CURRENTS ON THE SURFACE OF AN INFINITE CYLINDER EXCITED BY AN AXIAL SLOT*

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1. Summary. The distribution of surface-current density is determined on an axially infinite, perfectly conducting cylinder. A constant distribution of electric field across a narrow axial slot is assumed given.

2. Introduction. The analysis of the electromagnetic field radiated by a perfectly conducting circular cylinder of finite length fed by an axial slot becomes quite involved

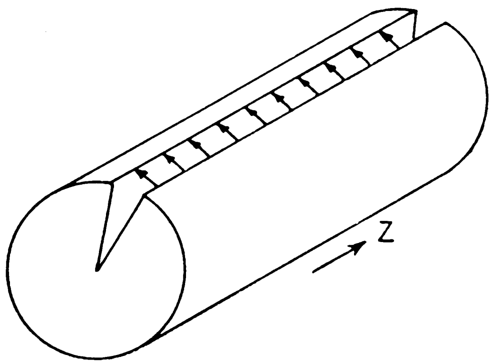

FIG. A. Infinite cylinder with axial slot. Constant E across mouth of slot.

especially when the end surfaces are rigorously treated. However, if the slot is situated half way between the ends of the cylinder, the peripheral surface current distribution and the radiation pattern in the plane through the slot and perpendicular to the axis of the cylinder is approximately the same as for an infinite cylinder with an infinitely long axial slot. Therefore, in so far as this peripheral surface current and radiation pattern is concerned, it is only necessary to solve the following problem.

The problem investigated is the distribution of currents on the surface of an axially infinite cylinder of radius a fed from an axial slot across which is maintained a constant distribution of electric field defined by

$$E = \theta_1 E_\theta(a, \theta, Z) = \theta_1 E_0 G(\theta) e^{-i\omega t}, \tag{1}$$

where $G(\theta)$ is defined as follows:

$$G(\theta) = \begin{cases} 1 \text{ for } -\theta_0/2 < \theta < \theta_0/2 \\ 0 \text{ for } \mid \theta \mid > \theta_0/2 \end{cases}$$

and θ_1 is the unit vector in the θ-direction of the cylindrical co-ordinates r, θ, z.

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Since the cylinder has infinite conductivity, all the components of the E-field for $r = a$ and $|\theta| > \theta_0/2$ are zero except the radial component E_r. Therefore, for $r = a$, E_θ

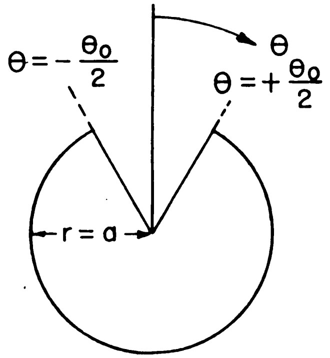


FIG. B. Cross-section of cylinder with slot.

is constant in the gap and zero outside it. The applied field, E_θ, maintains currents in the θ-direction. The field outside the cylinder has only the components E_θ, E_r, and H_z so that it is TM (transverse magnetic) and may be regarded as a superposition of cylindrical waves traveling radially outward from the surface of the cylinder. At great distances from the cylinder, E_r is negligible compared to E_θ, so that the field is predominantly TEM (transverse electromagnetic). As r approaches a, E_θ of the radiated field must approach the E_θ of the applied field, and H_z becomes numerically equal to the θ-component of the surface-current density J_θ.

3. Fourier expansion of the applied field. To obtain an analytical expression for the applied field as given by (1), it is expanded in a complex Fourier series; that is,

$$E_0 G(\theta)\, e^{-i\omega t} = \sum_{n=-\infty}^{n=+\infty} a_n\, e^{in\theta - i\omega t} \tag{2}$$

where a_n are the Fourier coefficients given by

$$a_n = E_0 (n\pi)^{-1} \sin\,(n\theta_0/2) \tag{3}$$

By substituting (3) in (2), the Fourier expansion for the applied field is

$$E_\theta = E_0 G(\theta)\, e^{-i\omega t} = E_0 \pi^{-1} \sum_{n=-\infty}^{n=+\infty} n^{-1} \sin\,(n\theta_0/2)\, e^{in\theta - i\omega t} \tag{4}$$

This infinite series represents E_θ for all values of θ between $-\pi$ and π.

