VIBRATIONAL PROBLEMS IN ELLIPTICAL COORDINATES*

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1. Introduction. We commence with the differential equation

$$\frac{\partial^2 \zeta_0}{\partial x^2} + \frac{\partial^2 \zeta_0}{\partial y^2} - K \frac{\partial^2 \zeta_0}{\partial t^2} = 0, \tag{1.1}$$

K being a constant. Writing $\zeta_0 = e^{i\omega t}\zeta(x, y)$ in (1.1), it becomes

$$\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + k_1^2 \zeta = 0, \tag{1.2}$$

with $k_1^2 = \omega^2 K$. This is the familiar two-dimensional wave equation for sinusoidal time-displacement, expressed in rectangular coordinates. The ordinary differential equations used herein, into which (1.2), expressed in elliptical coordinates, may be decomposed, are identical in form with those obtained by Mathieu[1] when he solved the problem of the elliptical membrane. Their preferred canonical forms are

$$\frac{d^2 y}{dz^2} + (a - 2q \cos 2z)y = 0, \tag{1.3}$$

and

$$\frac{d^2 y}{dz^2} - (a - 2q \cosh 2z)y = 0, \tag{1.4}$$

with $q > 0$. In dealing with an elliptical plate, these equations are used with $q \gtrless 0$. (1.3) is known as the ordinary Mathieu equation, and (1.4) as the *modified* Mathieu equation. They are derivable from each other by the substitution $\pm iz$ for z. Recently lists of solutions of (1.3), (1.4), and the corresponding equations for $q < 0$, have been published.[2,3,4] By aid of these we shall derive formal solutions pertaining to the vibrational modes of:

(a) A uniform, homogeneous, loss-free stretched membrane in the form of an elliptical ring;[5]

(b) Water in a lake of uniform depth whose plan view is an elliptical ring;

(c) A uniform, homogeneous, loss-free, elastic elliptical plate;

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[1] E. Mathieu, *Mémoire sur le mouvement vibratoire d'une membrane de forme elliptique*, Jour. de Math. Pures et Appliquées. **13**, 137 (1868).

[2] W. G. Bickley, and N. W. McLachlan, *Mathieu functions of integral order and their tabulation*, MTAC, **2**, 1, 1946.

[3] N. W. McLachlan, *Mathieu functions and their classification*, Jour. Math. Phys., **25**, 209, 1946; *Theory and application of Mathieu functions*, Oxford Press, New York, 1947.

[4] J. A. Stratton, P. M. Morse, L. J. Chu, and A. Hutner, *Elliptic cylinder and spheroidal wave functions*, Wiley, & Sons New York, 1942.

[5] An elliptical ring is that portion of the ellipse enclosed by the inner and outer confocal bounding ellipses.

(d) A uniform, homogeneous, loss-free, elastic elliptical ring plate.
In all cases, as the outer bounding ellipse tends to a circle, the formulae degenerate to those already known for a circular boundary.

2. Elliptical ring membrane. In this case $k_1^2 = \omega^2 \rho_1/\tau$, where ρ_1 = mass of membrane per unit area, τ = radial tension per unit arc length,[6] both of the latter being constant. To derive the appropriate formal solutions, we first transform (1.1) to elliptical coordinates,[7] where

$$x = h \cosh \xi \cos \eta, \qquad y = h \sinh \xi \sin \eta, \tag{2.1}$$

$2h$ being the interfocal distance. Then if ζ is the displacement of a point (ξ, η) on membrane, we get

$$\frac{\partial^2 \zeta}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial \eta^2} + \tfrac{1}{2} k^2 (\cosh 2\xi - \cos 2\eta)\zeta = 0. \tag{2.2}$$

with $k^2 = \frac{1}{4} k_1^2 h^2 = \omega^2 h^2 \rho_1 / 4\tau$.

