GENERATION OF SURFACE WAVES BY A MOVING PARTITION*

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Artificial waves in water are frequently generated by imparting rhythmic motion to a boundary of some sort. Two cases of wave production in deep water have been treated, for harmonic motion only, by Havelock.[1] The more interesting of these cases will be treated here by a different method, and expressions will be obtained for the initial phases as well as for the permanent regime; in addition, the solution will be extended to liquids of limited depth.

The harmonic case requires, for the most part, another two-dimensional solution of the Laplace equation, subject to a special set of boundary conditions. Besides the familiar condition at the free surface expressing the action of gravity, and perhaps a condition on a parallel bottom, the horizontal motion is here prescribed over a vertical boundary. The functions found are interesting in that they constitute an infinite variety of characteristic solutions, all asymptotic at infinity to the same functional form. This asymptotic form is the sine function that is familiar in the treatment of gravity waves and represents the only bounded solution when the free surface is unlimited in extent.

I. Case of Infinite Depth

Consider a semi-infinite body of incompressible, frictionless liquid having a free surface and of infinite depth. Let it be limited at the left by a boundary that executes infinitesimal displacements from an initial vertical plane position. Initially at time $t = 0$, let the liquid be at rest with a horizontal free surface.

Draw the y-axis downward in the initial plane of the vertical boundary and the x-axis perpendicular to this plane and in the horizontal plane that is occupied initially by the free surface. Then the motion excited by the motion of the boundary will be two-dimensional, occurring in planes parallel to the xy-plane, so that nothing is a function of z.

Let the infinitesimal horizontal displacement s of a point of the boundary at time t be

$$s = SF(y, t), \tag{1}$$

where S is a constant having the dimensions of length. Thus the function $F(y, t)$ is dimensionless. Initially, $F(y, 0) = 0$. It will be assumed that the function $F(y, t)$ is sufficiently smooth and vanishes sufficiently rapidly as $y \to \infty$ to justify the operations that are to be performed.

Since the liquid is assumed frictionless and incompressible, there will be a velocity potential satisfying Laplace's equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

A solution is required for $x \geqq 0$, $y \geqq 0$, subject to two boundary conditions. Over the

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[1]Havelock, Phil. Mag. 8, 569 (1929).

plane of the moving boundary the vertical component of velocity of the water is unrestricted, but the horizontal component must be the same as that of the boundary. Hence for $x = 0$,

$$v_x = -\frac{\partial \phi}{\partial x} = \frac{\partial s}{\partial t} = SF_t(y, t). \tag{2}$$

At the free surface the boundary condition for infinitesimal motions may be written[2] in terms of the elevation η of the surface:

$$\eta = \frac{1}{g}\left(\frac{\partial \phi}{\partial t}\right)_{y=0} \quad \text{(3a)}, \qquad \frac{\partial \eta}{\partial t} = \left(\frac{\partial \phi}{\partial y}\right)_{y=0}, \quad \text{(3b)}$$

where g is the acceleration due to gravity.

A solution of the differential equation satisfying these boundary conditions will be constructed in two steps. For the moment, imagine gravity to be absent. Then, since the displacement of the boundary remains small, the motion of the liquid can be regarded as due to a suitable distribution of horizontal line sources along the positive y-axis, continued above the free surface along negative y as negative image sources in order to keep the pressure constant on the free surface, or in the xz-plane. The velocity potential due to a line source located at $(0, \pm\beta)$ can be written $-A \log r^2 = -A \log [x^2 - (y \mp \beta)^2]$, where A is a constant and the flux toward one side is $2\pi A$; the desired flux from dy, however, is $v_x dy = SF_t(y, t)\, dy$ by (2). It is thus found that the desired velocity potential ϕ_1 and the corresponding horizontal velocity component v_{x1} are

$$\phi_1 = \frac{S}{2\pi}\int_0^\infty F_t(\beta, t) \log \frac{x^2 + (\beta + y)^2}{x^2 + (\beta - y)^2}\, d\beta, \tag{4}$$

$$v_{x1} = -\frac{\partial \phi_1}{\partial x} = \frac{S}{\pi}\int_0^\infty F_t(\beta, t)\left[\frac{x}{x^2 + (\beta - y)^2} - \frac{x}{x^2 + (\beta + y)^2}\right] d\beta,$$

where $F_t(\beta, t) = \partial F(\beta, t)/\partial t$.