4. General expression for field surrounding cylinder. The most general expressions for E_θ and H_z for $r \geq a$ are the following:[1,2]

$$E_\theta = \frac{-i\mu\omega}{k} \sum_{n=-\infty}^{n=+\infty} H_n^{(1)\prime}(kr) b_n\, e^{in\theta - i\omega t} \tag{5}$$

and

$$H_z = \sum_{n=-\infty}^{n=+\infty} H_n^{(1)}(kr) b_n\, e^{in\theta - i\omega t} \tag{6}$$

where $k = 2\pi/\lambda$, λ being the free space wavelength, and where $H_n^{(1)}(kr)$ are the Hankel functions of the first kind and order n, $H_n^{(1)\prime}(kr)$ are the derivatives $[d/d(kr)]H_n^{(1)}(kr)$, and b_n are constant coefficients. E_r is not involved in the determination of the unknown coefficients. Expressions (5) and (6) hold for all values $r \geq a$ and for $-\pi \leq \theta \leq \pi$.

5. Matching at surface of cylinder. At the surface of the cylinder, $r = a$, expressions (4) and (5) must be identically equal for all values of θ between $-\pi$ and π. Therefore,

$$E_0(n\pi)^{-1} \sin (n\theta_0/2) = -(i\mu\omega/k)H_n^{(1)\prime}(ka)b_n . \tag{7}$$

This equation determines b_n for all values of n. Explicitly,

$$b_n = \{ikE_0 \sin (n\theta_0/2)\}/\{\mu\omega\pi n H_n^{(1)\prime}(ka)\}. \tag{8}$$

By substituting (8) into (6), H_z becomes

$$H_z = (ikE_0/\mu\omega\pi) \sum_{n=-\infty}^{n=+\infty} n^{-1} \sin (n\theta_0/2) \frac{H_n^{(1)}(ka)}{H_n^{(1)\prime}(ka)} e^{in\theta - i\omega t} \tag{9}$$

at the surface of the cylinder.

6. Surface-current density. Since J_θ is numerically equal to H_z, the surface-current density around the periphery of the cylinder is

$$J_\theta = (ikE_0/\mu\omega\pi) \sum_{n=-\infty}^{n=+\infty} n^{-1} \sin (n\theta_0/2) \frac{H_n^{(1)}(ka)}{H_n^{(1)\prime}(ka)} e^{in\theta - i\omega t} \tag{10}$$

The limits of summation in (10) may be changed by using the formulas

$$\begin{aligned} H_{-n}^{(1)}(x) &= e^{n\pi i} H_n^{(1)}(x) \\ H_{-n}^{(1)\prime}(x) &= e^{n\pi i} H_n^{(1)\prime}(x) \end{aligned} \quad . \tag{11}$$

Substituting (11) in (10), noting that $k/(\mu\omega) = 1/(120\pi)$ mhos, the following expression is obtained for J_θ.

$$J_\theta = \frac{E_0 e^{-i\omega t}}{120\pi^2} \left[\frac{i\theta_0}{2} \frac{H_0^{(1)}(ka)}{H_0^{(1)\prime}(ka)} + 2i \sum_{n=1}^{\infty} \frac{\sin (n\theta_0/2)}{n} \frac{H_n^{(1)}(ka)}{H_n^{(1)\prime}(ka)} \cos n\theta \right] \tag{12}$$

Let α be the real part and β the imaginary part of the bracketed expression in (12) so that this becomes

$$J_\theta = (120\pi^2)^{-1} E_0 e^{-i\omega t}(\alpha + i\beta) \tag{13}$$

For any given values of the parameters θ_0 and ka, α and β are functions of the angle θ. The absolute value of J_θ is given by

