—NOTES—

ON BERNOULLI'S METHOD FOR SOLVING ALGEBRAIC EQUATIONS*

By BERNARD DIMSDALE (*Ballistic Research Laboratories, Aberdeen Proving Ground, Md.*)

1. Introduction. Both Graeffe's and Bernoulli's method for the solution of algebraic equations have the considerable advantage that no initial approximation to a root is required. Graeffe's method, although the convergence is quadratic, leads to factors[1] whose roots are large powers of the roots of the original equation, and for which the determination of angle is therefore quite difficult. Bernoulli's method, as amplified below, leads to factors of the original equation and therefore avoids this difficulty. While it is true that the convergence is slower (doubling the number of steps is roughly equivalent to one step of the Graeffe process) this is not a consideration of first importance if high speed computing machinery is available. Also, there is the possibility of applying rapidly convergent processes when a "sufficiently good" approximation is obtained. Furthermore, the method applies equally well to equations with complex coefficients.

Bernoulli's method consists of defining a sequence s_n in terms of the coefficients of the original equation, and taking quotients s_{n+1}/s_n . If these quotients converge, they converge to the root of largest modulus. If not, there are at least two roots of largest modulus. It will be seen that in this case successive systems of linear equations can be defined in terms of the s_n whose solutions will tend under certain conditions to the coefficients of a factor of the original equation.

2. Preliminary remarks. Let

$$p_N(z) = \sum_{\nu=0}^{N} a_\nu z^{N-\nu}, \qquad a_0 = 1 \tag{1}$$

with unequal roots $z_1, z_2, \cdots, z_{N'}$, of multiplicities $c_1, c_2, \cdots, c_{N'}$, respectively. The notation is chosen so that no modulus exceeds any preceding modulus. Let

$$P_M(z) = \sum_{\nu=0}^{M} \alpha_\nu z^{M-\nu}, \qquad \alpha_0 = 1, \qquad M \leq N' \tag{2}$$

with roots $z_1, z_2, \cdots, z_M$, and $M < N'$ if $z_{N'} = 0$. Let

$$s_n = \sum_{\nu=1}^{N'} c_\nu z_\nu^n, \qquad s_n' = \sum_{\nu=1}^{M} z_\nu^n$$

for all[2] n. We then have Newton's identities for s_n :

$$s_0 = N,$$

$$\sum_{\nu=0}^{n-1} a_\nu s_{n-\nu} + na_n = 0, \qquad n = 1, \cdots, N-1,$$

$$\sum_{\nu=0}^{N} a_\nu s_{n-\nu} = 0, \qquad n \geq N.$$

---

*Received Aug. 8, 1947.

[1]Bodewig, E. *On Graeffe's method for solving algebraic equations*, Q. Appl. Math. 4, 177-190 (1946). The methods described there apply only to equations with real coefficients.

[2]If $z_{N'} = 0$, then s_0 is not defined. Setting $s_0 = N$, as we are about to do, however, leads to no error in the rest of the sequence.

by which s_n can be calculated directly from the coefficients of (1) for all[3] n. The same identities hold for $s_{n'}$ when N, a, and s are replaced throughout by M, α, s'.

Consider the system of equations linear in $\alpha_{\nu n}$

$$\sum_{\nu=0}^{M} \alpha_{\nu n} s_{p-\nu} = 0, \qquad \alpha_{0n} = 1, \tag{3}$$

where $p = n, n + 1, \cdots, n + M - 1; n \geq M$. It will be shown in the next section that if $| z_M | > | z_{M+1} |$, then the system has a unique solution for all sufficiently large n and that $\alpha_{\nu n} - \alpha_\nu = O(z_{M+1}/z_M)^n$, so that $\alpha_{\nu n} \to \alpha_\nu$ as $n \to \infty$. Hence for all such values of M a factor of (1) in the form (2) is obtained, and (1) can be separated into factors each of which has roots of equal modulus. Then for each of these factors a shift of origin will result in further factorization.

