial x^{2i+2}} \right\}.$$

Using theorem 3

$$(-1)^n S \left\{ \frac{\partial^{(2i+2)} Y(x, \overline{A}_{2r}(t, v))}{\partial x^{2i+2}} \right\} = -C \left\{ \frac{\partial^{(2i+2)} Y(y, \overline{A}_{2r}(t, v))}{y^{2i+2}} \right\},$$

$$y = \pi - x$$

so that the solution of problem III reads

$$U(x, t) = \int_0^t \exp\left[-\bar{A}_0(t, v)\right] \frac{\partial^{(2i+2)} Y(y, \bar{A}_{2r}(t, v))}{\partial y^{2i+2}} \left[\sum_{r=0}^{m} (-1)^r A_{2r}(v) D_{2r-2i-2}(v)\right] dv,$$

$$y = \pi - x.$$

Problem IV. $L(U)=0,\ 0<x<\pi,\ 0<t,\ A_{2r}(t)>0,$

$$\frac{\partial^{(2m-2s-1)} U(0+, t)}{\partial x^{2m-2s-1}} = 0, \qquad \frac{\partial^{(2m-2s-2)} U(\pi-, t)}{\partial x^{2m-2s-2}} = 0,$$

$$s = 0, 1, 2, \cdots, m-1, \qquad U(x, 0+) = F(x).$$

According to (24) the transformed problem reads

$$\frac{du_c(n-\tfrac{1}{2}, t)}{dt} + \left[A_0(t) + \sum_{r=1}^{m} A_{2r}(t)(n-\tfrac{1}{2})^{2r}\right] u_c(n-\tfrac{1}{2}, t) = 0,$$

$$u_c(n-\tfrac{1}{2}, 0+) = f_c(n-\tfrac{1}{2}).$$

The solution thereof in the notation of problem I can be written as

$$u_c(n-\tfrac{1}{2}, t) = f_c(n-\tfrac{1}{2}) \exp\left[-\bar{A}_0(t, 0)\right] \exp\left[-\sum_{r=1}^{m} \bar{A}_{2r}(t, 0)(n-\tfrac{1}{2})^{2r}\right]$$

$$= \exp\left[-\bar{A}_0(t, 0)\right] C\{F(x)\} C\left\{\frac{\partial Y(x, \bar{A}_{2r}(t, 0))}{\partial x}\right\},$$

since

$$\exp\left[-\sum_{r=1}^{m} \bar{A}_{2r}(t, 0)(n-\tfrac{1}{2})^{2r}\right] = (n-\tfrac{1}{2}) S\{Y(x, \bar{A}_{2r}(t, 0))\}$$

$$= C\left\{\frac{\partial Y(x, \bar{A}_{2r}(t, 0))}{\partial x}\right\}$$

according to (15) with $r=1$ and $k=n-\frac{1}{2}$.

By theorem 6 the solution of problem IV can be written as

$$U(x, t) = \exp\left[-\bar{A}_0(t, 0)\right] \left[\frac{1}{2} \frac{\partial Y(x, \bar{A}_{2r}(t, 0))}{\partial x} * F_0(x)\right],$$

where $F_0(x)=F(x)$ in $(0, \pi)$ and $F_0(-x)=F_0(x)$ in $(-\pi, \pi)$.

It can be seen now that the solution of each problem is expressed in closed form in terms of the solution of the basic problem of section 8. Thus *the solution of the given problem* being the sum of the 2m+2 problems *is expressed in closed form* in terms of the solution of the basic problem.

10. A problem in heat conduction. As a particular case ($m=1$) of section 9 the solution of the following problem can be obtained.

To find the temperature distribution $U(x, t)$ in a slab of length π with a heat source inside, The end $x=\pi$ is kept at a temperature $D(t)$ and radiation through the end $x=0$ at a rate $B(t)$ takes place. Furthermore there is radiation through the lateral surface and the thermal diffusivity $K(t)$ depends on time. The initial temperature of the slab is $F(x)$.(Compare also [1].)