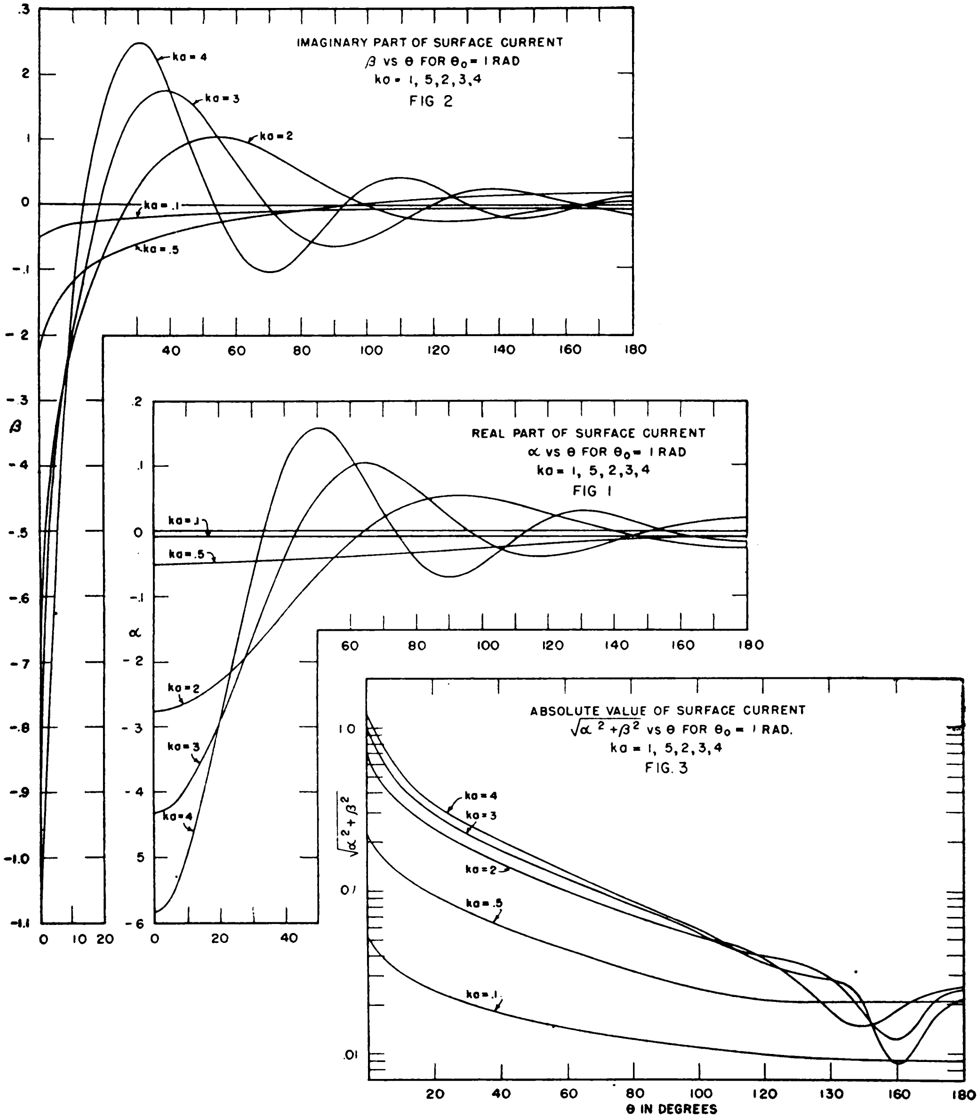
$$| J_\theta | = (120\pi^2)^{-1} E_0(\alpha^2 + \beta^2)^{1/2} \tag{14}$$

7. Radiation pattern. The radiation pattern is the distribution of $| E_\theta |$ about the cylinder along a circle whose radius r is so large that it is in the far zone defined by $kr \gg 1$. The formula for E_θ is obtained by substituting (8) in (5). The result, which is valid for $kr \geq ka$, is

$$E_\theta = \pi^{-1} E_0 e^{-i\omega t} \sum_{n=-\infty}^{n=+\infty} \frac{\sin (n\theta_0/2)}{n} \frac{H_n^{(1)\prime}(kr)}{H_n^{(1)\prime}(ka)} e^{in\theta} \tag{15}$$

Since in (15) a very good approximation to the infinite series is obtained by summing from $n = -M$ to $n = +M$ where $M = 40$, it is permissible to use the following asymptotic form when $kr \gg 1$ and $kr \gg M$.