3. Convergence. Let

$$\Delta_{0n}(s) = \begin{vmatrix} s_{n-1} & s_{n-2} & \cdots & s_{n-M} \\ s_n & s_{n-1} & \cdots & s_{n-M+1} \\ & \vdots & & \\ s_{n+M-2} & s_{n+M-3} & \cdots & s_{n-1} \end{vmatrix}$$

and let $\Delta_{\nu n}(s)$ for $\nu = 1, 2, \cdots, M$ be the same determinant with the column headed by $s_{n-\nu}$ replaced by $s_n, s_{n+1}, \cdots, s_{n+M-1}$, so that if $\Delta_{0n}(s) \neq 0$, then $\alpha_{\nu n} = -\Delta_{\nu n}(s)/\Delta_{0n}(s)$. Let

$$\delta_\nu(z_{\lambda 1}, z_{\lambda 2} \cdots, z_{\lambda M}) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_{\lambda 1} & z_{\lambda 2} & \cdots & z_{\lambda M} \\ z_{\lambda 1}^M & z_{\lambda 2}^M & \cdots & z_{\lambda M}^M \end{vmatrix}, \quad \nu = 0, 1, \cdots, M,$$

where the row with $z^{M-\nu}$ is absent. For brevity let $\delta_\nu = \delta_\nu(z_1, z_2, \cdots, z_M)$. We note that δ_0^2 is the discriminant of (2) and is therefore not zero, since all the roots of (2) are different. Then, by factoring,

$$\Delta_{\nu n}(s') = \pm(z_1, z_2 \cdots z_M)^{n-M} \delta_0 \delta_\nu \tag{4}$$

and, using the theorem for addition of determinants

$$\Delta_{\nu n}(s) = \pm \sum (c_{\lambda 1} c_{\lambda 2} \cdots c_{\lambda M})(z_{\lambda 1} \cdots z_{\lambda M})^{n-M} \delta_0(z_{\lambda 1}, \cdots, z_{\lambda M})$$

$$\delta_\nu(z_{\lambda 1}, \cdots, z_{\lambda M})$$

$$= (c_1 c_2 \cdots c_M) \Delta_{\nu n}(s') \pm \sum{}',$$

where $\sum$ denotes summation over all combinations of the M integers $\lambda_1 < \lambda_2 < \cdots < \lambda_M$ chosen from $1, 2, \cdots, N'$, and $\sum'$ is the same sum with $(1, 2, \cdots, M)$ omitted.

From (4), we have $\Delta_{0n}(s') \neq 0$ for all allowed values of M. Since the z's have been

[3]The original Bernoulli method permits an arbitrary selection of the first N of the s_n. In case there are two equal roots of largest modulus and the remaining roots have smaller moduli, it can be shown that for an arbitrary selection the convergence of s_{n+1}/s_n will in general be $O(1/n)$ whereas for this choice it will be in our present notation, $O(z_2/z_1)^n$.

arranged in order of non-increasing moduli, it follows that if $|z_{M+1}| < |z_M|$, then $\Delta_{0n}(s)/\Delta_{0n}(s') = (c_1c_2 \cdots c_M) + O(z_{M+1}/z_M)^n$, and therefore that for sufficiently large n, $\Delta_{0n}(s) \neq 0$. Hence, for such n, a unique solution of (3) exists. If δ_ν is not zero, we can also write $\Delta_{\nu n}(s)/\Delta_{\nu n}(s') = (c_1c_2 \cdots c_M) + O(z_{M+1}/z_M)^n$ from which

$$\Delta_{\nu n}(s)/\Delta_{0n}(s) = [1 + O(z_{M+1}/z_M)^n]\Delta_{\nu n}(s')/\Delta_{0n}(s'),$$

that is

$$\alpha_{\nu n} = [1 + O(z_{M+1}/z_M)^n]\alpha_\nu .$$

Therefore, $\alpha_{\nu n} - \alpha_\nu = O(z_{M+1}/z_M)^n$. This equation is easily established if δ_ν, that is, α_ν, is zero.

It might be remarked finally that if $M > N'$, the system (3) is dependent for every n; if $M = N'$, then $\alpha_{\nu n} = \alpha_\nu$ for all n if $z_{N'} \neq 0$, and if $z_{N'} = 0$ then in this case (3) is again dependent for all n.