The mathematical formulation of this problem is the problem of section 9 with $m=1$, $A_2(t)=K(t)$, $A_0(t)=A(t)$, $B_1(t)=B(t)$, $D_0(t)=D(t)$, i.e.

$$\frac{\partial U}{\partial t} - K(t)\frac{\partial^2 U}{\partial x^2} A(t)U(x, t) = Q(x, t), \qquad 0 < x < \pi, \qquad 0 < t,$$

$$\frac{\partial U(0+, t)}{\partial x} = B(t) \qquad U(\pi-, t) = D(t),$$

$$U(x, 0+) = F(x).$$

Put

$$\overline{K}(t, v) = \int_v^t K(x)dx, \qquad \overline{A}(t, v) = \int_v^t A(x)dx.$$

The solution of the basic problem of section 8 with $m=1$ is

$$Y(x, t) = (2/\pi)\sum_1^\infty (n - \tfrac{1}{2})^{-1} \exp\left[-(n - \tfrac{1}{2})^2 t\right] \sin(n - \tfrac{1}{2})x.$$

The solution of the heat conduction problem can be written in closed form using Jacobi's ϑ_2-function.
For

$$\frac{\partial Y(x, t)}{\partial x} = (2/\pi)\sum_1^\infty \exp\left[-(n - \tfrac{1}{2})^2 t\right] \cos(n - \tfrac{1}{2})x$$

$$= (1/\pi)\vartheta_2(x/2, \exp[-t]), \quad \text{(see [12] p. 464.)}$$

Thus the solution reads

$$U(x, t) = \int_0^t \exp\left[-\overline{A}(t, v)\right]\left[(1/2\pi)\vartheta_2(x/2, \exp\{-\overline{K}(t, v)\}) * Q_0(x, v)\right]dv$$

$$- (1/\pi)\int_0^t \exp\left[-\overline{A}(t, v)\right]\vartheta_2(x/2, \exp\{-\overline{K}(t, v)\})K(v)B(v)dv$$

$$- (1/\pi)\int_0^t \exp\left[-\overline{A}(t, v)\right]\frac{\partial\vartheta_2}{\partial y}(y/2, \exp\{-\overline{K}(t, v)\})K(v)D(v)dv$$

$$y = \pi - x$$

$$+ \exp\left[-\overline{A}(t, 0)\right]\left[(1/2\pi)\vartheta_2(x/2, \exp\{-\overline{K}(t, 0)\}) * F_0(x)\right].$$

Remarks. Since the ϑ-functions are tabulated the above form of the solution can be used to determine by mechanical integration the numerical values of $U(x, t)$ for given values x and t.

For questions on uniqueness and existence of solutions in heat conduction problems the reader is referred to [9].

11. An illustration of the method in the case of double characteristic values. Summary of the operational method. A formal solution of the following boundary value problem is to be found.

$$L(U) = U_t(x, t) - K(t)U_{xx}(x, t) + A(t)U(x, t) = Q(x, t),$$
$$0 < x < \pi, \qquad 0 < t, \qquad A(t) > 0, \qquad K(t) > 0,$$
$$L_1(U) = U_x(0+, t) + U_x(\pi-, t) = B(t),$$
$$L_2(U) = U(0+, t) + U(\pi-, t) = D(t),$$
$$U(x, 0+) = F(x).$$

As kernel of the transformation (9) to be applied to this problem the characteristic functions of $y''(x)+k^2y(x)=0$, $y'(0)+y'(\pi)=0$, $y(0)+y(\pi)=0$, are used. In this case the determinant $D(k)$ is of rank zero. The double characteristic values are $2n-1$, $n=1, 2, \cdots$, and $\sin(2n-1)x$ as well as $\cos(2n-1)x$ are characteristic functions. This means the sine as well as the cosine coefficients of the solution of the above problem have to be found. The formulae to be used are:

$$S\{F''(x)\} = -(2n-1)^2 f_s(2n-1) + (2n-1)[F(0) + F(\pi)], \tag{25}$$
$$C\{F''(x)\} = -(2n-1)^2 f_c(2n-1) - [F'(0) + F'(\pi)], \tag{26}$$
$$C\{F'(x)\} = (2n-1)f_s(2n-1) - [F(\pi) + F(0)]. \tag{27}$$