3. Solutions of (2.2). The appropriate type take the form

$$\zeta(\xi, \eta) = \chi(\xi)\psi(\eta) \tag{3.1}$$

where ψ and χ are functions of η, ξ alone, respectively, which satisfy the ordinary equations

$$\frac{d^2\psi}{d\eta^2} + (a - 2q \cos 2\eta)\psi = 0, \tag{3.2}$$

and

$$\frac{d^2\chi}{d\xi^2} - (a - 2q \cosh 2\xi)\chi = 0. \tag{3.3}$$

a is the arbitrary separation constant, while $q = k^2$, $k > 0$. These equations have the same forms as (1.3), (1.4), and we designate their first and second linearly independent solutions by

$$\psi_1(\eta, q), \qquad \psi_2(\eta, q); \tag{3.4}$$

$$\chi_1(\xi, q), \qquad \chi_2(\xi, q). \tag{3.5}$$

As shown in the papers mentioned in Footnotes 2, 3, 4 these solutions have a plurality of forms, so the next step is to select those appropriate to the problem of the ring membrane.

4. Physical conditions. Consider the maximum displacement of all points on a confocal ellipse on the membrane. If we start at $\eta = 0$ and move round the ellipse counterclockwise, the maximum displacement varies continuously. If *may* be repeated between $\eta = \pi$ and $\eta = 2\pi$, but it is *always* repeated at interval 2π. Thus ζ is single-valued and periodic in the coordinate η, the period being either π or 2π, ac-

[6] To obtain uniform tension, the membrane may be stretched with uniform radial tension over a circular frame, the elliptical rings then being clamped in position within the frame. The plane of the membrane, when at rest, is horizontal.

[7] A simple method of transformation is exemplified in Phil. Mag. **36,** 600, (1945).

cording to the mode of vibration. Consequently $\psi_1(\eta)$ and $\psi_2(\eta)$ must be functions such that either

$$\psi(\eta, q) = \psi(\eta + \pi, q), \tag{4.1}$$

or

$$\psi(\eta, q) = \psi(\eta + 2\pi, q). \tag{4.2}$$

According to the references in footnotes 2, 3, the only functions which satisfy these conditions are, respectively,

$$\psi_1(\eta, q) = ce_{2n}(\eta, q), \qquad se_{2n+2}(\eta, q), \qquad [a_{2n}, b_{2n+2}],^8 \tag{4.3}$$

and

$$\psi_1(\eta, q) = ce_{2n+1}(\eta, q), \qquad se_{2n+1}(\eta, q), \qquad [a_{2n+1}, b_{2n+1}],^8 \tag{4.4}$$

$n=0, 1, 2, \cdots$ The functions in (4.3), (4.4) are first solutions of (3.2). Linearly independent second solutions are *non-periodic* and, therefore, inadmissible here. There is no physical reason for discriminating between independent solutions of (3.3) so we may include both. Then,[2,3]

$$\chi_1(\xi, q) = Ce_m(\xi, q), \qquad Se_m(\xi, q), \qquad [a_m, b_m], \tag{4.5}$$

and

$$\chi_2(\xi, q) = Fey_m(\xi, q), \qquad Gey_m(\xi, q), \qquad [a_m, b_m], \tag{4.6}$$

where $m=2n, 2n+1, 2n+2$, as the case may be. The second alternative solutions $Fe_m(\xi, q)$, $Ge_m(\xi, q)$ may be used for $\chi_2(\xi, q)$, but for reasons stated in the references mentioned in Footnote 3, the solutions (4.6) are preferable.

5. The formal solution. This is to be constructed from (3.1), and (4.3)–(4.6). We arrange the $\chi\psi$ in groups, the functions in each group having the same characteristic number. Thus, introducing the real part of the time factor from Sec. 1 with an arbitrary phase angle, we have the component solutions of order m, corresponding to a_m and b_m, namely,

$$\zeta_m(\xi, \eta, t) = [C_m Ce_m(\xi, q) + F_m Fey_m(\xi, q)] ce_m(\eta, q) \cos(\omega_m t + \epsilon_m), \tag{5.1}$$

and

$$\bar{\zeta}_m(\xi, \eta, t) = [S_m Se_m(\xi, q) + G_m Gey_m(\xi, q)] se_m(\eta, q) \cos(\bar{\omega}_m t + \bar{\epsilon}_m). \tag{5.2}$$

