

In general, the system (1) has non-trivial solutions for n values $\lambda_1, \lambda_2, \dots, \lambda_n$ of the parameter λ , which are called the *characteristic values* or *roots* of the system; these are the roots of the determinantal equation

$$\begin{vmatrix} a_{11} - \lambda & a_{21} & \cdots & a_{1n} \\ a_{12} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0. \quad (2)$$

Corresponding to any characteristic value λ_i there exists a solution $(x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(n)})$ which may be termed a *characteristic vector* of the system.

The vector $(x_i^{(1)}, \dots, x_i^{(n)})$ may be regarded as an n by 1 matrix. Such a matrix is called a *column vector*. Similarly, a matrix having but one row is called a *row vector* and will be written in the form $\{y^{(1)}, y^{(2)}, \dots, y^{(n)}\}$. In general, if V is any given column vector, we shall denote by V' the row vector having the same elements.

If $C = [c_{ij}]$ is an m by n matrix and $D = [d_{ij}]$ is an n by p matrix, their *product* CD is defined to be the m by p matrix E having elements of the form

$$e_{ij} = \sum_{k=1}^n c_{ik}d_{kj} \quad (i = 1, 2, \dots, m; j = 1, \dots, p).$$

Note that DC is not necessarily equal to CD . In particular, a row vector R and a column vector K (with the same number of elements) have two products, a *scalar product* RK and a *matrix product* KR .

The system (1) may now be written in the form

$$AX = \lambda X, \quad (3)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \cdot \\ x^{(n)} \end{bmatrix}. \quad (4)$$

By a natural extension of meaning, the number λ_i and vectors $X_i = (x_i^{(1)}, \dots, x_i^{(n)})$ satisfying (3) are called the *characteristic roots* and *characteristic column vectors* of the matrix A . (Note that A is necessarily square.) Also any row vector Y such that

$$YA = \lambda Y \quad (5)$$

for some value of λ is a *characteristic row vector* of A . Clearly the values of λ for which (5) is satisfied must be the same as before, for (2) must hold in both cases.

In the following we shall suppose *either that A has n distinct roots, or that to each root of multiplicity r there correspond r linearly independent characteristic vectors of each type* (i.e., A has linear elementary divisors), as is almost always the case in practice.* The sets of characteristic roots and corresponding vectors will be denoted by

* For a discussion of the case of non-linear elementary divisors, see the paper mentioned in Footnote 2.

$\lambda_1, \lambda_2, \dots, \lambda_n; X_1, X_2, \dots, X_n; Y_1, Y_2, \dots, Y_n$. If any root λ_i is simple, the vectors X_i and Y_i are uniquely determined except for a constant factor; if two roots are equal, any linear combination of two corresponding characteristic vectors is a characteristic vector. In the following we shall understand by λ_i, X_i, Y_i , ($i=1, 2, \dots, n$) the i th root and corresponding column and row vectors of some given matrix A of order n .

Since $(\lambda_j Y_j)X_i = Y_j A X_i = Y_j (\lambda_i X_i)$, we have $(\lambda_i - \lambda_j) Y_j X_i = 0$, and it follows that the characteristic column vector X_i is orthogonal to the characteristic row vector Y_j if $\lambda_i \neq \lambda_j$. We may clearly choose the vectors corresponding to any multiple roots in such a way that $Y_j X_i = 0$ whenever $i \neq j$, and we shall suppose this done in future.

In what follow, unless stated otherwise, we shall assume that *all the roots are distinct* and shall take λ_1 to be the root of *largest absolute value*.

2. Some important relations. Since the X_i 's are linearly independent, any column vector V of order n may be expressed in the form

$$V = a_1 X_1 + a_2 X_2 + \dots + a_n X_n, \quad (6)$$

where the a_i 's are constants. Multiplying (2) by A , we get

$$AV = a_1 A X_1 + \dots + a_n A X_n = a_1 \lambda_1 X_1 + \dots + a_n \lambda_n X_n, \quad (7)$$

and by induction

$$A^k V = a_1 \lambda_1^k X_1 + a_2 \lambda_2^k X_2 + \dots + a_n \lambda_n^k X_n \quad (8)$$

for any positive integer k . More generally, if $P(\lambda)$ is any polynomial in the variable λ , we have

$$P(A)V = a_1 P(\lambda_1) X_1 + a_2 P(\lambda_2) X_2 + \dots + a_n P(\lambda_n) X_n. \quad (9)$$

By setting $V = X_i$, we see that the matrix $P(A)$ has the characteristic vectors X_1, X_2, \dots, X_n , and the characteristic roots $P(\lambda_1), P(\lambda_2), \dots, P(\lambda_n)$. (The same relations, of course, hold for row vectors.)

A further useful relation may be derived from equation (2), viz.,

$$a_{11} + a_{22} + \dots + a_{nn} = \lambda_1 + \lambda_2 + \dots + \lambda_n, \quad (10)$$

i.e., the sum of the roots is equal to the sum of the elements of the principal diagonal.

3. Symmetric matrices. A square matrix is *symmetric* if it is unchanged by turning its rows into columns and vice versa. The characteristic row vectors of a symmetric matrix have the same elements as the corresponding column vectors ($Y_i = X_i'$ for $i=1, 2, \dots, n$); hence the characteristic vectors may be regarded as a *single mutually orthogonal set*.

It follows that all the roots of a symmetric matrix are *real*, for two conjugate complex roots would correspond to two characteristic vectors having conjugate complex elements, and such vectors could not be orthogonal.

II. THE METHOD OF ITERATION

We now take up the problem of finding the characteristic roots and vectors of a given matrix A . For the sake of simplicity, we shall confine ourselves at first to the case where A is symmetric.

4. Expansion of the determinant. The most obvious procedure for solution would be to expand the determinant in Eq. (2) into a polynomial and then solve the resulting algebraic equation for λ . This method is sometimes the most expeditious, especially if use is made of the techniques of expansion described by Wayland.⁴ Nevertheless it is necessary to consider other methods, for the following reasons.

(a) Solving (2) gives only the roots; if the characteristic vectors are desired, they must be found by a separate process.

(b) Finding the complex roots of an algebraic equation may be as difficult as finding those of a matrix directly.

(c) In case approximate values of the roots are known, the work of expanding the determinant will not be affected, though that of solving the algebraic equation may be. On the other hand, we shall see in Sec. IV that such information can be used to great advantage when the problem is solved by iteration.

(d) The methods to be discussed are better adapted to machine calculations.

We therefore take up direct methods based on the process of iteration.

5. Iteration. This method consists in multiplying a suitable vector V repeatedly by A . Representing V in the form (6), and considering equation (8), we see that if $a_1 \neq 0$ (i.e., if V is not orthogonal to X) the first term on the right will predominate more and more as k increases. In fact

$$A^k V = \lambda_1^k (a_1 X_1 + O(\lambda_2/\lambda_1)^k), \quad (11)$$

where λ_2 is the root of second highest absolute value. Similarly

$$A^{k+1} V = \lambda_1^{k+1} (a_1 X_1 + O(\lambda_2/\lambda_1)^{k+1}).$$

Thus the ratio of corresponding elements of $A^k V$ and $A^{k+1} V$ approaches λ_1 as $k \rightarrow \infty$. And if $A^k V$ is divided by one of its elements, the resulting vector tends to a multiple of X_1 (which, of course, is determined only up to a constant factor anyway). These points will be clarified by the following concrete case.

Example 3.

$$A = \begin{bmatrix} 6 & 1 & -1 & 3 \\ 1 & 4 & 0 & -2 \\ -1 & 0 & 1 & 5 \\ 3 & -2 & 5 & 2 \end{bmatrix} \begin{array}{l} 9 \\ 3 \\ 15 \\ 8 \end{array}.$$

(The extra column on the right is the sum of the others; its purpose will be explained in the following.)

Choosing

$$V = (4, -1, 2, 4),$$

(more will be said later about the method of choice), we obtain

$$AV = (33, -8, 18, 32),$$

$$A^2 V = (268, -63, 145, 269),$$

$$A^3 V = (2207, -522, 1222, 2193),$$

$$A^4 V = (18077, -4267, 9980, 18161).$$

Successive ratios of the leading elements are

$$8.25, 8.12, 8.235, 8.191.$$

(The true value of the root is 8.22557331.) The vectors obtained by dividing the foregoing by their leading elements are (with the true vector for comparison)

$$(1.000, -.242, .545, .970),$$

$$(1.000, -.235, .541, 1.004),$$

$$(1.000, -.237, .554, .994),$$

$$(1.000, -.235, .552, 1.005);$$

$$X_1 = (1.00000000, -.24073464, .55955487, 1.00862094).$$

6. Use of powers of the matrix. It will be noticed that the convergence is not very rapid, and the multiplication might have to be repeated many times if λ_1 or the components of X_1 were wanted to many figures. The process may be shortened by first squaring A repeatedly:

$$A^2 = \begin{bmatrix} 47 & 4 & 8 & 17 \\ 14 & 21 & -11 & -9 \\ 8 & -11 & 27 & 12 \\ 17 & -9 & 12 & 42 \end{bmatrix} \begin{vmatrix} 76 \\ 5 \\ 36 \\ 62 \end{vmatrix},$$

$$A^4 = \begin{bmatrix} 2578 & 31 & 752 & 1573 \\ 31 & 659 & -604 & -631 \\ 752 & -604 & 1058 & 1063 \\ 1573 & -631 & 1063 & 2278 \end{bmatrix} \begin{vmatrix} 4934 \\ -545 \\ 2269 \\ 4283 \end{vmatrix}$$

$$A^8 = \dots$$

(The powers of a symmetric matrix are of course symmetric also.)