The last three equations follow from (12), (13) and (15) respectively with $r=1$ and $k=2n-1$, $n=1, 2, \cdots$. Relation (11) yields

$$S^{-1}\{f_s(2n-1)\} = (2/\pi)\sum_1^\infty f_s(2n-1)\sin(2n-1)x \quad \text{in} \quad (0, \pi). \tag{28}$$

Furthermore the solution of the following auxiliary problem will be useful.

$$Y_t(x, t) - Y_{xx}(x, t) = 0, \quad \text{in} \quad (0, \pi), \qquad Y_x(0+, t) + Y_x(\pi-, t) = 0,$$

$Y(0+, t)+Y(\pi-, t)=0$, $Y(x, 0+)=1$. From (25) it follows that $[dy_s(2n-1, t)/dt] +(2n-1)^2y_s(2n-1, t)=0$, $y_s(2n-1, 0+)=S\{1\}=2/(2n-1)$. Hence $y_s(2n-1, t) =2\exp[-t(2n-1)^2](2n-1)^{-1}$, which gives the sine coefficient of the solution of the auxiliary problem. The cosine coefficient is zero since $C\{1\}=0$ when $k=2n-1$; so that the solution of the auxiliary problem is according to (28)

$$Y(x, t) = (2/\pi)\sum_1^\infty 2(2n-1)^{-1}\exp[-(2n-1)^2t]\sin(2n-1)x, \tag{29}$$

$$Y_x(x, t) = (2/\pi)\vartheta_2(x, \exp[-4t]), \quad \text{see [18] p. 464.} \tag{30}$$

The following formulae by Kniess (see [12]) will also be used. If $K=\{n\}$ then a) if $F(x)$ is even and periodic with period 2π and $G(x)$ is odd, then

$$C\{F(x)\}S\{G(x)\} = \tfrac{1}{2}S\{F(x)*G(x)\}; \tag{31}$$

b) if $F(x)$ and $G(x)$ are even and $F(x)$ is periodic with period 2π, then

$$C\{F(x)\}C\{G(x)\} = \tfrac{1}{2}C\{F(x)*G(x)\}. \tag{32}$$

The solution of the given problem can be written as the sum of the solutions of the following four problems, where $0<x<\pi$, $0<t$, $A(t)>0$, $K(t)>0$.

$$\text{I. } L(U) = Q(x, t), \qquad L_1(U) = L_2(U), \qquad U(x, 0+) = 0.$$
$$\text{II. } L(U) = L_2(U), \quad U(x, 0+) = 0, \qquad L_1(U) = B(t).$$
$$\text{III. } L(U) = L_1(U), \quad U(x, 0+) = 0, \qquad L_2(U) = D(t).$$
$$\text{IV. } L(U) = L_1(U) = L_2(U) = 0, \qquad U(x, 0+) = F(x).$$

The sine and the cosine transforms using (25) and (26) respectively of the solution of each of these problems have to be found. The sine transform of the solution of II and the cosine transform of the solution of III are zero.

With the notation of section 10 it follows from (25) that the sine transform of the solution of I is

$$u_s(2n-1, t) = \int_0^t \exp\left[-\bar{A}(t, v)\right] \exp\left[-(2n-1)^2\bar{K}(t, v)\right] q_s(2n-1, v)dv.$$

Relations (29) and (27) yield

$$\exp\left[-(2n-1)^2t\right] = \left[(2n-1)/2\right]S\{Y(x, t)\} = \tfrac{1}{2}C\{Y_x(x, t)\}.$$

Using (31) and (30) the sine part of the solution of I can be written as

$$(1/2\pi)\int_0^t \exp\left[-\bar{A}(t, v)\right]\left[\vartheta_2(x, \exp(-4\bar{K}(t, v)\}) * Q_0(x \;\; v)\right]dv,$$

where Q is extended so as to be odd with respect to x.