In (5.1), (5.2), C_m, F_m, S_m, G_m, are arbitrary constants determinable from the displacement and velocity over the surface of the membranal ring at $t=0$, while ω_m, $\bar{\omega}_m$ are the respective pulsatances of the mth free modes of vibration, and ϵ_m, $\bar{\epsilon}_m$ their relative phase angles. ω_m and $\bar{\omega}_m$ are different as will be evident from the next section. For *Ce*, *ce*, *Fey*, $m=0, 1, 2, \cdots$, while for *Se*, *se*, *Gey*, $m=1, 2, \cdots$. The complete solution, without the time factor, maybe written

$$\zeta(\xi, \eta) = \sum_{m=0}^{\infty} \zeta_m(\xi, \eta) + \sum_{m=1}^{\infty} \bar{\zeta}_m(\xi, \eta). \tag{5.3}$$

[8] These are the characteristic numbers corresponding to the Mathieu functions *ce*, *se* of integral order $2n$, $2n+1$, $2n+2$.

6. Vibrational modes of ring membrane. We use the boundary conditions $\xi=\xi_0$, $\zeta=0$ at the outer clamp, $\xi=\xi_1$, $\zeta=0$ at the inner clamp. For modes of order m corresponding to characteristic number a_m, by (5.1) at the outer clamp

$$[C_m Ce_m(\xi_0, q) + F_m Fey_m(\xi_0, q)]ce_m(\eta, q) = 0. \tag{6.1}$$

At the inner clamp

$$[C_m Ce_m(\xi_1, q) + F_m Fey_m(\xi_1, q)]ce_m(\eta, q) = 0. \tag{6.2}$$

These equations are independent of η at the clamps, so for C_m, F_m to be non-zero, we must have

$$Ce_m(\xi_0, q)Fey_m(\xi_1, q) - Ce_m(\xi_1, q)Fey_m(\xi_0, q) = 0, \qquad (m \geqq 0) \tag{6.3}$$

which is the pulsatance equation. Moreover, corresponding to each value of $q_{m,p}$, $p=1, 2, 3, \cdots$ for which (6.3) is satisfied, the elliptical ring has a vibrational mode of order m and rank p. Since $q_{m,p}=k_{m,p}^2=\omega_{m,p}^2 h^2\rho_1/4\tau$, the pulsatances of the modes are given by

$$\omega_{m,p}^2 = 4\tau q_{m,p}/ h^2\rho_1, \tag{6.4}$$

$$= 4\tau q_{m,p}/a^2e^2\rho_1, \tag{6.5}$$

where $h=ae$, a being the semi-major axis, and e the eccentricity of the outer bounding ellipse. $q_{m,p}$ are the parametric zeros of (6.3), and determine the positions of the nodal ellipses, of which there are $(p-1)$ within the clamping rings. The nodal hyperbolae are determined for any $q=q_{m,p}$ by

$$ce_m(\eta, q) = 0, \qquad (m > 0). \tag{6.6}$$

For the set of modes corresponding to characteristic number b_m, the pulsatance equation defining the nodal ellipses is

$$Se_m(\xi_0, q)Gey_m(\xi_1, q) - Se_m(\xi_1, q)Gey_m(\xi_0, q) = 0, \qquad (m > 0), \tag{6.7}$$

where $q=\bar{q}_{m,p}$. The nodal hyperbolae, for any $q=\bar{q}_{m,p}$, are given by

$$se_m(\eta, q) = 0, \qquad (m > 0). \tag{6.8}$$

7. Elliptical ring lake. The formal analysis is identical with that for the ring membrane, but the condition at the inner and outer boundaries is zero velocity normal to them. Thus if ζ represents the tide height or vertical displacement of the water from its equilibrium level, we have[9]

$$\partial\zeta/\partial\xi = 0 \quad \text{at} \quad \xi = \xi_0, \quad \text{and also at} \quad \xi = \xi_1. \tag{7.1}$$

The conditional equations have the forms (6.1), (6.2), but the functions are replaced by their first derivatives with respect to ξ. Hence by (6.3) the pulsatance equation is

$$Ce_m(\xi_0, q)Fey_m'(\xi_1, q) - Ce_m'(\xi_1, q)Fey_m'(\xi_0, q) = 0, \qquad (m \geqq 0), \tag{7.2}$$

q having those values $q_{m,p}'$, $p=1, 2, \cdots$ for which (7.2) is satisfied. These determine the nodal ellipses, while the nodal hyperbolae for the same values of q are given by

[9] H. Jeffreys, *Free oscillations of water in an elliptical lake*, Proc. Lond. Math. Soc. **23**, 455 (1924).