The *sum columns* are useful as checks at this point, for the sum column of A^2 should be the same as the result of operating with A on its own sum column, and similarly for the higher powers. That this is so can be seen by noting that the sum column of A is equal to AH , where $H = (1, 1, 1, 1)$. Then $A^2H = A(AH)$, $A^4H = A^2(A^2H)$, etc.

We can now multiply V directly by some higher power of A and thus save many steps in the iteration. For example

$$A^4(A^4V) = A^8V = (82542442, -19739077, 46035155, 83107096)$$

$$A(A^8V) = A^9V = (678801708, -162628058, 379028193, 683495447)$$

$$A(A^9V) = A^{10}V = (5581640338, -1338701418, 311703720, 5623793099)$$

These are proportional to

$$(1.000, -.2391, -.5577, 1.0068), \quad (1.000, -.2396, .5584, 1.0069),$$

$$(1.0000, -.2398, .5586, 1.0076),$$

and the ratios of the leading elements are

$$8.224, \quad 8.224.$$

Note, however, that forming each power of A requires about as much computation as n vectors $A^k V$, so this procedure should not be carried too far.

For example, to compute $A^{16}V$ by operating directly with A on V sixteen times requires $16n$ multiplications; forming A^2 first and then operating eight times on V requires $\frac{1}{2}n(n+1)+8n$ multiplications. Similarly, using A^4 requires $n(n+1)+4n$; using A^8 , $\frac{3}{2}n(n+1)+2n$; and using A^{16} , $2n(n+1)+n$. For $n=4$, it will be seen that working with A^4 is the most economical.

7. Finding the smaller roots. So far we have obtained only λ_1 . What about the other roots?

Clearly these can be found if we can determine the products $\lambda_1\lambda_2$, $\lambda_1\lambda_2\lambda_3$, etc. We shall now see how to modify the iteration process in order to do this.

Suppose we have two sequences having k th terms of the form

$$\begin{aligned}\beta_k &\equiv \tilde{b}_1\lambda_1^k + b_2\lambda_2^k + \cdots + b_n\lambda_n^k \\ \beta'_k &\equiv b'_1\lambda_1^k + b'_2\lambda_2^k + \cdots + b'_n\lambda_n^k \quad (k = 0, 1, 2, \dots).\end{aligned}\quad (12)$$

Then we define

$$\begin{aligned}\beta_{2,k} &= \begin{vmatrix} \beta_k & \beta_{k+1} \\ \beta'_k & \beta'_{k+1} \end{vmatrix} \\ &= \lambda_1^k\lambda_2^k(b_2b'_1 - b_1b'_2)(\lambda_1 - \lambda_2) + \lambda_1^k\lambda_3^k(b_3b'_1 - b_1b'_3)(\lambda_1 - \lambda_3) + \cdots,\end{aligned}\quad (13)$$

whence

$$\beta_{2,k+1}/\beta_{2,k} = \lambda_1\lambda_2 + O(\lambda_3/\lambda_2)^k \quad (k = 0, 1, 2, \dots).\quad (14)$$

To be concrete, if we have a sequence of vectors of the type $V, AV, A^2V, \dots, A^kV, \dots$ we could write $\beta_k = (A^kV)^{(1)}$, $\beta'_k = (A^kV)^{(2)}$, where $(A^kV)^{(i)}$ is the i th element of A^kV ($i=1, \dots, n$), so that

$$\beta_{2,k} = \begin{vmatrix} (A^kV)^{(1)} & (A^{k+1}V)^{(1)} \\ (A^kV)^{(2)} & (A^{k+1}V)^{(2)} \end{vmatrix}$$

(of course, any other pair of elements would work equally well). Consider the following case.

Example 2.

$$A = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & -3 & 1 \\ 2 & -3 & -3 & 4 \\ -1 & 1 & 4 & -2 \end{bmatrix}.$$

It is found that $\lambda_1 = -8.075320860$. Operating with A upon the vector V below yields

$$\begin{aligned}
V &= (5, -9, -2, 8), \\
AV &= (-12, 5, 75, -38), \\
A^2V &= (188, -258, -417, 393), \\
A^3V &= (-225, 1383, 3970, -2896), \\
A^4V &= (10836, -13423, -30093, 24280), \\
A^5V &= (-84466, 101136, 249340, -193191).
\end{aligned}$$

Then

$$\begin{vmatrix} -2 & 75 \\ 8 & -38 \end{vmatrix} = -524, \quad \begin{vmatrix} 75 & -416 \\ -38 & 393 \end{vmatrix} = 13667, \text{ etc.}$$

Values of successive determinants and their ratios are

$$\begin{array}{cccccc}
-524 & 13667 & -355474 & 9242272 & -240278437 \\
-26.08 & -26.01 & -25.9985 & -25.9978 &
\end{array}$$

Thus $\lambda_1\lambda_2 = -26.00$ and $\lambda_2 = 3.219$.

On the other hand, we may take $\beta'_k = \beta_{k+1}$, in which case

$$\beta_{2,k} = \begin{vmatrix} \beta_k & \beta_{k+1} \\ \beta_{k+1} & \beta_{k+2} \end{vmatrix}. \quad (15)$$

That is, we can set up a sequence converging to $\lambda_1\lambda_2$ given any single sequence of the form (12)

If we wish to find λ_3 , we select three sequences of the form (12), set up the corresponding three-rowed determinant $\beta_{3,k}$, and get as before

$$\beta_{3,k+1}/\beta_{3,k} = \lambda_1\lambda_2\lambda_3 + O(\lambda_4/\lambda_3)^k. \quad (16)$$

Note however, that the characteristic vectors X_2, X_3, \dots, X_n cannot be found by this means.

III. DEVICES FOR IMPROVING CONVERGENCE

If we employ the method of iteration in the crude form, described above, we find that the convergence often leaves much to be desired in terms of time consumed. In this section we consider various ways of improving the efficiency of the method.

8. Scalar products. Consider the scalar product (see Sec. I, 1) $(V'A^k) \cdot (A^lV)$, $(k, l = 0, 1, 2, 3, \dots)$.

This may be written (in view of (6) and the orthogonality of X_i 's)

$$V'A^kV^lV = V'A^{k+l}V = a_1\lambda_1^{k+l}X'_1X_1 + a_2\lambda_2^{k+l}X'_2X_2 + \dots + a_n\lambda_n^{k+l}X'_nX_n. \quad (17)$$

In particular

$$V'A^{k-1} \cdot A^kV = a_1\lambda_1^{2k-1}X'_1X_1 + a_2\lambda_2^{2k-1}X'_2X_2 + \dots + a_n\lambda_n^{2k-1}X'_nX_n,$$

and

$$V'A^k \cdot A^kV = a_1\lambda_1^{2k}X'_1X_1 + a_2\lambda_2^{2k}X'_2X_2 + \dots + a_n\lambda_n^{2k}X'_nX_n,$$

so that

$$\frac{V'A^k \cdot A^k V}{V'A^{k-1} \cdot A^k V} = \lambda_1 + O(\lambda_2/\lambda_1)^{2k-1}, \quad (k = 1, 2, 3, \dots). \quad (18)$$

It follows from (18) that by taking suitable scalar products the number of steps required to find λ_1 to a given degree of accuracy is *approximately halved*.

To illustrate this method, let us apply it to the vector sequence of Example 1. Working with the last two vectors, we get

$$\begin{aligned} V'A^{18}V &= 1098048041, & V'A^{19}V/V'A^{18}V &= 8.2255708, \\ V'A^{19}V &= 9032871992, & V'A^{20}V/V'A^{19}V &= 8.2255717. \\ V'A^{20}V &= 74293955655, \end{aligned}$$

(The last figures of the scalar products have been omitted.) The improvement in the approximation is evident.

The method can also be applied to finding the products of the roots, by making proper use of (15).