From (26), (29) and (27) it follows that

$$u_c(2n - \tfrac{1}{2}, t) = \frac{1}{2}\int_0^t \exp\left[-\bar{A}(t, v)\right]C\{Y_x(x \;\; \bar{K}(t, v))\}C\{Q(x \;\; v)\}dv.$$

According to (32) and (30) the cosine part of the solution of I can be written as

$$(1/2\pi)\int_0^t \exp\left[-\bar{A}(t, v)\right]\left[\vartheta_2(x, \exp\{-4\bar{K}(t, v)\}) * Q_e(x, v)\right]dv,$$

where Q is extended so as to be even with respect to x.

It may be left to the reader to show that in a similar way the sine and cosine parts of the problems II–IV can be expressed in closed form in terms of ϑ_2 by using relations (25) to (32). Thus the solution of the given problem can be expressed in closed form in terms of Jacobi's ϑ_2 function as follows:

$$\begin{aligned}
U(x, t) = {} & (1/2\pi)\int_0^t \exp\left[-\bar{A}(t, v)\right]\left[\vartheta_2(x, \exp\{-4\bar{K}(t, v)\}) * Q_0(x, v)\right]dv \\
& + (1/2\pi)\int_0^t \exp\left[-\bar{A}(t, v)\right]\left[\vartheta_2(x, \exp\{-4\bar{K}(t, v)\}) * Q_e(x, v)\right]dv \\
& - (1/\pi)\int_0^t \exp\left[-\bar{A}(t, v)\right]\vartheta_2(x, \exp\{-4\bar{K}(t, v)\})B(v)K(v)dv \\
& - (1/\pi)\int_0^t \exp\left[-\bar{A}(t, v)\right]\vartheta_2(x, \exp\{-4\bar{K}(t, v)\})D(v)K(v)dv \\
& + (1/2\pi)\exp\left[-\bar{A}(t, 0)\right]\left[\vartheta_2(x, \exp\{-4\bar{K}(t, 0)\}) * F_0(x)\right] \\
& + (1/2\pi)\exp\left[-\bar{A}(t, 0)\right]\left[\vartheta_2(x, \exp\{-4\bar{K}(t, 0)\}) * F_e(x)\right].
\end{aligned}$$

Summary of the operational method. The method is applicable to the following type of boundary value problem:

$$\sum_{r=0}^{m} (-1)^r A_{2r} \frac{d^{2r}F(x)}{dx^{2r}} = G(x) \quad \text{in} \quad (0, \pi),$$

where $G(x)$, the quantities

$$a_1F^{(2r-2s-2)}(0) + a_2F^{(2r-2s-1)}(0) + a_3F^{(2r-2s-2)}(\pi) + a_4F^{(2r-2s-2)}(\pi)$$

and

$$b_1F^{(2r-2s-2)}(0) + b_2F^{(2r-2s-1)}(0) + b_3F^{(2r-2s-2)}(\pi) + b_4F^{(2r-2s-2)}(\pi)$$

and the constants A_{2r} are assigned, $s=0, 1, 2, \cdots, r-1$. The constants a_i, b_i, $i=1, 2, 3, 4$ are allowed to be different for each s. The transformation is also useful in boundary value problems in partial differential equations, where one or more variables behave like x in the above problem. In order to obtain a formal solution quickly, and possibly a closed form solution, the following procedure is recommended:

1° Set up (3) corresponding to the given problem.

2° Find the roots of $D(k)=0$ (see (7)).

3° Find the rank of the determinant $D(k)$.

4° If the rank is greater than zero set up $T\{F\}$ using (9). If the rank is zero set up $T_1\{F\}$ and $T_2\{F\}$, where T_1 and T_2 have the two independent characteristic functions of (3) as respective kernels.

5° Find $T\{F^{2r}\}$ using (16).

6° Apply $T\{F^{2r}\}$ to the boundary value problem (which may be resolved into several problems each of which has only one non-homogeneous condition) and find $f(k_n)$, the transform of the solution.

7° Try to obtain the inverse $T^{-1}\{f(k_n)\}$ in closed form by application of the theorems of sections 3, 5, 7 or by use of tables of transforms or a combination of both.

8° If 7° does not lead to the solution, use (11) to find $T^{-1}\{f(k_n)\}$ and obtain the solution in series form.

Remark. Formulae for a few special cases of T can be found in the tables section 12.