(6.6). In the present problem $k^2 = q = \omega^2 h^2/gd$, g being the acceleration due to gravity, and d the uniform depth of the lake.

For the set of modes corresponding to characteristic number b_m, we have

$$Se'_m(\xi_0, q)Gey'_m(\xi_1, q) - Se'_m(\xi_1, q)Gey'_m(\xi_0, q) = 0, \qquad (m > 0), \tag{7.3}$$

and

$$se_m(\eta, q) = 0, \qquad (m > 0), \tag{7.4}$$

with $q = \bar{q}'_{m,p}$

8. Elliptical plate. Let ρ be the density, t the uniform thickness, $\sigma < 1$ Poisson's ratio, E the modulus of elasticity, and $c^2 = Et^2/12\rho(1-\sigma)$, whose dimensions are (velocity).[2] Then if ζ varies as $e^{i\omega t}$ in the various vibrational modes, $k_1^2 = \omega^2/c^2$, and rotation effects are ignored, the differential equation of motion expressed in rectangular coordinates, may be shown to be

$$\frac{\partial^4\zeta}{\partial x^4} + \frac{\partial^4\zeta}{\partial y^4} + \frac{2\partial^4\zeta}{\partial x^2\partial y^2} - k_1^4\zeta = 0, \tag{8.1}$$

or

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_1^2\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k_1^2\right)\zeta = 0. \tag{8.2}$$

If a function ζ_1 satisfies

$$\frac{\partial^2\zeta_1}{\partial x^2} + \frac{\partial^2\zeta_1}{\partial y^2} + k_1^2\zeta_1 = 0, \tag{8.3}$$

and another function ζ_2 satisfies

$$\frac{\partial^2\zeta_2}{\partial x^2} + \frac{\partial^2\zeta_2}{\partial y^2} - k_1^2\zeta_2 = 0, \tag{8.4}$$

their sum, with the appropriate arbitrary constants, is a solution of (8.1), (8.2). To obtain this solution, (8.3), (8.4) are expressed in elliptical coordinates, thereby yielding two equations of the form (2.2), with $+k^2$ in one and $-k^2$ in the other. These may be split up into pairs of equations like (3.2), (3.3), one pair being associated with $+q$ and the other with $-q$. Then omitting the time factor, the solutions of order m corresponding to (8.3) are

$$\zeta_1^{(m)}(\xi, \eta) = [C_m Ce_m(\xi, q) + F_m Fey_m(\xi, q)]ce_m(\eta, q), \qquad (a = a_m), \tag{8.5}$$

$$\bar{\zeta}_1^{(m)}(\xi, \eta) = [S_m Se_m(\xi, q) + G_m Gey_m(\xi, q)]se_m(\eta, q), \qquad (a = b_m). \tag{8.6}$$

The solutions corresponding to (8.4) are

$$\zeta_2^{(m)}(\xi, \eta) = [\bar{C}_m Ce_m(\xi, -q) + \bar{F}_m Fek_m(\xi, -q)]ce_m(\eta, -q) \tag{8.7}$$

for $a = a_{2n}$, $m = 2n$; $a = b_{2n+1}$, $m = 2n+1$, and

$$\bar{\zeta}_2^{(m)}(\xi, \eta) = [\bar{S}_m Se_m(\xi, -q) + \bar{G}_m Gek_m(\xi, -q)]se_m(\eta, -q) \tag{8.8}$$

for $a=b_{2n}$, $m=2n$; $a=2n+1$, $m=2n+1$. In these cases $q<0$ in (2.2) and the functions $Fek_m(\xi, -q)$, $Gek_m(\xi, -q)$ have been used in preference to $Fey_m(\xi, -q)$, $Gey_m(\xi, -q)$, since the former are real, while the latter are complex if ξ is real.[3]

On the interfocal line of the ellipse $\xi=0$. In crossing this line orthogonally from $(0, \eta)$ to $(0, -\eta)$, we must have

(a) continuity of displacement, so $\zeta(0, \eta)=\zeta(0, -\eta)$, and

(b) continuity of gradient, so

$$\frac{\partial}{\partial\xi}[\zeta(\xi, \eta)]_{\xi\to 0} = -\frac{\partial}{\partial\xi}[\zeta(\xi, -\eta)]_{\xi\to 0}.$$