9. Choice of initial vector. So far nothing has been said about the choice of the initial vector V . Clearly, the best V is one that is already "near" to X_1 , i.e., one having a_1 (equation (6)) large in comparison with the other a 's. A few suggestions may be helpful. The sum columns of A and even more of A^2, A^3 , etc., may be good indications, though sometimes they are misleading, as when H happens to be nearly orthogonal to X_1 ; in this case we may be led astray to one of the other vectors. The other columns of A , especially those having the largest elements, are also good first guesses. In any case, the first V 's should have small integers for elements and should be tested by trial multiplications by A .

10. Reduction of leading element to unity. Once V has been selected and the iteration is under way, it is clear (since only the *ratios* of the vector elements are significant) that the approximation to X_1 is not affected if each vector $A^k V$ is divided through by one of its elements—say the first if this is not zero. Then it will be clear from the extent of agreement of successive vectors how far the approximation has progressed, and time can be saved by retaining in the elements only as many digits as appear to be correct. (The choice of V is actually slightly altered at each step, but this is of no importance.) The following example shows the procedure.

Example 3.

$$A = \begin{bmatrix} -2 & -2 & 0 & 3 & -1 \\ -2 & 0 & -3 & 5 & 0 \\ 0 & -3 & -5 & 11 & 1 \\ 3 & 5 & 1 & -3 & -1 \\ -1 & 0 & 1 & -1 & -1 \end{bmatrix}.$$

Casual inspection suggests the vector

$$V = (1, 1, 1, -1, 0)$$

as a first approximation of X_1 . The vector AV resembles V sufficiently to indicate

that we are on the right track. Operating repeatedly with A and dividing through by the leading element each time yields the sequence below:

$$\begin{aligned} AV &= (-7 \ -10, -9, 12, 1)(69), \\ &\quad (1.00, 1.46, 1.28, -1.70, -.22)(-9.80), \\ &\quad (1.000, 1.463, 1.296, -1.724, -.224)(-9.874), \\ &\quad (1.000, 1.469, 1.298, -1.722, -.227)(-9.877), \\ &\quad (1.000, 1.468, 1.301, -1.725, -.227)(-9.884); \\ X_1 &= (1.000000, 1.469801, 1.302061, -1.724997, -.228106). \end{aligned}$$

The figure in parentheses after each vector is the number by which the elements of the next vector have to be divided to reduce its leading element to unity. These numbers converge, of course, to λ_1 . Note that while computing errors may slow up the convergence, they will not invalidate the final result.

To obtain a more accurate value of λ_1 , we should operate with A on the last vector in the sequence and take scalar products as before. The number of correct significant figures thus obtained should be about double that of the last number in parentheses.

11. Aitken's δ^2 process. Since the rapidity of convergence in all the above processes depends essentially on the ratio $|\lambda_2/\lambda_1|$, where λ_2 is the root of second largest absolute value, the convergence is slow if this ratio is near 1. In such cases the convergence may be speeded up by using the δ^2 process, due to Aitken, which we now explain.

Our approximating sequences are all of the type

$$\phi_k = \lambda_1 + c_1 q_1^k + c_2 q_2^k + \cdots \quad (k = 0, 1, 2, \cdots; 1 > |q_1| \geq |q_2| \geq \cdots) \quad (19)$$

where the q 's are of the form λ_2/λ_1 , λ_3/λ_1 , $(\lambda_2/\lambda_1)^2$, etc.

Writing

$$\psi_{2,k} = \begin{vmatrix} \phi_k & \phi_{k+1} \\ \phi_{k+1} & \phi_{k+2} \end{vmatrix}, \quad (k = 0, 1, 2, \cdots), \quad (20)$$

we readily see that

$$\psi_{2,k} = \lambda_1 [c_1(1 - q_1)^2 q_1^k + c_2(1 - q_2)^2 q_2^k + \cdots] + c_1 c_2 q_1^k q_2^k (q_1 - q_2)^2 + \cdots.$$

On the other hand

$$\delta^2 \phi_{k+1} \equiv \phi_{k+2} - 2\phi_{k+1} + \phi_k = c_1(1 - q_1)^2 q_1^k + c_2(1 - q_2)^2 q_2^k + \cdots.$$

Defining

$$\phi_{2,k} = \psi_{2,k} / \delta^2 \phi_{k+1} \quad (k = 0, 1, 2, \cdots), \quad (21)$$

we see that

$$\phi_{2,k} = \lambda_1 + O(q_2^k). \quad (22)$$

Thus, if $|q_2| < |q_1|$, the $\phi_{2,k}$'s will form a sequence converging to λ_1 more rapidly than the sequence of ϕ_k 's. Applying the same process to the $\phi_{2,k}$'s will then result in a still more rapidly convergent sequence, and so on.

However, the remainder term in the sequence obtained by applying the δ^2 process twice is of the form $O[(q_2^2/q_1)^k] + O(q_3^k)$. Thus the improvement in convergence will be slight if $|q_1|$ and $|q_2|$ are nearly equal, especially if $q_2 \sim -q_1$.

12. Extension of the δ^2 process. For this case, we shall develop a modified procedure which proves more efficacious.

We define a *second derived sequence*

$$\phi_{3,k} = \psi_{3,k}/\chi_{3,k} \quad (k = 0, 1, 2, \dots), \quad (23)$$

where

$$\psi_{3,k} = \begin{vmatrix} \phi_k & \phi_{k+1} & \phi_{k+2} \\ \phi_{k+1} & \phi_{k+2} & \phi_{k+3} \\ \phi_{k+2} & \phi_{k+3} & \phi_{k+4} \end{vmatrix} = \begin{vmatrix} \psi_{2,k+1} & \psi_{2,k+2} \\ \psi_{2,k+2} & \psi_{2,k+3} \end{vmatrix} / \phi_{k+2} \quad (24)$$

and

$$\chi_{3,k} = \begin{vmatrix} \delta^2\phi_{k+1} & \delta^2\phi_{k+2} \\ \delta^2\phi_{k+2} & \delta^2\phi_{k+3} \end{vmatrix}. \quad (25)$$

It can be verified that

$$\phi_{3,k} = \lambda_1 + O(q_3^k). \quad (26)$$

It should be noted that if the elements ϕ_k of the original sequence are replaced by $\phi_k - c$, where c is a constant, all the sequences will converge to $\lambda_1 - c$. Thus if the first few digits of the ϕ_k 's are the same for all k , we may simply ignore these digits in computing the derived sequences.

Example 4.

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 0 & 0 & -2 \\ -1 & 0 & -3 & 5 \\ 3 & -2 & 5 & -2 \end{bmatrix} \quad V = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix}.$$

Carrying out the foregoing procedure, we form the following table:

$V'A^kV$	ϕ_k	$\delta^2\phi_k$	$\psi_{2,k-1}$	$\phi_{2,k-1}$	$\phi_{3,k-2}$
20	-8.550000000			-8.59	
-171	-8.602339181	.066643012	-.006071541	1105433	-8.59
1471	-8.588035350	-.018447694	.001683322	1248386	120357
-12633	-8.592179213	.005452930	-.000497222	1184409	120331
108545	-8.590870146	-.001769071	.000161358	1210537	
-932496	-8.591330150				
8011381					

(In the above, $\phi_k - 8.5$, rather than ϕ_k , is used in computing the later columns.)

Further derived sequences, showing still better convergence, could be defined in a similar fashion. However, the extra computation required and the rapid accumulation of rounding-off errors make such a procedure of doubtful value.

The same process can be used in finding the elements of X_1 (see the paper quoted in Footnote 2).

IV. FINDING THE REMAINING ROOTS AND VECTORS

In Secs. II and III we saw how all the *roots* of A could be obtained, though with limited accuracy for the smaller roots in practice. Now we consider methods of getting both roots and vectors at once, at the same time improving the accuracy of the former.

13. λ_1 -differencing. Perhaps the simplest method is that of λ_1 -differencing the sequence (8). That is, we form the differences

$$A^{k+1}V - \lambda_1 A^k V = a_2(\lambda_2 - \lambda_1)\lambda_2^k X_2 + a_3(\lambda_3 - \lambda_1)\lambda_3^k X_3 + \cdots + a_n(\lambda_n - \lambda_1)\lambda_n^k X_n. \quad (27)$$

If λ_2 is larger numerically than the other roots and $a_2 \neq 0$ the first term predominates, and we find X_2 and λ_2 much as we originally found X_1 and λ_1 .

The disadvantage of this method is that nearly equal numbers have to be subtracted at each stage with a resulting loss of significant figures. The better the convergence of the original sequence is, the worse this becomes. Needless to say, the difficulty is increased if we try to find X_3 , X_4 , etc., by repeating the procedure.