4. Example. We apply the method to the equation

$$z^5 - 3z^4 - (2 + i)z^3 + (12 + 5i)z^2 - (8 + 8i)z + 4i = 0,$$

whose roots are

$$z_1 = -1 - (1 + i)^{1/2} = -2.098684113 - 0.4550898608i,$$

$$z_2 = 2, \qquad z_3 = 1,$$

$$z_4 = -1 + (1 + i)^{1/2} = .0986841134 + .4550898608i,$$

and $c_1 = c_3 = c_4 = 1$, $c_2 = 2$. Here $|z_1| = 2.147459380$, $|z_4| = .4656665498$. We observe that $|z_2|/|z_1| = .931$, $|z_3|/|z_2| = .5$ and so we would expect the quotients to converge much more slowly than the solutions of the second order system. We observe that as a matter of fact convergence of the quotient is not visible for $n \leq 30$, but that $n = 30$ gives us nine place accuracy in the second order system.

On solving the quadratic

$$z^2 + \alpha_{1,29}z + \alpha_{2,29} = 0,$$

we obtain

$$z_1 = -2.098684113 - 0.455089863i, \qquad z_2 = 1.999999998 + 0.000000004i.$$

The third order system is not computed, because significant digits are lost in the computation. If we carry ten significant figures throughout, we will have only one or two left toward the end of our computation for $n = 30$. This corresponds to the fact that the second order system has converged to eight or nine significant figures by this time. This is true in general of the behaviour of these systems, and therefore one would, in practice, only carry the method to the lowest order which converges with sufficient rapidity ("sufficient" in this case referring to the speed with which these computations can be performed). On removal of the factor obtained, one could repeat the operation with the reduced equation. It is interesting to observe in our present example that since the quotients converge, we will, if we go far enough, begin to lose significant digits in the second order system, but that in the meantime we will have obtained all the accuracy possible with ten digits.

For any n, the fourth order system has the solution $\alpha_{1,n} = -1$, $\alpha_{2,n} = -4 - i$,

n	$R(s_n)$	$I(s_n)$	$R(s_n/s_{n-1})$	$I(s_n/s_{n-1})$	$R(\alpha_{1,n})$	$I(\alpha_{1,n})$	$R(\alpha_{2,n})$	$I(\alpha_{2,n})$
0	5	0						
1	3	0	0.60000	0				
2	13	2	4.3333	0.66667	0.007417	0.641533	−2.60445	−0.784920
3	9	−6	0.60694	−0.55491	0.384898	0.587371	−4.27631	−0.801871
4	47	16	2.7949	3.6410	0.1263745	0.4572780	−4.047214	−0.8663713
5	43	−40	0.56024	−1.0418	0.1404347	0.4418591	−4.142651	−0.8744711
6	157	94	0.86721	2.9928	0.1125302	0.4637507	−4.149824	−0.9158061
7	241	−210	0.54045	−1.6612	0.1095630	0.4520286	−4.182332	−0.8967605
8	451	448	0.14299	1.9835	0.1026775	0.4572601	−4.187307	−0.9110257
9	1359	−912	0.50564	−2.5244	0.1009175	0.4543819	−4.192759	−0.9071229
10	933	1762	−0.12656	1.2116	0.0998953	0.4556626	−4.195222	−0.9103003
11	7241	−3190	0.28554	−3.9583	0.0991674	0.4549566	−4.196049	−0.9094951
12	143	5264	−0.25167	0.61610	0.0990177	0.4551808	−4.196893	−0.9101518
13	35675	−7384	−1.2177	−6.8102	0.09879357	0.45507276	−4.1970161	−0.91004092
14	−11075	6718	−0.33506	0.11896	0.09877110	0.45510405	−4.1972566	−0.91015585
15	160609	5854	−10.367	−6.8170	0.09871069	0.45508979	−4.1972786	−0.91015344
16	−65789	−55552	−0.42167	−0.33051	0.09870581	0.45509142	−4.1973402	−0.91016976
17	650015	206176	−7.3126	3.0408	0.09869098	0.45509074	−4.1973463	−0.91017495
18	−195899	−609214	−0.54393	−0.76470	0.09868935	0.45508986	−4.1973608	−0.91017645
19	2282777	1606298	−3.4816	2.6275	0.09868798	0.45508849	−4.1973630	−0.91017885
20	237967	−3932784	−0.74109	−1.2013	0.0986853501	0.4550898011	−4.197366237	−0.9101787855
21	6306379	9099768	−2.2087	1.7372	0.0986846123	0.4550898752	−4.197366968	−0.9101795943
22	8097245	−20058722	−1.0725	−1.6331	0.0986844093	0.4550898405	−4.197367716	−0.9101794241
23	8260177	42229518	−1.6674	1.0849	0.0986842476	0.4550899897	−4.197368040	−0.9101796673
24	70647235	−84750400	−1.6178	−1.9894	0.0986841815	0.4550898564	−4.197368083	−0.9101796616
25	−49306257	160983760	−1.4069	0.59098	0.0986841491	0.4550898673	−4.197368160	−0.9101797080
26	451798373	−284874718	−2.4037	−2.0703	0.0986841296	0.4550898600	−4.197368191	−0.9101797074
27	−527709591	453334314	−1.2884	0.19099	0.0986841228	0.4550898618	−4.197368211	−0.9101797164
28	2414035727	−589087984	−3.1839	−1.6188	0.0986841174	0.4550898607	−4.197368218	−0.9101797180
29	−3133922117	382030920	−1.2617	−0.14963	0.0986841158	0.4550898609	−4.197368223	−0.9101797197
30	11151899520	1113102974	−3.4637	−0.77741				

