NOTE ON THE THERMAL STRESSES IN A LONG CIRCULAR CYLINDER OF $m + 1$ CONCENTRIC MATERIALS*

By B. E. GATEWOOD (*AF Institute of Technology*)

It has been shown that the stresses and displacements in a long cylindrical body[1] can be expressed in terms of two functions U and V which satisfy the equations

$$\nabla^4 U = 0, \qquad \nabla^2 V = kT, \tag{1}$$

where T is the temperature change, and $k = Eq\alpha/(q - 1)$ with E Young's modulus, q the reciprocal of Poisson's ratio, and α the linear coefficient of thermal expansion. The stresses are as follows

$$\begin{gathered} \sigma_x = U_{yy} - V_{yy}, \qquad \sigma_y = U_{xx} - V_{xx}, \qquad \tau_{xy} = V_{xy} - U_{xy} \\ \tau_{zx} = 0, \qquad \tau_{yz} = 0, \qquad \sigma_z = Ee_z - kT + \nabla^2 U/q, \end{gathered} \tag{2}$$

where the constant strain e_z is determined from the requirement that the total normal force on the cross-section of the cylinder be zero. The function U can be expressed[1] in terms of two analytic functions $\phi(z)$ and $H(z)$ as

$$U = z\,\bar{\phi}\,(\bar{z}) + \bar{z}\,\phi(z) + H(z) + \bar{H}\,(\bar{z}), \tag{3}$$

and the function V may be taken as a particular integral of $\nabla^2 V = kT$ since the solution of $\nabla^2 V = 0$ can be included in $H(z)$.

The boundary conditions for the case of a circular cylinder composed of $m + 1$ different materials concentric to each other, each of which has different E, q, and α, can be obtained by taking the displacements and normal stresses continuous on the junction surfaces and the normal stress zero on the outside surface (let S_m be the inside material, S_{m-1} the adjoining material concentric to S_m, and S_0 the outside material; let C_0 of radius r_0 be the outside boundary and C_j of radius r_j the boundary between S_j and S_{j-1} with $j = 1, 2, \cdots, m$):

$$\phi_0(t) + t\,\overline{\phi_0'(t)} + \overline{\psi_0(t)} = [Y]_0, \text{ on } C_0, \tag{4}$$

$$\begin{aligned} (1 + M_j)\phi_j(t) + N_j\phi_{j-1}(t) + t\,\overline{\phi_j'(t)} + \overline{\psi_j(t)} \\ = R_j t + [Y]_j, \text{ on } C_j \text{ with } j = 1, 2, \cdots, m, \end{aligned} \tag{5}$$

$$\begin{aligned} M_j\phi_j(t) + (1 + N_j)\phi_{j-1}(t) + t\,\overline{\phi_{j-1}'(t)} + \overline{\psi_{j-1}(t)} \\ = R_j t + [Y]_{j-1}, \text{ on } C_j, \end{aligned} \tag{6}$$

where $z = t$ on C_j, $Y = V_x + iV_y$, $G = Eq/2(q + 1)$, $\psi(z) = H'(z)$, and

$$\begin{aligned} M_j &= 4G_{j-1}(q_j - 1)/q_j(G_j - G_{j-1}), \qquad j = 1, 2, \cdots, m, \\ -N_j &= 4G_j(q_{j-1} - 1)/q_{j-1}(G_j - G_{j-1}), \\ R_j &= 2e_z G_j G_{j-1}(q_{j-1} - q_j)/q_j q_{j-1}(G_j - G_{j-1}). \end{aligned} \tag{7}$$

---

*Received Sept. 22, 1947.

[1] B. E. Gatewood, *Thermal stresses in long cylindrical bodies*, Phil. Mag. (7) **32**, 282-301 (1941).