14. Deflation.^{1,2} We may also proceed by *deflation*, which consists in replacing the matrix A by a matrix A_1 having the same characteristic roots and vectors as A , except that λ_1 is replaced by zero. Applying the method of iteration to A_1 will then yield λ_2 and X_2 . Deflating again give λ_3 and X_3 , and so on.

We set up the matrix A_1 as follows:

$$A_1 = A - \frac{\lambda_1}{X_1' X_1} X_1 X_1'. \quad (28)$$

Note that $X_1' X_1$ is the *scalar* product (sum of squares) while $X_1 X_1'$ is the *matrix* product

$$\begin{bmatrix} \begin{matrix} (1)2 \\ x_1 \end{matrix} & \begin{matrix} (1) (2) \\ x_1 x_1 \end{matrix} & \cdots & \begin{matrix} (1) (n) \\ x_1 x_1 \end{matrix} \\ \begin{matrix} (2) (1) \\ x_1 x_1 \end{matrix} & \begin{matrix} (2)2 \\ x_1 \end{matrix} & \cdots & \begin{matrix} (2) (n) \\ x_1 x_1 \end{matrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{matrix} (n) (1) \\ x_1 x_1 \end{matrix} & \begin{matrix} (n) (2) \\ x_1 x_1 \end{matrix} & \cdots & \begin{matrix} (n)2 \\ x_1 \end{matrix} \end{bmatrix}.$$

That A_1 has the desired properties can be seen by substitution:

$$A_1 X_1 = A X_1 - \frac{\lambda_1}{X_1' X_1} X_1 X_1' X_1 = \lambda_1 X_1 - \lambda_1 X_1 = 0,$$

$$A_1 X_i = A X_i - \frac{\lambda_1}{X_1' X_1} X_1 X_1' X_i = \lambda_i X_i \quad (i \neq 1).$$

Let us apply this procedure to Example 1:

$$X_1' X_1 = 2.38837102, \quad \lambda_1 / X_1' X_1 = 3.44400985$$

$$A_1 = \begin{bmatrix} 2.55599015 & 1.82909247 & -2.92711248 & - .47370045 \\ 1.82909247 & 3.71445974 & .46392273 & -1.16375997 \\ -2.92711248 & .46392273 & - .07832518 & 3.05627400 \\ - .47370045 & -1.16375997 & 3.05627400 & -1.50364702 \end{bmatrix} \begin{bmatrix} .98426969 \\ 4.84371497 \\ .51475907 \\ -.08483344 \end{bmatrix}.$$

Starting with the vector V_1 below and proceeding as in Sec. III, 10, we get in succession the vectors

$$\begin{aligned} V_1 &= (2, 2, -1, -1)(12.216), \\ &\quad (1.00, 0.96, -0.65, -0.40)(6.4040), \\ &\quad (1.00, 0.87, -0.57, -0.46)(6.0337), \\ &\quad (1.00, 0.88, -0.64, -0.42)(6.2379), \\ &\quad (1.00, 0.85, -0.60, -0.45)(6.0802) \end{aligned}$$

whereas the vector and root to four places are

$$X_2 = (1.0000, 0.8681, -0.6198, -0.4404), \lambda_2 = 0.16666.$$

We see that the difficulty of losing significant figures by subtraction is largely avoided, while the convergence toward X_2 is fairly rapid. On the other hand, the extra figures in the elements of A_1 make the further computations somewhat laborious, and in any case the accuracy attainable in X_2 is limited by that of λ_1 and X_1 .

Some saving of time may be affected by noting that

$$A_1^k = A^k - \frac{\lambda_1^k}{X_1' X_1} X_1 X_1',$$

whence

$$A_1^k V = A^k V = \frac{\lambda_1^k}{X_1' X_1} X_1 X_1' V,$$

i.e., the vector $A_1^k V$ can be obtained from $A^k V$ without setting up the matrix A_1 . But this again brings a loss of significant figures; thus this device is of limited applicability.

15. Use of matrix polynomials. In view of what has been said above, it would seem that there is room for a different approach. Now, upon reviewing our work up to this point, it will be seen that we have tacitly assumed that we had no prior notion of the location of the roots of our matrix A , and we have indicated no way of using such information if we had it. However, in a practical case growing out of, say, a vibration problem, we could usually make at least a guess at where the roots lie. In any event, we could get an approximation to the largest root by a couple of steps of iteration, and we shall see shortly how the other roots might be estimated.

Suppose, then, that we have approximate values $\lambda_1', \lambda_2', \dots, \lambda_n'$ of the n roots $\lambda_1, \lambda_2, \dots, \lambda_n$. Now consider the polynomial.

$$P_1(\lambda) \equiv (\lambda - \lambda_2')(\lambda - \lambda_3') \cdots (\lambda - \lambda_n') \quad (28)$$

and the corresponding *matrix*

$$P(A) = (A - \lambda_2')(A - \lambda_3') \cdots (A - \lambda_n'). \quad (29)$$

It follows from equation (9) that $P(A)$ has the characteristic roots $P_1(\lambda_1), P_1(\lambda_2), \dots, P_1(\lambda_n)$. But by definition

$$P_1(\lambda_2') = P_1(\lambda_3') = \cdots = P_1(\lambda_n') = 0,$$

and since $P(\lambda)$ is continuous one would expect $P_1(\lambda_2)$, $P_1(\lambda_3)$, \dots , $P_1(\lambda_n)$ to be small, how small depending on how good the approximations were. On the other hand, the roots being supposed distinct, $P_1(\lambda_1)$ would not in general be near zero, and therefore

$$|P_1(\lambda_1)| \gg |P_1(\lambda_i)|, \quad i \neq 1.$$

If we now choose a vector V of the form (6), with $a_1 \neq 0$, we have by (9)

$$P_1(A) \cdot V = a_1 P_1(\lambda_1) X_1 + a_2 P_1(\lambda_2) X_2 + \dots + a_n P_1(\lambda_n) X_n. \quad (30)$$

The first term on the right will predominate strongly, so that if we proceed by iteration, i.e., multiply repeatedly by $P_1(A)$, *the convergence to X_1 will be very rapid.* (Of course it is all the better if V is already an approximation to X_1 .)

Once X_1 is obtained, λ_1 can be determined to the same degree of accuracy by multiplying X_1 by A , and to greater accuracy by forming scalar products.

Note that nothing is really altered if λ_1 is replaced by one of the other roots. For example, to find λ_2 and X_2 we could use

$$P_2(A) \equiv (A - \lambda'_1)(A - \lambda'_3)(A - \lambda'_4) \dots (A - \lambda'_n).$$

Example 5. Suppose that for a given fourth-order matrix we have

$$\lambda_1 \sim 6.1 = \lambda'_1, \quad \lambda_2 \sim -4.3 = \lambda'_2, \quad \lambda_3 \sim -2.6 = \lambda'_3, \quad \lambda_4 \sim 0.4 = \lambda'_4.$$

Then

$$P_1(\lambda) = (\lambda + 4.3)(\lambda - 2.6)(\lambda - 0.4) = \lambda^3 + 1.3\lambda^2 - 11.9\lambda + 4.5,$$

where the coefficients have been rounded off to one decimal place, as there is clearly no advantage in retaining more. In fact, we should in practice make a further simplification, and replace $P_1(\lambda)$ as written above by a polynomial like $\lambda^3 + \lambda^2 - 12\lambda + 4$ or $2\lambda^3 + 3\lambda^2 - 24\lambda + 9$, whose coefficients are *small integers*. In this way, the subsequent computations would be simplified without slowing down the convergence much. Where to strike the balance in practice is largely a matter of experience and "feel" on the part of the computer.

16. Application of orthogonality. Let us return to Sec. I, 2, but suppose that V , instead of being a completely arbitrary vector, is *orthogonal to X_1* , i.e., $a_1 = 0$. (We shall see later how to select such a V .) Then

$$\begin{aligned} A^k V &= a_2 \lambda_2^k X_2 + a_3 \lambda_3^k X_3 + \dots + a_n \lambda_n^k X_n, \\ P(A)V &= a_2 P(\lambda_2) X_2 + a_3 P(\lambda_3) X_3 + \dots + a_n P(\lambda_n) X_n. \end{aligned} \quad (31)$$

To see the application of this, consider Example 5 again and suppose that X_1 has been found to the desired accuracy. From what has been said before, our first step in getting X_2 would be to set up the polynomial

$$P_2(\lambda) = (\lambda - 6.1)(\lambda - 2.6)(\lambda - 0.4).$$

However, if we are to work with a vector V orthogonal to X_1 , we might just as well use the simpler polynomial

$$Q_2(\lambda) = (\lambda - 2.6)(\lambda - 0.4).$$

The fact that $Q_2(\lambda_1)$ is not small compared with $Q_2(\lambda_2)$ has no effect on the iteration, for

$$Q_2(A) \cdot V = a_2 Q_2(\lambda_2) X_2 + a_3 Q_2(\lambda_3) X_3 + a_4 Q_2(\lambda_4) X_4.$$

How can we select such a vector V ? Consider any vector $V_0 = (v_0^{(1)}, v_0^{(2)}, \dots, v_0^{(n)})$. Then *the vector*

$$\left(v_0^{(1)}, v_0^{(2)}, \dots, v_0^{(n-1)} - \frac{v_0^{(1)} x_1^{(1)} + v_0^{(2)} x_1^{(2)} + \dots + v_0^{(n-1)} x_1^{(n-1)}}{x_1^{(n)}} \right) \quad (32)$$

will be orthogonal to X_1 and can serve as V (provided $x_1^{(n)} \neq 0$).