$$H_n^{(1)\prime}(kr) \cong i(2/\pi kr)^{1/2} e^{i(kr-[(2n+1)/4]\pi)} \tag{16}$$

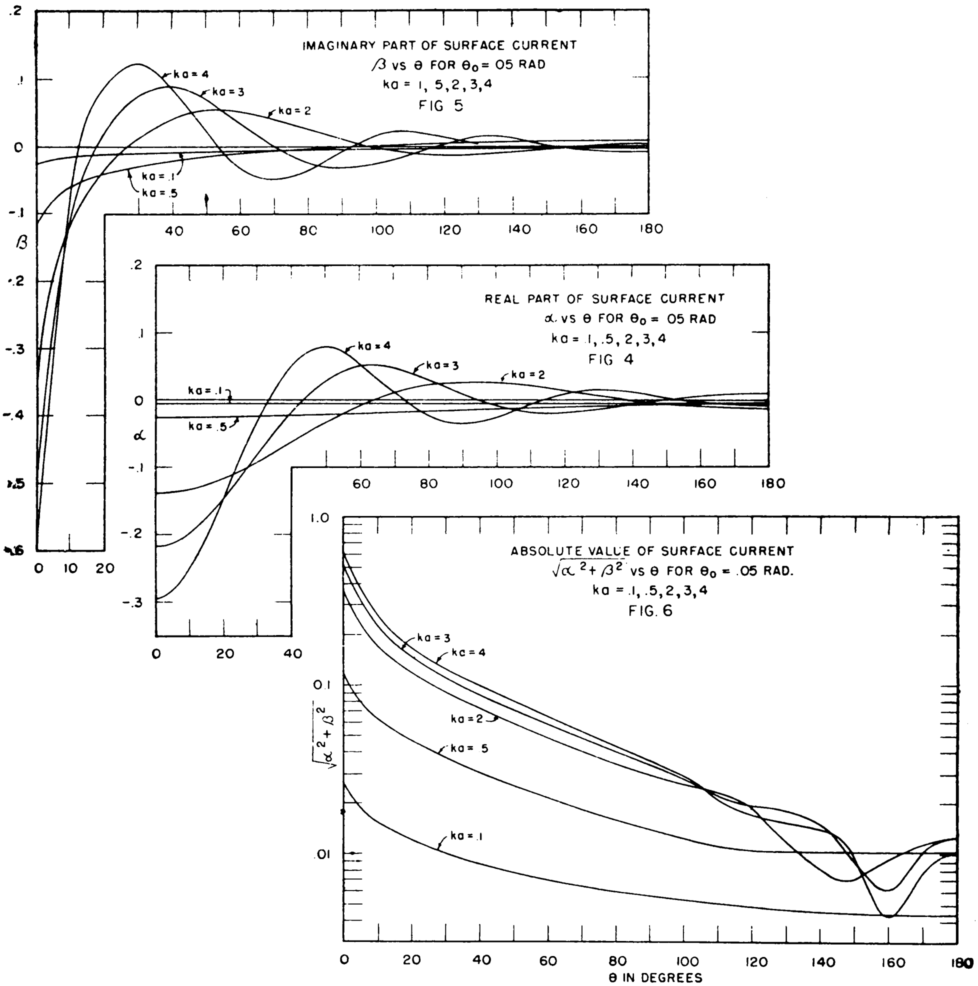


Substituting (16) into (15) and changing the limits of summation, E_θ (for far zone) becomes

$$E_\theta = \pi^{-1} E_0 e^{-i\omega t} (2/\pi kr)^{1/2} e^{ikr} e^{i/(\pi/4)} [\theta_0 \{2H_0^{(1)\prime}(ka)\}^{-1} + 2 \sum_{n=1}^{n=M} \sin(n\theta_0/2) e^{-in\pi/2} \{nH_n^{(1)\prime}(ka)\}^{-1} \cos n\theta] \tag{17}$$

The factor in front of the first square bracket is a complex constant independent of θ. Denoting the complex quantity in the square brackets by $\delta + i\epsilon$, and taking the absolute value of both sides of (17), the following expression is obtained for E_θ which defines the radiation pattern

$$E_\theta \sim [\delta + i\epsilon]; \qquad |E_\theta| \sim (\delta^2 + \epsilon^2)^{1/2} \tag{18}$$