Since the functions ϕ_j and ψ_j are analytic, they can be represented by a Laurent series in S_j $(j = 0, 1, \cdots, m-1)$ and a power series in S_m:

$$\begin{aligned} \phi_j(z) &= \sum_{n=0}^{\infty} (a_{jn} + ib_{jn})z^n + \sum_{n=1}^{\infty} (c_{jn} + id_{jn})z^{-n}, \\ \psi_j(z) &= \sum_{n=0}^{\infty} (e_{jn} + if_{jn})z^n + \sum_{n=1}^{\infty} (g_{jn} + ih_{jn})z^{-n}, \end{aligned} \tag{8}$$

with $c_{mn} = d_{mn} = g_{mn} = h_{mn} = 0$. Also, Y_j may be expressed in a series of powers of z such that on C_j $(j = 0, 1, \cdots, m)$

$$\begin{aligned} [Y]_j &= \sum_{n=0}^{\infty} (A_{jn} + iB_{jn})t^n + \sum_{n=1}^{\infty} (C_{jn} + iD_{jn})t^{-n}, \\ [Y]_{j-1} &= \sum_{n=0}^{\infty} (E_{jn} + iF_{jn})t^n + \sum_{n=1}^{\infty} (G_{jn} + iH_{jn})t^{-n}. \end{aligned} \tag{9}$$

If these expressions be substituted in the boundary conditions (4), (5), and (6) and the resulting expressions together with their conjugates be integrated over the respective circles by use of the Cauchy integral formula, then a system of equations for determining the desired analytic functions is obtained. In some cases the functions can be determined directly while in others they can be obtained by equating coefficients of like powers of z. For the general case the coefficients in Eq. (8) can be expressed in terms of the coefficients in Eq. (9) by means of recursion formulas. The coefficients $a_{j0} + ib_{j0}$, $e_{j0} + if_{j0}$, b_{j1}, and h_{j1} may be neglected since they contribute nothing to the stresses. a_{j1} and g_{j1} are determined by

$$\begin{gathered} 2a_{01} + g_{01}/r_0^2 = A_{01}, \qquad g_{m1} = 0, \\ (2 + M_j)a_{j1} + N_j a_{j-1,1} + g_{j1}/r_j^2 = R_j + A_{j1}, \qquad j = 1, \cdots, m, \\ M_j a_{j1} + (2 + N_j)a_{j-1,1} + g_{j-1,1}/r_j^2 = R_j + E_{j1}. \end{gathered} \tag{10}$$

Define $w_{jn} = a_{jn} + ib_{jn}$, $x_{jn} = c_{jn} + id_{jn}$, $y_{jn} = e_{jn} + if_{jn}$, $z_{jn} = g_{jn} + ih_{jn}$, $W_{jn} = A_{jn} + iB_{jn}$, etc., whence w_{j2} and z_{j2} are given by

$$\begin{gathered} w_{02} + \bar{z}_{02}/r_0^4 = W_{02}, \qquad z_{m2} = 0, \\ (1 + M_j)w_{j2} + N_j w_{j-1,2} + \bar{z}_{j2}/r_j^4 = W_{j2}, \qquad j = 1, \cdots, m, \\ M_j w_{j2} + (1 + N_j)w_{j-1,2} + \bar{z}_{j-1,2}/r_j^4 = Y_{j2}. \end{gathered} \tag{11}$$

The remaining coefficients w_{jn}, $x_{j,n-2}$, $y_{j,n-2}$, and z_{jn} $(n \geq 3)$ are given by $(p_j = r_{j+1}/r_j)$

$$\begin{aligned} -M_{j+1}p_j^{2n}w_{j+1,n} &+ [1 + M_j - (1 + N_{j+1})p_j^{2n}]w_{jn} + N_j w_{j-1,n} \\ &- (n-2)r_j^{2-2n}\bar{x}_{j,n-2}(1 - p_j^2) = W_{jn} - p_j^{2n}Y_{j+1,n}, \\ -M_{j+1}\bar{x}_{j+1,n-2} &+ [(1 + M_j)p_j^{2n-4} - (1 + N_{j+1})]\bar{x}_{j,n-2} + N_j p_j^{2n-4}\bar{x}_{j-1,n-2} \\ &+ nr_{j+1}^{2n-2}(p_j^{-2} - 1)w_{jn} = p_j^{2n-4}\bar{X}_{j,n-2} - \bar{Z}_{j+1,n-2}, \qquad j = 1, \cdots, m, \end{aligned}$$