Note that we have altered only the last element $v_0^{(n)}$ of V_0 . We could, of course, have altered any other element $v_0^{(i)}$ instead, provided $x_1^{(i)} \neq 0$. In general it is best to alter an element corresponding to one of the larger elements of X_1 so that V may differ as little as possible from V_0 .

In order to avoid the accumulation of rounding-off errors, it is generally desirable to repeat the orthogonalization at each step of the iteration. Also, it is well to determine the elements of X_1 to one more significant figure than is required in those of the other vectors. (Of course we could use any of the vectors X_i as a starting point.)

17. Obtaining the first approximations. It is well to give some attention to the question of getting rough values of the roots when these are not available beforehand. Since the procedure is rough and ready rather than fixed, it is best shown by an example. For this purpose we return to the matrix A of Example 3.

We have already seen that

$$\lambda_1 = -9.886, X_1 = (1.000, 1.470, 1.302, -1.725, -0.228).$$

Looking back, we see that one or two steps in the iteration would have sufficed to give us one-place accuracy, which is all we need at this stage.

Referring to equation (10), we see that

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = -2 + 0 - 5 - 3 - 1 = -11, \quad (33)$$

so that A must certainly have a root λ_2 greater than -1 . It follows that the matrix $A_1 = A + 6$ will have a root $\lambda_2 + 6 > 5$, which is greater in absolute value than its root $\lambda_1 + 6$. Trying out simple vectors like $(1, 0, 0, 0, 0)$, $(0, 1, 0, 0, 0)$, etc., we see that the combination

$$V_1 = (0, 1, 0, 1, 0)$$

looks hopeful. In fact

$$A_1 V_1 = (1, 11, -2, 8, -1), \quad A_1^2 V_1 = (7, 110, -28, 81, -16),$$

and by taking scalar products we get $\lambda_2 + 6 \sim 10.2$, $\lambda_2 \sim 4.2$, while $A_1^2 V_1$ will stand as a first approximation to X_2 .

To proceed further, we note that $\lambda_1 + 3 \sim -6.9$, $\lambda_2 + 3 \sim 7.2$. Therefore the matrix $(A + 3)^2 - 50$ will have two roots near 0 and three between -50 and 0. This matrix could be used to obtain another root λ_3 of A . However, it probably has two or three roots near -50 , which would be difficult to separate, so we used instead the matrix $A_2 = (A^2 + 6A + 9) - 35 = A(A + 6) - 26$.

After a few trials we decide on the vector $V_2 = (2, 0, 0, 1, 4)$. Then $A_2 V_2 = (-70, -4, -6, -27, -128)$, and taking scalar products yields

$$(\lambda_3 + 3)^2 - 35 \sim -32.5, \quad (\lambda_3 + 3)^2 \sim 2.5, \quad \lambda_3 + 3 \sim \pm 1.6,$$

so that $\lambda_3 \sim -1.4$ or -4.6 . Since $A V_2 = (-5, 1, 5, -1, -7)$ and $A_1 V_2 = (7, 1, 5, 5, 17)$, the value $\lambda_3 \sim -1.4$ must be the correct one.

To get approximations to λ_4 and λ_5 , we take the sum of the principal diagonal terms in A_2 :

$$\sum_{i=1}^5 (\lambda_i^2 + 6\lambda_i - 26) = -55,$$

so that, from (33)

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_5^2 = 141. \quad (34)$$

Combining (33) and (34) with the known values of λ_1 , λ_2 , and λ_3 , we get $\lambda_4 \sim 0.9$, $\lambda_5 \sim -4.8$.

18. Examples. Two more examples show how to carry through the process.

Example 6. Find to six decimal places the roots of

$$A = \begin{bmatrix} 2 & 2 & 0 & 4 \\ 2 & -1 & -1 & 3 \\ 0 & -1 & 0 & -2 \\ 4 & 3 & 2 & 0 \end{bmatrix} \begin{vmatrix} 8 \\ 3 \\ -3 \\ 5 \end{vmatrix}.$$

Using the methods of the last example, we conclude that $\lambda_1 \sim 7.06$, $\lambda_2 \sim -4.19$, $\lambda_3 \sim 0.45$, $\lambda_4 \sim -2.32$, $X_1 \sim (106, 67, -38, 100)$, $X_2 \sim (53, 53, -41, -106) = V_2$. We next compute X_1 to greater accuracy in order to obtain λ_1 to six places. Since we want λ_1 to seven significant figures, four figures in the elements of X_1 should be sufficient, as the formation of scalar products about doubles the number of significant figures, as explained above. In order to avoid rounding-off errors, both here and in the later calculations, we carry an extra figure.

If we were to follow rigidly the procedure outline above (paragraph 18), we should begin by writing the polynomial $(\lambda + 4.19)(\lambda - 0.45)(\lambda + 2.32)$. However, we note that the simple *linear* matrix polynomial $A_2 = A_1 + 2$ has very favorable root ratios, and since the number of figures required is not great anyway, it seems worthwhile to go ahead with this matrix rather than taking the time to set up a more complicated expression. Applying A_2 to our first approximation and dividing by the leading element at each step yields successively

$$\begin{aligned} &(106, 67, -38, 100)(958), \\ &(1.000, .644, -.358, .941)(9.052), \\ &(1.0000, .6435, -.3582, .9423)(9.0562), \\ &(1.00000, .64360, -.35826, .96206)(9.05544), \\ &(1.00000, .64360, -.35826, .94213)(9.05572), \\ &(9.05572, 5.82825, -3.24438, 8.53158). \end{aligned}$$

The last vector was obtained by applying A_2 once more without division. Taking scalar products of the last two vectors, we get

$$\begin{array}{rcl} & 2.4301801245 & \\ & 22.0069707442 & 9.055695307 \\ & 199.2884216617 & 9.055695307 \end{array}$$

We conclude that $\lambda_1 = 7.055695307$, to greater accuracy than was required.

We could now find λ_2 in the same manner by starting with $A - 2$, say. However the convergence would be rather slow because the ratios of the roots would be less favorable than before. Instead, we proceed as in Example 5. Since

$$(\lambda + 2.32)(\lambda - .45) = \lambda^2 + 1.87\lambda - 1.04,$$

we set up the matrix $A_3 = A^2 + 2A - 1$. Now if we start with a vector orthogonal to X_1 , we shall get a sequence converging to X_2 . We therefore start with the vector V_2 , but modify its last element so as to make it orthogonal to X_1 . The modified last element is

$$- [53(1.00000) + 53(.64360) - 41(-.35826)]/.94213 = -108.05. \quad (35)$$

We now apply A_3 to the modified vector several times, repeating the orthogonalization process at each step. We get successively

$$\begin{aligned} & (53, 53, -41, -108.05)(417.90), \\ & (1.000, 1.130, -.739, -2.11438)(8.21364), \\ & (1.0000, 1.1257, -.7384, -2.11122)(8.19976), \\ & (1.00000, 1.12581, -.73843, -2.11130)(8.20028), \\ & (1.00000, 1.12579, -.73842, -2.11128). \end{aligned}$$

Applying $A_4 = A - 2$ (of which $\lambda_2 - 2$ is the numerically largest root) and taking scalar products yields $\lambda_2 - 2 = -6.1937207$, $\lambda_2 = -4.1937207$. It follows by direct calculation that $\lambda_3 = 0.464791$, $\lambda_4 = -2.326766$.

Example 6. Find to five significant figures the characteristic vectors of the matrix

$$B = \begin{bmatrix} 2 & 1 & 0 & 4 \\ 1 & -1 & -1 & 3 \\ 0 & -1 & 8 & -2 \\ 4 & 3 & -2 & 0 \end{bmatrix} \begin{vmatrix} 7 \\ 2 \\ -3 \\ 5 \end{vmatrix}$$

First we compute B^2 and B^3 .

Proceedings as in the previous example, we shortly conclude that $X_1 \sim (76, 44, -30, 76)$, $X_2 \sim (67, 84, -37, -129)$, $\lambda_1 \sim 6.56$, $\lambda_2 \sim -4.53$, $\lambda_3 \sim 0.69$, $\lambda_4 \sim -1.72$.