It is readily seen that ϕ_1 is a solution of Laplace's equation and that $\partial\phi_1/\partial t = 0$ when $y = 0$. Consequently, the pressure remains zero at $y = 0$, as it must for infinitesimal motions in the absence of gravity. As $x \to 0$, the second term in the expression for v_{x1} vanishes. In the first term, however, the integrand is small except near $y = \beta$, so that in the limit as $x \to 0$, $F_t(\beta, t)$ may be replaced by $F_t(y, t)$ and the lower limit of the integral may be replaced by $-\infty$. Thus

$$\lim_{x \to 0} v_{x1} = \frac{S}{\pi} F_t(y, t) \int_{-\infty}^\infty \frac{x\, d\beta}{x^2 + (\beta - y)^2} = SF_t(y, t),$$

as is required by (2).

The corresponding upward velocity of the surface of the liquid is

$$v_{n1} = \left(\frac{\partial \phi_1}{\partial y}\right)_{y=0} = \frac{2S}{\pi}\int_0^\infty F_t(\beta, t) \frac{\beta\, d\beta}{x^2 + \beta^2}\,. \tag{5}$$

The effect of gravity may now be introduced by the following device. During each element of time dt the motion already assumed adds an increment to the elevation of

[2]Lamb, *Hydrodynamics*, Cambridge University Press; 6 ed., Sec. 227.

the surface represented by $v_{n1}\,dt$. In the absence of gravity these increments remain superposed upon each other without alteration. Under gravity, on the other hand, each increment undergoes transformation in the manner characteristic of an "initial elevation" of the same magnitude (the relevant theory is given in Lamb's *Hydrodynamics*,[3] y being there measured upward). The usual Fourier integral may be omitted here, however, since it may be verified by carrying out the integration in k that (5) can be written

$$v_{n1} = \left(\frac{\partial \phi_1}{\partial y}\right)_{y=0} = \frac{2S}{\pi} \int_0^\infty \cos kx\, dk \int_0^\infty F_t(\beta,\, t)\, e^{-k\beta}\, d\beta. \tag{6}$$

Now a standing wave with a surface elevation $\eta = \cos \sigma\, t \cos kx$, where $\sigma^2 = gk$, has an associated potential $g(\sin \sigma t/\sigma)e^{-ky} \cos kx$ (cf. Lamb, Sec. 238). Hence the potential for an oscillation reducing at time $t = t'$ to a surface elevation $v_{n1}\,dt'$ is, using (6),

$$\begin{aligned} d\phi &= \frac{2}{\pi}\, g\, S\, dt' \int_0^\infty \sigma^{-1} \sin \sigma(t - t')\, e^{-ky} \cos kx\, Q(k,\, t')\, dk, \\ Q(k,\, t) &= \int_0^\infty F_t(\beta,\, t)\, e^{-k\beta}\, d\beta. \end{aligned} \tag{7}$$

The total potential ϕ is then found by integrating (7) with respect to t', and adding ϕ_1 :

$$\phi = \phi_1 + \frac{2}{\pi}\, g\, S \int_0^t dt' \int_0^\infty \sigma^{-1} \sin \sigma(t - t')\, e^{-ky} \cos kx\, Q(k,\, t')\, dk. \tag{8}$$

The corresponding surface elevation is, from (3a), since $\partial\phi_1/\partial t = 0$ at $y = 0$,

$$\eta = \frac{2S}{\pi} \int_0^t dt' \int_0^\infty \cos \sigma(t - t') \cos kx\, Q(k,\, t')\, dk. \tag{9}$$

It is easily verified that ϕ as given by (8) satisfies Laplace's equation and the required boundary conditions (2) and (3b). At $x = 0$, $\partial\phi/\partial x = \partial\phi_1/\partial x$ and (2) is satisfied; in $\partial\eta/\partial t$, the term obtained from the upper limit equals $\partial\phi_1/\partial y$ as expressed in (6), and $gk = \sigma^2$.