12. Tables. Tables A–D contain a few examples of transforms. It would be desirable to have extensive tables of transforms, since they would help in obtaining closed form solutions of boundary value problems (see the example of section 6).

Table E below contains a list of transformations which are special cases of the transformation T (see (9)) of section 1. In each case the formula for the transform of an even derivative (see (16)) as well as the inverse transformation (see (11)) are given. The completion of this list of special transformations is left to the reader.

A. Tables of S-transforms.

	$f_s(k)=S\{F(x)\}$	$F(x)$
1.	$(1-\cos k\pi)k^{-1}$	1
2.	$k^{-2}\sin k\pi-k^{-1}\pi\cos k\pi$	x
3.	$[(-1)^r(2r)!(1-\cos k\pi)/k^{2r+1}]+\cos k\pi\sum_{s=0}^{r-1}(-1)^{r+s}[(2r)!\pi^{2s+2}/(2s+2)!k^{2r-2s-1}]$	x^{2r}
	$+\sin k\pi\sum_{s=0}^{r-1}(-1)^{r+s+1}[(2r)!\pi^{2s+1}/(2s+1)!k^{2r-2s}]$	
4.	$\sin k\pi\sum_{s=0}^{r-1}(-1)^{r+s+1}[\pi^{2s}(2r-1)!/(2s)!k^{2r-2s}]$	x^{2r-1}
	$+\cos k\pi\sum_{s=0}^{r-1}(-1)^{s+r}[\pi^{2r+1}(2r-1)!/(2s+1)!k^{2r-2s-1}]$	
5.	$(k^2-a^2)^{-1}(a\cos a\pi\sin k\pi-k\sin a\pi\cos k\pi),\ a\neq k$	$\sin ax$
6.	$(k^2-a^2)^{-1}(k-k\cos k\pi\cos a\pi-a\sin k\pi\sin a\pi),\ a\neq k$	$\cos ax$

B. Tables of C-transforms.

	$f_c(k)=C\{F(x)\}$	$F(x)$
1.	$k^{-1}\sin k\pi$	1
2.	$k^{-1}\pi\sin k\pi-(1-\cos k\pi)k^{-2}$	x
3.	$\sin k\pi\sum_{s=0}^{r}(-1)^{r+s}[\pi^{2s}(2r)!/(2s)!k^{2r-2s+1}]$	x^{2r}
	$+\cos k\pi\sum_{s=0}^{r-1}(-1)^{r+s+1}[\pi^{2s+1}(2r)!/(2s+1)!k^{2r-2s}]$	
4.	$[(-1)^r(2r-1)!/k^{2r}]+\cos k\pi\sum_{s=0}^{r-1}(-1)^{r+s+1}[(2r-1)!\pi^{2s}/(2s-2)!k^{2s-2r}]$	x^{2r-1}
	$+\sin k\pi\sum_{s=0}^{r-1}(-1)^{r+s+1}[(2r-1)!\pi^{2s+1}/(2s-1)!k^{2r-2s-1}]$	
5.	$(k^2-a^2)^{-1}(-a+k\sin k\pi\sin a\pi+a\cos k\pi\cos a\pi),\ a\neq k$	$\sin ax$
6.	$(k^2-a^2)^{-1}(k\cos a\pi\sin k\pi-a\sin a\pi\cos k\pi),\ a\neq k$	$\cos ax$

C. Table of S-transforms when $K=\{n-\frac{1}{2}\}$, $n=1, 2, \cdots$.

	$f_s(n-\frac{1}{2})=S\{F(x)\}$	$F(x)$
1.	$(n-\frac{1}{2})^{-1}$	1
2.	$(-1)^{n+1}(n-\frac{1}{2})^{-2}$	x
3.	$(n-\frac{1}{2})^{-3}$	$x(2\pi-x)/2$
4.	$(n-\frac{1}{2})^{-1}(1-(n-\frac{1}{2})^2)^{-1}$	$1-\cos x$
5.	$(n-\frac{1}{2})[(n-\frac{1}{2})^2-a^2]^{-1}$	$\cos a(\pi-x)/\cos a\pi$
6.	$(-1)^{n+1}a[a^2+(n-\frac{1}{2})^2]^{-1}$	$\operatorname{sh} ax/\cosh a\pi$