Next we find X_1 to the desired accuracy. Since more accuracy is required than in the previous example, it is worth while to go to more trouble to secure rapid convergence. We have

$$(\lambda - 0.69)(\lambda + 1.72)(\lambda + 4.53) = \lambda^3 + 5.56\lambda^2 + 3.48\lambda - 5.39.$$

Thus $B_1 = 2B^3 + 11B + 7B - 11$ will have three roots near zero. We get successively

$$\begin{aligned} &(76, 44, -30, 76)(81506), \\ &(1.0000000, .5789512, -.3916276, .9944053)(1069.511549), \\ &(1.0000000, .5789483, -.3916329, .9944266)(1069.520900), \\ &(1.0000000, .5789483, -.3916329, .9944265). \end{aligned}$$

The rapidity of convergence is to be noted. (As before, the extra digits are retained to avoid rounding-off errors both here and in the orthogonalization that follows.)

To obtain the vector X_2 , we write $(\lambda - 0.69)(\lambda + 1.72) = \lambda^2 + 1.03\lambda - 1.19$, and set up the matrix $B_2 = B^2 + B - 1$. As in the previous example, we must operate on vectors orthogonal to X_1 . Orthogonalizing at each step we get in succession

$$\begin{aligned} &(67, 84, -37, -129)(1048), \\ &(1.0000, 1.1698, -.5821, -1.915902)(14.877570), \\ &(1.0000, 1.1894, -.5893, -1.930148)(15.003080), \\ &(1.000000, 1.189683, -.589403, -1.9303534)(15.0048880), \\ &(1.000000, 1.189687, -.589404, -1.9303561). \end{aligned}$$

The convergence could have been improved by setting up a cubic matrix polynomial as was done with X_1 , but the extra time needed for doing so would have more than offset that saved later.

We could now obtain X_3 by starting with $B + 2$, say, and working with vectors orthogonal to both X_1 and X_2 . The double orthogonalization, however, would be time-consuming, so we start again with $(\lambda + 1.72)(\lambda + 4.53) = \lambda^2 + 6.25\lambda + 7.79$.

Use of $B^2 + 6B + 8$ suggests itself. However, a quick estimate of the roots indicates that this will not lead to very rapid convergence, and as yet we have not even a first approximation to X_3 . So we use instead $B_3 = 4B^2 + 25B + 31$.

Starting with a vector that turns out to be a rather poor guess and orthogonalizing with respect to X_1 at each step yields the following rapidly convergent sequence:

$$\begin{aligned} &(1, 0, 0, -1), \\ &(21, -10, 26, -5)(1036), \\ &(1.0000, -.5097, 1.2838, -.203265)(50.266140), \\ &(1.000000, -.516826, 1.293370, -.195347)(50.513110), \\ &(1.000000, -.516805, 1.293339, -.1953714). \end{aligned}$$

The vector X_4 can now be determined from the fact that it is orthogonal to X_1 , X_2 , and X_3 . We set

$$X_4 = V + a_1X_1 + a_2X_2 + a_3X_3, \quad (36)$$

where V is any vector not orthogonal to X_1' . Multiplying (36) by X_1' gives $0 = X_1'V + a_1X_1'X_1$, whence

$$a_1 = -X_1'V/X_1'X_1 \quad (37)$$

and similarly for a_2 and a_3 . Choosing $V = (1, 0, 0, 0)$, we obtain $a_1 = -0.4036424$, $a_2 = -0.1541062$, $a_3 = -0.3357978$. Substituting in (36) and dividing through by the leading element of the resulting vector yields $X_4 = (1.000000, -2.287233, -1.741508, -0.359851)$.

The characteristic roots can be found, if desired, simply by computing the leading elements of BX_1 , BX_2 , BX_3 , and BX_4 . Greater accuracy can be achieved by forming scalar products as in previous examples.

19. Summary. We now summarize our procedure.

(a) Obtain rough values of all the roots. This may be done by operating on properly chosen vectors with matrices of the forms $A + n$, $(A + m)^2 + n$, etc. After all but two of the roots have been located in this way, the last two can be found by solving a quadratic equation.

The examples indicate that three significant figures in the roots are amply sufficient at this stage, and one could often get along with fewer. Certainly a disproportionate amount of time should not be spent in getting these first approximations, as their sole purpose is the saving of time in the later steps. Fortunately the question is beside the point in many practical cases, where rough values of the roots are known to begin with; this step can then be omitted.

Naturally, if only a few figures are wanted in the final results, it may prove most economical to carry the work to a conclusion at this stage, rather than taking the time to set up the matrix polynomials of the succeeding steps.

(b) Select a polynomial $P(\lambda)$ such that, for some λ_i , $|P(\lambda_i)| > |P(\lambda_j)|$, $j \neq i$, and operate on a suitably chosen vector with $P(A)$ until X_i is obtained to the desired degree of accuracy.

$P(A)$ will ordinarily be chosen as in the examples. It will be noted that one or two figures in the coefficients are enough for rapid convergence, and more should not be used as time will be wasted in computing the elements of $P(A)$. It should be borne in mind that the coefficients of the higher powers are the most important.

Care should be taken to avoid errors at this stage, as much time can be lost if $P(A)$ is incorrectly chosen or written.

An extra place should be retained in the elements of X_i to avoid rounding-off errors, and it may be well to keep two such places if vectors orthogonal to X_i are to be constructed in step (c).

(c) Proceed similarly to obtain the other characteristic vectors, making use of the orthogonality of the vectors as in the examples given. If only the roots are desired, two of the vectors need not be computed. If all the vectors are wanted, the last one may be obtained from the others by the condition of orthogonality.

(d) Once the vectors are obtained, the corresponding roots may be found to about twice as many significant figures as the vectors by the scalar product method. In case the order of the matrix is greater than four, it may be necessary to use a matrix of the form $(A + m)^2 + n$ to do this, getting first the square of the root and then finding the root itself by taking a square root. The last two roots may be obtained from the others by solving a quadratic as before.

In case A has multiple roots, this fact should become evident while carrying out step (a). We can then proceed much as before, except that we should not, of course, expect to find unique vectors corresponding to the multiple roots.

V. GENERAL MATRICES; COMPLEX ROOTS

We now approach the problem of the general (unsymmetric) matrix. Unlike the symmetric case, the characteristic roots are no longer necessarily real, and there are two sets of characteristic vectors instead of one, having less convenient orthogonality properties.

Most of the methods of the preceding section remain applicable. However, we can no longer make such effective use of the orthogonality relation. Also, since the elements of the row vector $V'A$ need not be the same as those of the column vector AV , the method of scalar products no longer effects a great saving of time. On the other hand, new methods are required to handle complex roots.

20. Real roots. The case where all the roots are real presents few difficulties, and the procedure is very similar to that for the symmetric case. Some use may even be made of the orthogonality properties.

The same can be said if only two of the roots are complex, and the real roots can be found first, since the complex pair can then be obtained by solving a quadratic equation. The following example illustrates these points.

Example 7.

$$A = \begin{bmatrix} 2 & 0 & -1 & -3 \\ 1 & -3 & 0 & -2 \\ -2 & 1 & 2 & 1 \\ 3 & 4 & 0 & -1 \end{bmatrix} \begin{vmatrix} -2 \\ -4 \\ 2 \\ 6 \end{vmatrix}.$$

Find all the roots correct to six places, given that $\lambda_1 \sim 2.49$, $\lambda_2 \sim -1.77$.

We compute A^2 and A^3 and deduce that $\lambda_{3,4} \sim -0.36 \pm 3.28i$. Since

$$(\lambda + .36 - 3.28i)(\lambda + .36 + 3.28i)(\lambda + 1.77) = \lambda^3 + 2.45\lambda^2 + 12.2\lambda + 19.3,$$

we can get λ_1 by using the matrix $A_1 = 4A^3 + 10A^2 + 49A + 77$. After a few steps we obtain

$$X_1 = (-.3637407, .0336918, 1.0000000, -.2743017),$$

whence (multiplying by A once) $\lambda_1 = 2.4868715$.

To find λ_2 , we make use of the fact that the characteristic row vector Y_2 is orthogonal to X_1 . We have

$$(\lambda + .36 - 3.28i)(\lambda + .36 + 3.28i) = \lambda^2 + .72\lambda + 10.9.$$

Writing

$$A_2 = 10A^2 + 7A + 109 = \begin{bmatrix} 93 & -130 & -47 & -61 \\ -63 & 98 & -10 & 36 \\ -54 & 37 & 183 & 57 \\ 91 & -132 & -30 & -58 \end{bmatrix} \begin{vmatrix} -145 \\ 61 \\ 223 \\ -129 \end{vmatrix},$$

we get the following sequence of row vectors by multiplying on the right by A_2 and orthogonalizing with respect to X_1 at each step:

$$\begin{aligned} &\{0, 1, 0, 0\}, \\ &\{1.00, -1.56, -.46, -0.57\}, \\ &\{1.0000, -1.5795, 0.26538, -0.5526\}, \\ &\{1.000000, -1.579690, 0.2653589, -0.552692\}, \\ &\{1.0000000, -1.5796883, 0.2653587, -0.5526927\}. \end{aligned}$$

Operating once with A yields $\lambda_2 = -1.7684837$. It follows that $\lambda_{3,4} = -0.3591939 \pm 3.2840604i$.