Remembering that $\zeta(\xi, \eta)=\chi(\xi)\psi(\eta)$, it may be demonstrated that with the product pairs $Fey_m ce_m$, $Gey_m se_m$, $Fek_m ce_m$, $Gek_m se_m$, conditions (a), (b) cannot *both* be satisfied. Hence in (8.5)–(8.8) we put $F_m=G_m=\overline{F}_m=\overline{G}_m=0$. Thus the complete formal solution is

$$\zeta(\xi, \eta) = \sum_{m=0}^{\infty} C_m Ce_m(\xi, q)ce_m(\eta, q) + \overline{C}_m Ce_m(\xi, -q)ce_m(\eta, -q)$$

$$+ \sum_{m=1}^{\infty} S_m Se_m(\xi, q)se_m(\eta, q) + \overline{S}_m Se_m(\xi, -q)se_m(\eta, -q). \tag{8.9}$$

9. Boundary conditions. When the periphery of the plate is clamped, $\zeta=\partial\zeta/\partial\xi=0$ at $\xi=\xi_0$. Thus for mode m corresponding to characteristic number a_{2n}, we have

$$C_{2n}Ce_{2n}(\xi_0, q)ce_{2n}(\eta, q) + \overline{C}_{2n}Ce_{2n}(\xi_0, -q)ce_{2n}(\eta, -q) = 0, \tag{9.1}$$

and

$$C_{2n}Ce'_{2n}(\xi_0, q)ce_{2n}(\eta, q) + \overline{C}_{2n}Ce'_{2n}(\xi_0, -q)ce_{2n}(\eta, -q) = 0. \tag{9.2}$$

For C_{2n}, $\overline{C}_{2n}$ to be non-zero, we must have

$$[Ce_{2n}(\xi_0, q)Ce'_{2n}(\xi_0, q) - Ce_{2n}(\xi_0, -q)Ce'_{2n}(\xi_0, q)]ce_{2n}(\eta, q)ce_{2n}(\eta, -q) = 0. \tag{9.3}$$

Hence the pulsatance equation is

$$Ce_{2n}(\xi_0, -q)Ce'_{2n}(\xi_0, q) - Ce'_{2n}(\xi_0, -q)Ce_{2n}(\xi_0, q) = 0, \qquad (n \geqq 0), \tag{9.4}$$

or

$$\frac{d}{d\xi}[Ce_{2n}(\xi, q)/Ce_{2n}(\xi, -q)]_{\xi=\xi_1} = 0. \tag{9.5}$$

This equation is satisfied when $q=q'_{2n,s}$ say, where $s=1, 2, 3, \cdots$, and these roots define a system of confocal nodal ellipses. For the same values of q, by (9.3) we have the equations whose roots define two systems of nodal hyperbolae, namely,

$$ce_{2n}(\eta, q) = 0, \quad \text{and} \quad ce_{2n}(\eta, -q) = 0 \qquad (n > 0). \tag{9.6}$$

For modes associated with the characteristic numbers a_{2n+1}, b_{2n+1}, we take the respective equations, with $m=(2n+1)$,

$$C_m Ce_m(\xi,\, q) ce_m(\eta,\, q) + \bar{S}_m Se_m(\xi,\, -q) se_m(\eta,\, -q) = 0, \tag{9.7}^*$$

and

$$S_m Se_m(\xi,\, q) se_m(\eta,\, q) + \bar{C}_m Ce_m(\xi,\, -q) ce_m(\eta,\, -q) = 0. \tag{9.8}^*$$

Using the boundary conditions in (9.7) yields for $a = a_{2n+1}$, the equations

$$\frac{d}{d\xi}\left[Ce_m(\xi,\, q)/Se_m(\xi,\, -q)\right]_{\xi=\xi_0} = 0, \tag{9.9}$$

and

$$ce_m(\eta,\, q) = 0, \qquad se_m(\eta,\, -q) = 0, \tag{9.10}$$

for $q = q'_{m,s}$, these being the roots of (9.9).