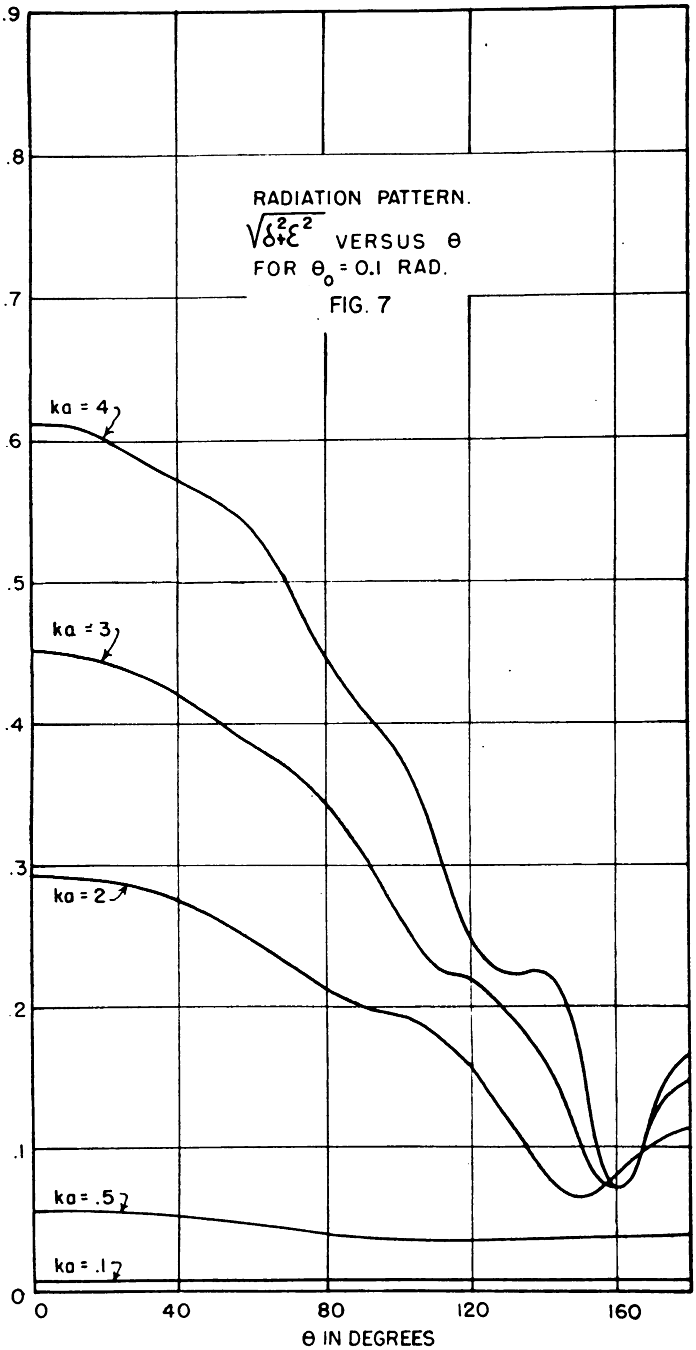


8. Computations. For suitably chosen values of θ_0 and ka, α and β were computed using tables of Hankel functions [3] for orders $n = 0$ to $n = 20$. For n greater than 20, use was made of the asymptotic form

$$\frac{H_n^{(1)}(ka)}{H_n^{(1)\prime}(ka)} \to -\left(\frac{ka}{n}\right), \tag{19}$$

which holds for $(ka)^2 \ll 2n + 1$. Forty terms were used in the calculation.

9. Results. The current density on the periphery of cylinders of various sizes ($ka = 1/10, 1/2, 2, 3$, and 4) and for two sizes of gap ($\theta_0 = 1/10$ and $1/20$) were calculated


RADIATION PATTERN.
$\sqrt{\delta^2+\varepsilon^2}$ VERSUS θ
FOR θ_0 = 0.1 RAD.
FIG. 7
ka = 4
ka = 3
ka = 2
ka = .5
ka = .1
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
0
40
80
120
160
θ IN DEGREES

using (13). The results are in Figs. 1-6. Figures 1, 2 and 3 show plots of α, β and $(\alpha^2 + \beta^2)^{1/2}$ as functions of the angle θ for $\theta_0 = 1/10$. Note that α and β are proportional, respectively, to the real and imaginary parts of the surface-current density J_θ ; $(\alpha^2 + \beta^2)^{1/2}$ is proportional to the magnitude $| J_\theta |$. Figures 4, 5, and 6 show corresponding plots of α, β, and $(\alpha^2 + \beta^2)^{1/2}$ for $\theta_0 = 1/20$.