How to find the characteristic vectors corresponding to such complex roots will be explained below.

21. Complex roots. When the complex roots cannot be approached from behind in this fashion, a new method of attack is necessary. Since such roots come in conjugate pairs of equal modulus, applying the matrix repeatedly to an arbitrary vector will not result in a convergent vector sequence (in fact such behaviour is sometimes the best indication that the largest roots are complex).

However, if $\lambda_1 = r_1 e^{i\theta_1}$, $\lambda_2 = r_1 e^{-i\theta_1}$, $|\lambda_1| = |\lambda_2| > |\lambda_3| > \dots$, we can still find the product $\lambda_1 \lambda_2 = r_1^2$ by the method used in Example 2; indeed the convergence should be relatively better than when $|\lambda_1| > |\lambda_2|$, since the loss of accuracy due to subtracting almost equal numbers will not arise. Having r_1^2 , we can find λ_1 and λ_2 as follows.

First by operating repeatedly on a vector V with A (or better, with some power of A) we obtain a vector W of the form

$$W = a_1 X_1 + a_2 X_2 + \Delta, \quad (38)$$

where Δ is as small as we please. It can now be seen that

$$r_1 \cos \theta_1 = \frac{r^2 W^{(i)} + (A^2 W)^{(i)}}{2(AW)^{(i)}} + 0(\Delta) \quad (j = 1, 2, 3, \dots, n). \quad (39)$$

Thus λ_1 and λ_2 are determined to any desired accuracy.

Furthermore,

$$\begin{aligned} AW + i(r^2 W - A^2 W)/2r_1 \sin \theta_1 \\ = a_1 \lambda_1 X_1 + a_2 \lambda_2 X_2 + i \frac{\lambda_1 \lambda_2 (a_1 X_1 + a_2 X_2) - a_1 \lambda_1^2 X_1 - a_2 \lambda_2^2 X_2}{i(\lambda_1 - \lambda_2)} + 0(\Delta) \\ = 2a_1 \lambda_1 X_1 + 0(\Delta) \end{aligned} \quad (40)$$

so that X_1 and X_2 can be found once λ_1 and λ_2 are known.

The problem can also be approached by operating with a matrix of the form $A + ni$, whose roots will ordinarily have distinct moduli, but in practice this method will usually prove more laborious than the preceding one.

If all the roots are wanted, we begin by locating them roughly, as in the symmetric case. Then matrix polynomials can be constructed to secure rapid convergence to more accurate values. These points will become clearer in the following example.

Example 8.

$$A = \begin{bmatrix} 1 & -2 & 0 & -4 \\ 3 & 0 & 1 & 2 \\ -1 & 3 & -1 & 1 \\ 1 & 0 & 4 & 0 \end{bmatrix} \begin{vmatrix} -5 \\ 6 \\ 2 \\ 5 \end{vmatrix}.$$

Find the roots of A to seven decimal places and its characteristic column vectors to seven significant figures.

As usual, we write down A^2 and A^3 :

$$A^2 = \left[\begin{array}{cccc|c} -9 & -2 & -18 & -8 & -37 \\ 4 & -3 & 7 & -11 & -3 \\ 10 & -1 & 8 & 9 & 26 \\ -3 & 10 & -4 & 0 & 3 \end{array} \right] \quad A^3 = \left[\begin{array}{cccc|c} -5 & -36 & -16 & 14 & -43 \\ -23 & 13 & -54 & -15 & -79 \\ 8 & 4 & 27 & -34 & 5 \\ 31 & -6 & 14 & 28 & 67 \end{array} \right]$$

Thus $\sum_{i=1}^4 \lambda_i = 0$, $\sum_{i=1}^4 \lambda_i^2 = -4$, and the largest root must be complex.

Starting with the vector $V = (0, 0, 1, 0)$, we operate repeatedly with A^3 , rounding off to three or four figures at each step to avoid accumulating useless digits. After several such multiplications we come to the sequence of vectors given below:

$$\begin{aligned} &(-458, 752, -68, -432), (-2974, 3046, 1220, -3176), \\ &(1588, 898, 1293, -1823), (-7060, 572, 8778, -8756). \end{aligned}$$

As we proceed, we form two-rowed determinants from the first and second elements of successive vectors, and likewise from the second and third elements, and then take the ratios of the values of successive determinants. The terms of these sequences corresponding to the above vectors are

$$\begin{array}{ccc} 841380 & 2166396 & 5431544 \\ & 257 & 251 \\ 1224568 & 2482918 & 7143048 \\ & 253 & 251 \end{array}$$

(The sequences of ratios above show convergence to r_1^6 , aside from the location of the decimal point, which is immaterial at this stage.)

The agreement of the last numbers suggest that if we now operate on the last vector with A and form ratios of determinants as before, we shall obtain r_1^2 to about three figures:

$$\begin{aligned} &(-7060, 572, 8778, -8756), (26820, -29914, -8758, 28052), \\ &(-25560, 127806, -79752, -8212), (-248324, -172856, 480518, -344568), \\ &19585 \quad 266316 \quad 3615550 \\ &13.60 \quad 13.58 \\ &25758 \quad 350503 \quad 4762747 \\ &13.61 \quad 13.59 \end{aligned}$$

Thus $r_1^2 \sim 13.6$. Using equation (39) with W as the second vector above yields

$$\begin{aligned} r_1 \cos \theta_1 &\sim -2.27, & r_1 \sin \theta_1 &\sim 2.91, \\ \lambda_1 &\sim -2.27 + 2.91i, & \lambda_1 &\sim -2.27 - 2.91i. \end{aligned}$$

It follows that $\lambda_{3,4} = 2.27 \pm 1.96i$.

In order to get the roots more exactly, we note that

$$(\lambda - 2.27 + 1.96i)(\lambda - 2.27 - 1.96i) = \lambda^2 - 4.54\lambda + 8.99.$$

Thus the matrix $A_1 = 2A^2 - 9A + 18$ will have two roots near zero. Rounding off the elements of the last vector above, we multiply it by A_1 and obtain ratios of determinants formed from the first and second, and second and fourth elements:

$$\begin{array}{cccc} (-2483, -1729, 4805, -3446)(-243759, 188294, 153735, -27083), \\ (-6120173, 18490944, -8355979, -4216169), \\ (53044664, 46504148, -111097501, 75339351), \\ (56799532, -40188604, -39094634, 63788114), \\ -888993313 & -335494216 & -126545934 & -477320484 \\ 3774 & & 37719260 & 37719148 \\ 111704493 & 421315396 & 158916507 & 599419524 \\ 3771 & 37719131 & & 37719148 \end{array}$$

We now multiply the last vector above by A and form ratios of determinants, getting

$$\begin{array}{cc} (56799532, -40188604, -39094634, 63788114), \\ (-117975716, 258880190, -74482596, -99579004), \\ (-237420080, -627567752, 869519878, -415906100), \\ 9962994304 & 1355011183 \\ & 13.6004404 \\ 1311417783 & 1783585959 \\ & 13.6004405 \end{array}$$

Taking $r_1^2 = 13.6004405$, we obtain, by using (39), $\lambda_{1,2} = -2.26774878 \pm 2.90822213i$.

Using equation (40) with W as the first vector above gives

$$\begin{aligned} 2a_1X_1 &= (-117975716 + 173631636i, 258880190 + 13923461i, \\ &\quad -74482596 - 240907341i, -00579004 + 220659305i) \\ X_1 &= (1.00000000, -0.63822188 - 1.05732751i, \\ &\quad -0.74982611 + 0.93844573i, 1.13604812 - 0.19839177i). \end{aligned}$$

The remaining roots must be $\lambda_{3,4} = 2.26774878 \pm 1.95642866i$.