D. Tables of C-transforms when $K=\{n-\frac{1}{2}\}$, $n=1, 2, \cdots$.

	$f_c(n-\frac{1}{2})=C\{F(x)\}$	$F(x)$
1.	$(-1)^{n+1}(n-\frac{1}{2})^{-1}$	1
2.	$(-1)^{n+1}\pi(n-\frac{1}{2})^{-1}-(n-\frac{1}{2})^{-2}$	x
3.	$(n-\frac{1}{2})^{-2}$	$\pi-x$
4.	$(n-\frac{1}{2})^{-2}[1-(n-\frac{1}{2})^2]^{-1}$	$\pi-x-\sin x$
5.	$[(n-\frac{1}{2})^2-a^2]^{-1}$	$\sin a(\pi-x)/\cos a\pi$
6.	$(-1)^{n+1}(n-\frac{1}{2})[(n-\frac{1}{2})^2-a^2]^{-1}$	$\cos ax/\cos a\pi$
7.	$[a^2+(n-\frac{1}{2})^2]^{-1}$	$\operatorname{sh} a(\pi-x)/\cosh a\pi$
8.	$(-1)^{n+1}(n-\frac{1}{2})[a^2+(n-\frac{1}{2})^2]^{-1}$	$\cosh ax/\cosh a\pi$

E. Table of some special transformations.

$$S\{F(x)\} = \int_0^\pi F(x) \sin nx dx = f_s(n), \qquad n = 1, 2, \cdots \tag{33}$$

$$S^{-1}\{f_s(n)\} = (2/\pi) \sum_1^\infty f_s(n) \sin nx \quad \text{in} \quad (0, \pi)$$

$$S\{F^{(2r)}(x)\} = (-n^2)^r f_s(n) + \sum_{s=0}^{s=r-1} (-1)^s n^{2s+1} [F^{(2r-2s-2)}(0) - (-1)^n F^{(2r-2s-2)}(\pi)]$$

$$C\{F(x)\} = \int_0^\pi F(x) \cos nx dx = f_c(n), \qquad n = 0, 1, 2, \cdots \tag{34}$$

$$C^{-1}\{f_c(n)\} = (1/\pi) f_c(0) + (2/\pi) \sum_1^\infty f_c(n) \cos nx \quad \text{in} \quad (0, \pi)$$

$$C\{F^{(2r)}(x)\} = (-n^2)^r f_c(n) + \sum_{s=0}^{s=r-1} (-1)^s n^{2s} [(-1)^n F^{(2r-2s-1)}(\pi) - F^{(2r-2s-1)}(0)]$$

$$S\{F(x)\} = \int^\pi F(x) \sin (n - \tfrac{1}{2}) x dx = f_s(n - \tfrac{1}{2}), \qquad n = 1, 2, \cdots \tag{35}$$

$$S^{-1}\{f_s(n - \tfrac{1}{2})\} = (2/\pi) \sum_1^\infty f_s(n - \tfrac{1}{2}) \sin (n - \tfrac{1}{2}) x \quad \text{in} \quad (0, \pi)$$

$$S\{F^{(2r)}(x)\} = (-(n - \tfrac{1}{2})^2)^r f_s(n - \tfrac{1}{2})$$
$$+ \sum_{s=0}^{s=r-1} (-1)^s (n - \tfrac{1}{2})^{2s} [(-1)^{n+1} F^{(2r-2s-1)}(\pi) + (n - \tfrac{1}{2}) F^{(2r-2s-2)}(0)]$$

$$C\{F(x)\} = \int_0^\pi F(x) \cos (n - \tfrac{1}{2}) x dx = f_c(n - \tfrac{1}{2}), \quad n = 1, 2, \cdots \tag{36}$$

$$C^{-1}\{f_c(n - \tfrac{1}{2})\} = (2/\pi) \sum_1^\infty f_c(n - \tfrac{1}{2}) \cos (n - \tfrac{1}{2}) x \quad \text{in} \quad (0, \pi)$$