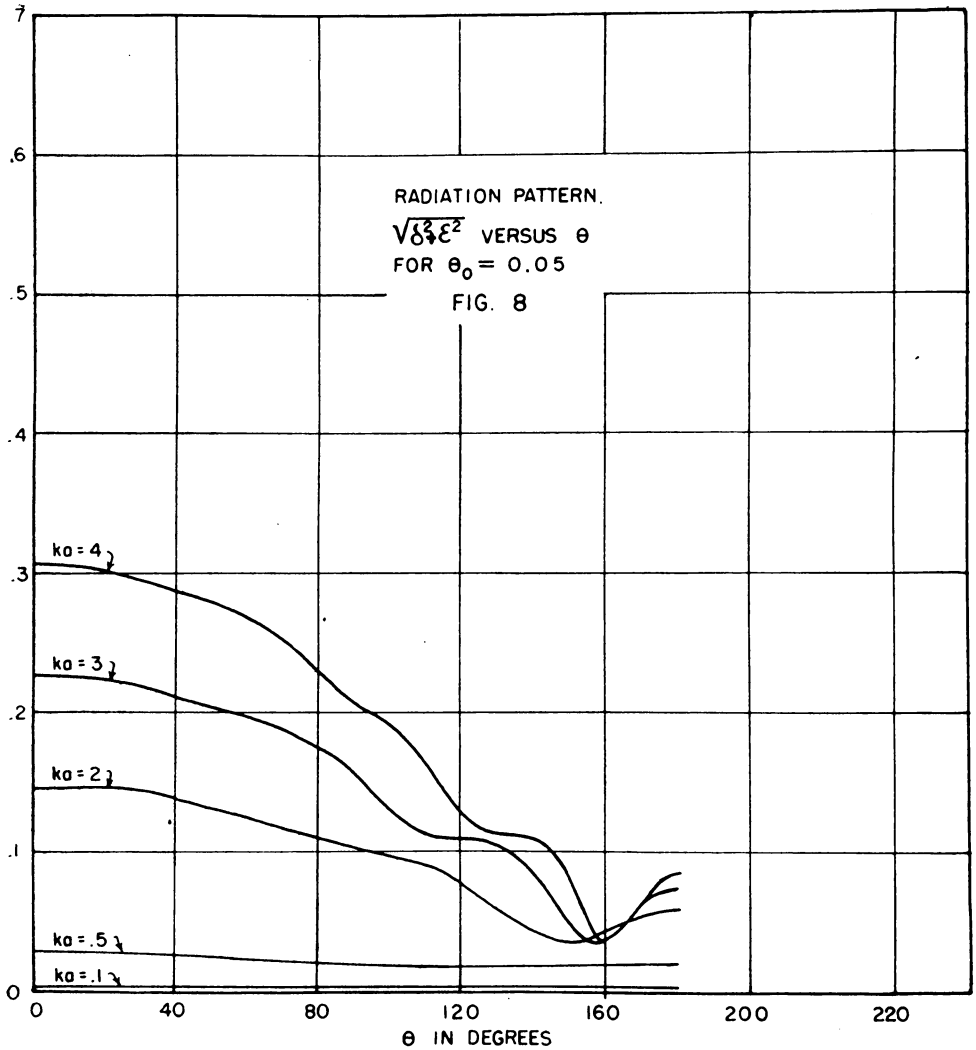


FIG. 8

For $ka = 1/10$, the current density about the periphery of the cylinder is almost constant. For $ka = 1/2$, 2, 3, and 4 there is a standing-wave pattern about the periphery. In all the cases considered, the current near the gap is larger than the current near $\theta = 180°$. By narrowing the gap from $\theta = 1/10$ to $\theta_0 = 1/20$, the overall level of current density was decreased, since reducing the size of the gap is essentially tantamount to decreasing the strength of the source.

The radiation patterns for the two gap sizes $\theta_0 = 0.1$ and $\theta_0 = 0.05$ (Figs. 7 and 8) show that the radius of the cylinder has a marked effect. For $ka = 0.1$, the pattern is uniform. With increasing values of ka the patterns become nonuniform. This is to be expected since when ka is small the current distribution has a uniform amplitude about the periphery of the cylinder, whereas when ka is increased, standing waves obtain.

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References

1. A. Sommerfeld in Riemann-Weber, *Differentialgleichungen der Physik*, Vol. II, 7th ed., 1927, pp. 507-511.
2. J. A. Stratton, *Electromagnetic theory*, McGraw-Hill Book Company, 1941, pp. 524-525.
3. *Scattering and radiation from circular cylinders and spheres*, Math. Tables Project and M. I. T. Underwater Sound Laboratory, published by O. R. I., Navy Department.