22. Finding the remaining vectors. The vector X_3 cannot be determined as X_1 was, since there is no ready-made vector sequence available. However, the matrix polynomial

$$\begin{aligned} A_2 &= (A + 2.26774878 - 2.90822213i)(A + 2.26774878 + 2.90822213i) \\ &\quad \cdot (A - 2.26774878 - 1.95642866i) \\ &= (A^3 + 2.26774878A^2 + 3.31507144A - 30.8423824) \\ &\quad + i(1.95624866A^2 + 8.87337741A + 26.6082916) \end{aligned}$$

has three roots so near zero that a single multiplication of an arbitrary vector by A^2 should give us X_3 to about eight figures. To do this *it is not necessary to compute the elements of the matrix A_2* , for by (9)

$$A_2V = A^3V + 2.26774878A^2V + 3031507144AV - 30.8423824V \\ + i(1.95642866A^2V + 8.87337741AV + 26.6082916V).$$

Taking

$$V(1, 1, 1, 1),$$

we see that AV , A^2V , A^3V are the sum columns of A , A^2 , A^3 . Then

$$A_2V = (-174.3244445 - 90.1464559i, -96.7552001 + 73.9792701i, \\ 39.7292288 + 95.2221916i, 59.5362211 + 76.8444646i) \\ X_3 \sim (1.00000000, .26477276 - .56129590i, -.40277933 \\ -.33795067i, -.44932357 - .20845922i).$$

(The second vector above was obtained from the first by dividing by the leading element.) As a check, we perform the same operations with

$$V_1 = (1, 0, 0, 0)$$

(note that AV_1 , A^2V_1 , A^3V_1 are simply the first columns of A , A^2 , A^3) and get

$$(1.00000000, .26477273 - .56129590i, -.40277934 - .33795066i, \\ -.44932357 - .20845922i).$$

VI. SOLUTION OF ALGEBRAIC EQUATIONS BY MATRIX METHODS

23. The companion matrix. Before closing we consider briefly the solution of an algebraic equation

$$f(\lambda) = \lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - \dots - p_n = 0 \quad (41)$$

by the use of the "companion matrix"

$$C = \begin{bmatrix} p_1 & p_2 & \dots & p_{n-1} & p_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad (42)$$

(Bernoulli's method). It is easily seen that (42) has the same roots as (41) and that the characteristic column vector of (42) corresponding to the root λ_i ($i = 1, 2, \dots, n$) is

$$X_i = (\lambda_i^{n-1}, \dots, \lambda_i^2, \lambda_i, 1) \quad \text{or} \quad (1, \lambda_i^{-1}, \lambda_i^{-2}, \dots, \lambda_i^{-n+1}). \quad (43)$$

Thus the roots of (41) are determined if either the roots or the column vectors of (42) can be found.

24. Special methods of solution. While the methods employed in handling a gen-

eral matrix are applicable in this case, the special properties of C can be used to speed up the solution. First, equation (41) helps us in locating the roots, especially the real ones. Again, if λ'_i is an approximation of λ_i , the matrix polynomial

$$\frac{f(C) - f(\lambda'_i)}{\lambda - \lambda'_i} \quad (44)$$

(which may be obtained by synthetic division) will have only one large root, corresponding to λ_i . We recall that to set up the corresponding polynomial $P(A)$ in the general case approximations to all the roots are required. (However, (44) is not easily set up if λ_i is complex.)

Moreover, suppose we have a vector V that is an approximation of X_i and that $D = P(C)$ is a matrix having $P(\lambda_i)$ as its largest root. Then our next approximation, under our previous procedure, would be

$$W = (1, w_2, w_3, \dots, w_n) = \frac{DV}{(DV)_{(1)}} \quad (45)$$

Instead we may use the more easily computed

$$W^* = (1, w_2, w_2^2, w_2^{n-1}).$$

This can be done because X_i and W^* are both of the form (43), and because w_2 , being obtained from W , is an approximation to λ_i^{-1} . Unfortunately this process is not necessarily convergent (though in practice it generally is, especially when $|P(\lambda_i)| \gg |P(\lambda_j)|, j \neq i$). However, we can always try it and then fall back on the standard process if necessary.

Clearly, rather than determining W^* from the first two elements of DV , we could determine it from the last two. It can be shown that this will lead to better convergence if λ_i is the smallest root of (41), while the first process will work better if λ_i is the largest root of (42). For roots in between we should use one process or the other depending on their relative magnitude.

25. Examples. The following examples will make this procedure clear.

Example 9.

$$f(\lambda) = \lambda^4 - 4\lambda - 3 = 0. \quad (46)$$

Direct substitution shows that (46) has real roots between -1 and 0 and between 1 and 2 .

If we take 2 as a first approximation to λ_i , the matrix corresponding to (44) is $C_1 = C^3 + 2C^2 + 4C + 4$.

The vector corresponding to λ_1 can be written

$$X_1 = (1, \lambda_1^{-1}, \lambda_1^{-2}, \lambda_1^{-3}).$$

The first element is taken as unity since λ_1 is a relatively large root. Since $\lambda_1 \sim 2$, we start with the vector $V = (1.00, .50, .25, .12)$. Operating with C_1 yields the vectors

$$(1.000, .561, .315, .177), \quad (1.0000, .5604, .3140, .1760).$$

In forming these vectors, the first two elements of each vector were obtained in

the usual way, but the last two elements are simply the square and cube of the second element. Note that only the first two rows of C_1 are used in this procedure.

From the last vector we get

$$\lambda_1 \sim \frac{1}{0.5604} = 1.794.$$

In order to get λ_1 more exactly, we divide (46) by $\lambda - 1.784$ getting approximately

$$\lambda^3 + 1.784\lambda^2 + 3.18\lambda + 1.68.$$

Thus the matrix $C_2 = 5C^3 + 9C^2 + 16C + 8$ has only one large root, corresponding to λ_1 (cf. (44)). Operating as before (using only the first two rows of C_2) gives the rapidly convergent vector sequence

$$\begin{aligned} &(1.000, .560, .314, .176)(1.00000, .56043, .31408, .17602), \\ &\quad (1.0000000, .5604256, .3410769, .1760167), \\ &\quad (1.00000000, .56042566, .31407692, .17601677), \end{aligned}$$

whence $\lambda_1 = 1.7843580$.

Note that it was not necessary to approximate the other roots in order to find λ_1 .

The root λ_2 can be obtained in much the same way. Starting with the matrix

$$C_3 = C^3 - C^2 + C - 5$$

and the vector

$$V = (-1, 1, -1, 1)$$

we reach after a few steps the vector

$$(-.334, .482, -.694, 1.000).$$

(Since λ_2 is the smallest root, we take the last element as unity here and use the last two rows of C_3 in the computation.) Thus $\lambda_2 \sim -0.69$. Dividing (36) by $\lambda + 0.69$ gives

$$\lambda^3 - 0.69\lambda^2 + 0.48\lambda - 4.33.$$

Using the matrix

$$C_4 = -10C^3 + 7C^2 - 5C + 43$$

yields after a few more steps $\lambda_2 = -0.69250484$.

The complex roots are then $\lambda_{3,4} = -0.5459266 \pm 1.4593779i$.

Example 10.

$$\lambda^4 - 4\lambda + 4 = 0. \tag{37}$$

Here the roots are all complex. The companion matrix is

$$D = \begin{bmatrix} 0 & 0 & 4 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{vmatrix} 0 \\ 1 \\ 1 \\ 1 \end{vmatrix}.$$

By the methods employed in the general case, we get the rough values $\lambda_{1,2} \sim -1.05 \pm 1.43i$, $\lambda_{3,4} \sim 1.05 \pm 0.41i$.

Noting that

$$\begin{aligned} (\lambda - 1.05 - .41i)(\lambda - 1.05 + .41i)(\lambda + 1.05 + 1.43i) \\ = (\lambda^3 - 1.05\lambda^2 - .94\lambda + 1.33) + i(1.43\lambda^2 - 3.00\lambda + 1.82), \end{aligned}$$

we set up the matrix

$$\begin{aligned} D_1 &= (10D^3 - 11D^2 - 9D + 13) + i(14D^2 - 30D + 18) \\ &= \begin{bmatrix} 53 & -84 & 8 & 36 \\ -9 & 53 & -4 & 44 \\ . & . & . & . \\ . & . & . & . \end{bmatrix} + i \begin{bmatrix} 18 & 56 & -176 & 120 \\ -30 & 18 & 36 & -56 \\ . & . & . & . \\ . & . & . & . \end{bmatrix}. \end{aligned}$$

Since only the first two rows will be used, the others need not be written. Starting with a vector whose elements are power of $(-1.05 + 1.43i)^{-1}$, we get the vector sequence

$$\begin{aligned} (1.000, -.333 - .454i, -.095 + .302i, .169 - .057i), \\ (1.000000, -.332486 - .453252i, -.094890 + .301400i, .168160 - .057202i), \\ (1.00000000, -.33248603 - .45325413i, \text{—————}, \text{—————}), \end{aligned}$$

Thus

$$\lambda_1 = \frac{1}{-0.33248603 - 0.45325413i} = -1.0522167 + 1.4344109i$$

and $\lambda_{3,4} = 1.0522167 \pm 0.3959609i$.