$$C\{F^{(2r)}(x)\} = (-(n - \tfrac{1}{2})^2)^r f_c(n - \tfrac{1}{2})$$
$$+ \sum_{s=0}^{s=r-1} (-1)^{s+1} (n - \tfrac{1}{2})^{2s} [F^{(2r-2s-1)}(0) + (n - \tfrac{1}{2})(-1)^n F^{(2r-2s-2)}(\pi)]$$

$$S\{F(x)\} = \int_0^\pi F(x) \sin k_n x dx = f_s(k_n), \qquad n = 1, 2, \cdots, \tag{37}$$

where k_n are the roots of $\tan k\pi = -(k/h)$, $k > 0$, $h \neq 0$

$$S^{-1}\{f_s(k_n)\} = \sum_1^\infty N(k_n) f_s(k_n) \sin k_n x \quad \text{in} \quad (0, \pi) \quad \text{and}$$

$$1/N(k_n) = (\pi/2) - (4k_n)^{-1} \sin 2\pi k_n$$

$$S\{F^{(2r)}(x)\} = (-k_n^2)^r f_s(k_n) + \sum_{s=0}^{s=r-1} (-k_n^2)^s [\sin k_n\pi [F^{(2r-2s-1)}(\pi) + hF^{(2r-2s-2)}(\pi)] + k_n F^{(2r-2s-2)}(0)]$$

$$\tilde{S}\{F(x)\} = \int_0^\pi F(x) \sin k_n(\pi - x) dx = \tilde{f}_s(k_n), \qquad n = 1, 2, \cdots \tag{38}$$

$$\tilde{S}^{-1}\{\ {}_s(k_n)\} = \sum_1^\infty N(k_n)\tilde{f}_s(k_n) \sin k_n(\pi - x), \quad \text{in} \quad (0, \pi),$$

$$k_n \quad \text{and} \quad N(k_n) \quad \text{as in (37)}$$

$$\tilde{S}\{F^{(2r)}(x)\} = (-k_n^2)^r f_s(k_n) + \sum_{s=0}^{s=r-1} (-1)^{s+1} k_n^{2s} [\sin k_n\pi [F^{(2r-2s-1)}(0) - hF^{(2r-2s-2)}(0)] - k_n F^{(2r-2s-2)}(\pi)]$$

$$C\{F(x)\} = \int_0^\pi F(x) \cos k_n x dx = f_c(k_n), \qquad n = 1, 2, \cdots, \tag{39}$$

where k_n are the roots of $\tan k\pi = (h/k)$, $k > 0$, $h \neq 0$

$$C^{-1}\{f_c(k_n)\} = \sum_1^\infty N(k_n) f_c(k_n) \cos k_n x \quad \text{in} \quad (0, \pi) \quad \text{and}$$

$$1/N(k_n) = (\pi/2) + (4k_n)^{-1} \sin 2k_n\pi$$

$$C\{F^{(2r)}(x)\} = (-k_n^2)^r f_c(k_n) + \sum_{s=0}^{s=r-1} (-k_n^2)^s [\cos k_n\pi [F^{(2r-2s-1)}(\pi) + hF^{(2r-2s-2)}(\pi)] - F^{(2r-2s-1)}(0)]$$

$$\tilde{C}\{F(x)\} = \int_0^\pi F(x) \cos k_n(\pi - x) dx = \tilde{f}_c(k_n), \qquad n = 1, 2, \cdots \tag{40}$$

$$\tilde{C}^{-1}\{f_c(k_n)\} = \sum_1^\infty N(k_n)\tilde{f}_c(k_n) \cos k_n(\pi - x), \quad \text{in} \quad (0, \pi)$$

$$k_n \quad \text{and} \quad N(k_n) \quad \text{as in (39)}$$

$$\tilde{C}\{F^{(2r)}(x)\} = (-k_n^2)^r \tilde{f}_c(k_n) + \sum_{s=0}^{s=r-1} (-1)^{s+1} k_n^{2s} [\cos k_n\pi [F^{(2r-2s-1)}(0) - hF^{(2r-2s-2)}(0)] - F^{(2r-2s-1)}(\pi)]$$