$\alpha_{3,n} = 4 + 3i$, $\alpha_{4,n} = -2i$, and the corresponding equation has the roots z_1, z_2, z_3, z_4. This occurs because of the presence of the double root $z_2 = 2$. For the same reason, all fifth order systems will be dependent.

A practical difficulty may arise with fixed decimal point machinery in that s_n may go off scale either on the right or the left. This will happen for sufficiently large n when $|z_1| \neq 1$. This difficulty may be overcome by a transformation $z' = z/r$, where $r = |\alpha_{M,n}|^{1/M}$, M being the order of a convergent process, n being "sufficiently" large. In our example we have $r = |\alpha_{2,n}|^{1/2}$. For n as small as three, $|s_n|$ would remain in a very satisfactory region.

AERODYNAMIC FORCES ON A SLOTTED FLAT PLATE*

By EDMUND PINNEY (*University of California, Berkeley*)

The derivation of the flat plate approximation to the aerodynamic forces exerted by an incompressible fluid on an airfoil is simple and is to be found in most textbooks. The present note treats the less simple case in which the airfoil contains slots oriented parallel to its length. This problem is of importance in the theory of suspension bridges where slotted roadbeds are used to cut down aerodynamically driven oscillations.

The spaces between the slots will be called "lanes". Let V be the velocity of flow, and α the angle of attack, assumed to be small. Let the plate occupy the interval $-b \leq x \leq b$, $y = 0$ of a rectangular coordinate system, and let $u(z)$, $v(z)$ be the velocity components of the fluid parallel to the x and y axes, respectively, at the point $z = x + iy$. Then, if powers of α higher than the first are neglected,

$$u(z) \to V, \qquad v(z) \to V\alpha \qquad \text{as } |z| \to \infty. \tag{1}$$

By two-dimensional potential theory, $v(z) + iu(z)$ is an analytic function of z away from the plate. Consider

$$v(z) + iu(z) = iV + V\alpha \prod_1^N {}_n (z - t_n b)^{1/2}(z - l_n b)^{-1/2} \tag{2}$$

where $x = t_n b$ and $x = l_n b$ are the trailing and leading edges, respectively, of the n-th lane, and N is the total number of lanes. The lanes are numbered from left to right so that

$$l_1 = -1, \qquad t_{n+1} > t_n, \qquad l_{n+1} > l_n, \qquad l_{n+1} > t_n, \qquad t_N = 1. \tag{3}$$

As $z \to \infty$, it is seen that (1) is satisfied.

In the n-th lane $l_n b < x < t_n b$, and by (2),

$$v(x \pm i0) + iu(x \pm i0) = iV \pm iV\alpha \prod_1^N {}_n |x - t_n b|^{1/2} |x - l_n b|^{-1/2}.$$

Therefore

$$u(x \pm i0) = V \pm V\alpha \prod_1^N {}_n |x - t_n b|^{1/2} |x - l_n b|^{-1/2}, \qquad v(x \pm i0) = 0. \tag{4}$$

*Received Sept. 22, 1947.