Similarly for (9.8) we derive for $a = b_{2n+1}$, the equations

$$\frac{d}{d\xi}\left[Se_m(\xi,\, q)/Ce_m(\xi,\, -q)\right]_{\xi=\xi_0} = 0, \tag{9.11}$$

and

$$se_m(\eta,\, q) = 0, \qquad ce_m(\eta,\, -q) = 0, \tag{9.12}$$

for $q = \bar{q}'_{m,s}$, these being the roots of (9.11).

For characteristic number $a = b_{2n}$, the equations may be obtained from (9.5), (9.6) by writing Se for Ce, se for ce, $\bar{q}'_{2,ns}$ for $q'_{2,ns}$, $n \geqq 1$.

10. Elliptical ring plate. When an elliptical plate is clamped at its periphery and also at an internal confocal ellipse, it becomes a ring. Defining the inner and outer boundaries by ξ_1 and ξ_0, the conditions to be satisfied are $\zeta = \partial\zeta/\partial\xi = 0$ at $\xi = \xi_1$ and ξ_0, where $0 \leqq \xi_1 < \xi_0$. If ξ_1 were zero, the inner clamp would be on the interfocal line of length $2h$. By Sec. 8, the formal solution for characteristic number a_{2n}, is

$$\begin{aligned}\zeta^{(2n)} = {} & \left[C_{2n} Ce_{2n}(\xi,\, q) + F_{2n} Fey_{2n}(\xi,\, q)\right] ce_{2n}(\eta,\, q) \\ & + \left[\bar{C}_{2n} Ce_{2n}(\xi,\, -q) + \bar{F}_{2n} Fek_{2n}(\xi,\, -q)\right] ce_{2n}(\eta,\, -q).\end{aligned} \tag{10.1}$$

Then at $\xi = \xi_0$ and at $\xi = \xi_1$, we must have

$$\begin{aligned}& \left[C_{2n} Ce_{2n}(\xi,\, q) + F_{2n} Fey_{2n}(\xi,\, q)\right] ce_{2n}(\eta,\, q) \\ & \quad + \left[\bar{C}_{2n} Ce_{2n}(\xi,\, -q) + \bar{F}_{2n} Fek_{2n}(\xi,\, -q)\right] ce_{2n}(\eta,\, -q) = 0,\end{aligned} \tag{10.2}$$

and

$$\begin{aligned}& \left[C_{2n} Ce'_{2n}(\xi,\, q) + F_{2n} Fey'_{2n}(\xi,\, q)\right] ce_{2n}(\eta,\, q) \\ & \quad + \left[\bar{C}_{2n} Ce'_{2n}(\xi,\, -q) + \bar{F}_{2n} Fek'_{2n}(\xi,\, -q)\right] ce_{2n}(\eta,\, -q) = 0.\end{aligned} \tag{10.3}$$

Thus there are four equations from which we can eliminate the four arbitrary constants, and so derive the pulsatance equation, and those for the nodal hyperbolae. Similar equations may be derived corresponding to a_{2n+1}, b_{2n+1}, b_{2n+2}.

11. Examples. 1°. *Elliptical ring membrane.* The first pulsatance equation is (6.3),

* It should be observed that $ce_{2n+1}(\eta,\, q)$ and $se_{2n+1}(\eta,\, -q)$ have $a = a_{2n+1}$, while for $se_{2n+1}(\eta,\, q)$ and $ce_{2n+1}(\eta,\, -q)$, $a = b_{2n+1}$. Similarly for Ce and Se.

and we have to find those values $q_{m,p}=\omega_{m,p}^2 h^2\rho_1/4\tau$ which satisfy it. In the absence of tabular values of the two functions involved, calculation of the values q for the modes of lower order would be tedious. However, by aid of formulae asymptotic in $k=+q^{1/2}$, we can easily determine the approximate roots of (6.3). When $q=k^2$ is large enough[3]

$$\left.\begin{matrix} Ce \\ Fey \end{matrix}\right\}_m (\xi, q) \sim \frac{K_m}{(\cosh \xi)^{1/2}} \begin{matrix}\cos \\ \sin\end{matrix} (v_r - \theta_r), \tag{11.1}$$

where K_m is a constant dependent upon q, $v_r=2k \sinh \xi_r$, and $\theta_r=(2m+1)\tan^{-1}(\tanh \frac{1}{2}\xi_r)$. Using (11.1) in (6.3) leads to

$$\sin\left[(v_0 - v_1) - (\theta_0 - \theta_1)\right] = 0, \tag{11.2}$$

so

$$(v_0 - v_1) = p\pi + (\theta_0 - \theta_1), \tag{11.3}$$

p integral $\geqq 1$. Thus (11.3) gives

$$k_{m,p} \simeq (p\pi + \theta_0 - \theta_1)/2(\sinh \xi_0 - \sinh \xi_1). \tag{11.4}$$