$$\begin{aligned}
x_{mn} &= 0, \qquad z_{mn} = 0, \qquad \bar{z}_{0n} = r_0^{2n}[W_{0n} - w_{0n} + (n-2)r_0^{2-2n}\bar{x}_{0,n-2}],\\
\bar{z}_{0n} &= r_1^{2n}[Y_{1n} - M_1 w_{1n} - (1 + N_1)w_{0n} + (n-2)r_1^{2-2n}\bar{x}_{0,n-2}],\\
\bar{z}_{jn} &= r_j^{2n}[W_{jn} - (1 + M_j)w_{jn} - N_j w_{j-1,n} + (n-2)r_j^{2-2n}\bar{x}_{j,n-2}],\\
y_{0,n-2} &= r_0^{4-2n}[\overline{X}_{0,n-2} - nr_0^{2n-2}w_{0n} - \bar{x}_{0,n-2}],\\
y_{0,n-2} &= r_1^{4-2n}[\overline{Z}_{1,n-2} - M_1\bar{x}_{1,n-2} - (1 + N_1)\bar{x}_{0,n-2} - nr_1^{2n-2}w_{0n}],\\
y_{j,n-2} &= r_j^{4-2n}[\overline{X}_{j,n-2} - (1 + M_j)\bar{x}_{j,n-2} - N_j\bar{x}_{j-1,n-2} - nr_j^{2n-2}w_{jn}].
\end{aligned}$$

For the case of the temperature a function of the radius the functions become very simple with the series in Eqs. (8) and (9) reducing to one term. With the temperature as $T_j(r)$ in S_j , the particular integral of $\nabla^2 V = kT$ yields

$$z\,\overline{Y}_m = k_m \int_0^r rT_m(r)\,dr,$$

$$z\,\overline{Y}_j = k_m \int_0^{r_m} rT_m(r)\,dr + k_{m-1}\int_{r_m}^{r_{m-1}} rT_{m-1}(r)\,dr + \cdots + k_j \int_{r_{j+1}}^{r} rT_j(r)\,dr,$$

whence, using Eq. (9), $[Y]_j = L_j t = A_{j1}t = [Y]_{j-1} = E_{j1}t$ with $r_j^2 L_j = [z\overline{Y}_j]_{r=r_j}$. Further, $\phi_j(z) = a_{j1}z$ and $\psi_j(z) = g_{j1}/z$, where a_{j1} and g_{j1} are given by Eq. (10). The function U is $U_j = a_{j1}r^2 + g_{j1}\log r$ and the radial and tangential stresses in polar coordinates are

$$\begin{aligned}
\sigma_{rj} &= 2a_{j1} + (g_{j1} - z\overline{Y}_j)/r^2, \qquad \tau_{r\theta j} = 0,\\
\sigma_{\theta j} &= 2a_{j1} - (g_{j1} - z\overline{Y}_j)/r^2 - k_jT_j\ ,\\
\sigma_{zj} &= E_je_z - k_jT_j + 4a_{j1}/q_j\ .
\end{aligned}$$

ON THIRD-ORDER CORRELATION AND VORTICITY IN ISOTROPIC TURBULENCE*

By F. N. FRENKIEL (*Cornell University*)

The fundamental equation of the propagation of the correlation function obtained by Th. von Kármán and L. Howarth[1] for isotropic turbulence in an incompressible viscous fluid, is written

$$\frac{\partial(\overline{u'^2}f)}{\partial t} + 2(\overline{u'^2})^{3/2}\left(\frac{\partial h}{\partial r} + \frac{4}{r}h\right) = 2\nu\overline{u'^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right). \tag{1}$$

---

*Received Nov. 19, 1947.

[1] Th. von Kármán and L. Howarth. *On the statistical theory of isotropic turbulence*, Proc. Roy. Soc. London (A) **164**, 192 (1938).