$$T\{F(x)\} = \int_0^\pi F(x)(a \sin k_n x - k_n b \cos k_n x) dx = f(k_n), \qquad n = 1, 2, \cdots \tag{41}$$

where k_n are the roots of $\tan k\pi = k(bc - ad)/(ac + k^2 bd)$,

$$k > 0, \qquad ad - bc \neq 0$$

$$T^{-1}\{f(k_n)\} = \sum_{1}^{\infty} N(k_n)f(k_n)(a \sin k_n x - k_n b \cos k_n x) \quad \text{in} \quad (0, \pi) \quad \text{and}$$

$$1/N(k_n) = \tfrac{1}{2}(\pi a^2 + \pi k_n^2 b^2 - ab) + [(k_n^2 b^2 - a^2)/4k_n] \sin 2\pi k_n + (ab/2) \cos 2\pi k_n$$

$$T\{F^{(2r)}(x)\} = (-k_n^2)^r f(k_n) + \sum_{s=0}^{s=r-1} (-k_n^2)^s \{ k_n [aF^{(2r-2s-2)}(0) + bF^{(2r-2s-1)}(0)]$$

$$- k_n^2 b(c \sin k_n\pi + k_n d \cos k_n\pi)^{-1}[cF^{(2r-2s-2)}(\pi) + dF^{(2r-2s-1)}(\pi)]\}.$$

Bibliography

1. H. K. Brown, *Resolution of temperature problems by use of finite Fourier transformations*, Bull. Amer. Math. Soc. **50**, 376–385 (1944).

2. H. K. Brown, *The resolution of boundary value problems by means of the finite Fourier transformation: general vibration of a string*, J. Appl. Phys. **14**, 609–618 (1943).

3. H. K. Brown, *General vibration of a hinged beam*, J. Appl. Phys. **15**, 410–413 (1944).

4. R. V. Churchill, *Fourier series and boundary value problems*, McGraw-Hill, New York, 1941.

5. R. V. Churchill, *Modern operational mathematics in engineering*, McGraw-Hill, New York, 1944.

6. Courant-Hilbert, *Methoden der mathematischen Physik*, Berlin, 1931.

7. S. R. de Groot, *Sur l'intégration de quelques problèmes aux limites régis par l'équation de Fourier dite "de la chaleur" au moyen de la méthode des transformations fonctionnelles simultanées*, Nederl. Akad. Wetensch., Proc. **45**, 643–649, 820–825 (1942).

8. G. Doetsch, *Integration von Differentialgleichungen vermittels der endlichen Fourier Transformation*, Math. Ann. **112**, 52–68 (1935).

9. G. Doetsch, *Theorie und Anwendung der Laplace Transformation*, Dover Publications, New York, 1943.

10. Frank-v. Mises, *Differentialgleichungen der Physik*, 2nd ed. New York, 1943.

11. S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen*, Warsaw, 1935.

12. H. Kniess, *Lösung von Randwertproblemen bei Systemen gewöhnlicher Differentialgleichungen vermittels der endlichen Fourier Transformation*, Math. Zeit. **44**, 266–291 (1938).

13. L. Koschmieder, *Die endliche Fouriersche Abbildung und ihr Nutzen bei Aufgaben der Wärmeleitung in Stäben und Platten*, Deutsche Math. **5**, 521–545 (1945).

14. M. Picone, *Appunti di analisi superiore*, Naples, 1940. pp. 790–793.

15. I. Roettinger, *An operational approach to the solution of boundary value problems by generalized Fourier series*, Abstract, Bull. Amer. Math. Soc. **51**, 67 (1945).

16. A. G. Strandhagen, *Use of sine transform for non-simply supported beams*, Quart. Appl. Math. **1**, 346–348 (1944).

17. C. J. Tranter, *On a problem in heat conduction*, Philos. Mag. (7) **35**, 102–105 (1944).

18. E. T. Whittaker and G. N. Watson, *A course of modern analysis*, Cambridge University Press, 1940.

19. A. Zygmund, *Trigonometrical series*, Warsaw, 1935.