If e is the eccentricity of a confocal ellipse ξ, $e^{-1}=\cosh \xi$ and, therefore, $\sinh \xi=e^{-1}(1-e^2)^{1/2}$. Substituting this into (11.4) yields

$$\begin{aligned} q_{m,p} &= k_{m,p}^2 \\ &\simeq (p\pi + \theta_0 - \theta_1)^2 e_0^2 e_1^2/4\left[e_0^2 + e_1^2 - 2e_0^2 e_1^2 - 2e_1e_0\left\{(1 - e_1^2)(1 - e_0^2)\right\}^{1/2}\right], \end{aligned} \tag{11.5}$$

e_0, e_1 being the eccentricities of the outer and inner bounding ellipses, respectively. The accuracy of (11.5) improves with increase in e and p. The second pulsatance equation (6.7) may be treated in a similar manner.

2°. *Elliptical plate.* Referring to (9.5), we shall consider the modes of order $2n$ corresponding to the characteristic number a_{2n}. As before, owing to absence of tabular values, we shall assume that k is large enough for (11.1) to be used. This formula applies when $q>0$, but we also require one for $q<0$. To derive this we write $(\xi+\frac{1}{2}\pi i)$ for ξ in (11.1), since this substitution changes the sign of q in (3.3) of which (11.1) is an approximate solution when q is large and positive. As explained in the paper quoted in Footnote 3, it is also necessary to multiply by $(-1)^n$ and select the real part of the formula, since ξ is real. Then if $|\xi|\gg\frac{1}{2}\pi$, we obtain

$$Ce_{2n}(\xi, -q) \sim \overline{K}_{2n}\chi_1/(\sinh \xi)^{1/2}, \tag{11.6}$$

where $\chi_1=\cosh u \cos \phi_{2n}+\sinh u \sin \phi_{2n}$, $u=2k\cosh \xi$, $\phi_{2n}\simeq(4n+1)\tan^{-1}(\tanh \frac{1}{2}\xi)$, and $\overline{K}_{2n}$ is a constant dependent upon q. Thus from (11.6) and (11.1), we get

$$\left(\frac{\overline{K}_{2n}}{K_{2n}}\right)\frac{Ce_{2n}(\xi, q)}{Ce_{2n}(\xi, -q)} \sim \frac{(\tanh \xi)^{1/2}\cos \chi}{\chi_1}, \tag{11.7}$$

with $\chi=2k\sinh \xi-(4n+1)\tan^{-1}(\tanh \frac{1}{2}\xi)$. Performing the differentiation indicated in (9.5), and equating to zero, leads after a little reduction to the pulsatance equation

$$\tan \chi = (1/\chi' \sinh 2\xi) - (\chi_1'/\chi_1\chi'), \qquad \xi = \xi_0. \tag{11.8}$$

If ξ_0 is large enough, $\chi_1'/\chi_1\chi' \simeq -1$, while $\chi' \sinh 2\xi_0 \gg 1$. Then (11.8) reduces to

$$\tan \chi \simeq -1, \quad \text{or} \quad \chi \simeq (s - \tfrac{1}{4})\pi, \tag{11.9}$$

s being a positive integer such that χ is large enough for the above approximate asymptotic formulae to be valid. Thus (11.9) entails

$$2k \sinh \xi_0 \simeq (s - \tfrac{1}{4})\pi + (4n + 1) \tan^{-1} (\tanh \tfrac{1}{2}\xi_0). \tag{11.10}$$

If ξ_0 is such that $2k \sinh \xi_0 \simeq ke^{\xi_0}$, and $\tanh \frac{1}{2}\xi \simeq 1$, we obtain

$$q_{2n,s} = k_{2n,s}^2 \simeq (s + n)^2\pi^2e^{-2\xi_0}. \tag{11.11}$$