These equations represent the motion caused by an arbitrary motion of the vertical boundary. In the harmonic case the expressions become simpler.

1. The harmonic case. As a special case, let the displacement of the vertical boundary after $t = 0$ be

$$s = SF(y,\, t) = Sf(y) \sin \omega t, \tag{10}$$

so that $F_t(y,\, t) = \omega f(y) \cos \omega t$. Then, replacing t' by $\tau = t - t'$ as the variable of integration, (8), (9) and (7) are replaced by

$$\begin{aligned} \phi = \phi_1 + \frac{2}{\pi}\, \omega g S \int_0^t d\tau \int_0^\infty &(\cos \omega t \cos \omega\tau + \sin \omega t \sin \omega\tau) \\ &\times \sigma^{-1} \sin \sigma\tau\, e^{-ky} \cos kx\, K(k)\, dk, \end{aligned} \tag{11}$$

[3] Lamb, *Hydrodynamics*, Cambridge University Press, 6 ed., Sec. 238.

$$\eta = \frac{2}{\pi}\omega S \int_0^t d\tau \int_0^\infty (\cos \omega t \cos \omega\tau + \sin \omega t \sin \omega\tau) \times \cos \sigma\tau \cos kx\, K(k)\, dk, \tag{12}$$

$$K(k) = \int_0^\infty e^{-k\beta} f(\beta)\, d\beta. \tag{13}$$

These expressions represent a non-harmonic motion of the water, caused by a harmonic motion of the boundary after an initial state of complete rest. As $t \to \infty$, however, the integrals representing the coefficients of $\sin \omega t$ and $\cos \omega t$ will approach fixed limits, provided $f(y)$ satisfies the usual restrictions. Thus, the entire motion becomes more and more nearly harmonic.

There must exist, therefore, an ideal solution representing exactly harmonic motion, in which (10) holds at all times. To obtain the corresponding amplitude functions in ϕ, which may be written

$$\phi = \phi_1 + \psi_1(x, y) \cos \omega t + \psi_2(x, y) \sin \omega t,$$

we carry out the integrations in τ and take limits as $t \to \infty$. The amplitude functions thus found are:

$$\psi_1(x, y) = \lim_{t\to\infty} \frac{\omega g S}{\pi} \int_0^\infty \left(\frac{1 - \cos(\sigma + \omega)t}{\sigma + \omega} + \frac{1 - \cos(\sigma - \omega)t}{\sigma - \omega}\right) G\, dk + \frac{\phi_1}{\cos \omega t},$$

$$\psi_2(x, y) = \lim_{t\to\infty} \frac{\omega g S}{\pi} \int_0^\infty \left(-\frac{\sin(\sigma + \omega)t}{\sigma + \omega} + \frac{\sin(\sigma - \omega)t}{\sigma - \omega}\right) G\, dk,$$

$$G = \sigma^{-1} e^{-ky} \cos kx\, K(k).$$

The values of the limits are obtained at once from the formulas:

$$\lim_{t\to\infty} \int_a^b f(x) \frac{\sin tx}{x}\, dx = \pi f(0) \qquad \text{or} \qquad 0 \tag{14}$$

according as either $a < 0 < b$ or a and b have the same sign;

$$\lim_{t\to\infty} \int_{-a}^a f(x) \frac{1 - \cos tx}{x}\, dx = \int_{-a}^a f(x) \frac{dx}{x}, \tag{15}$$

where the principal value of the last integral is taken.

These formulas can be reduced to the familiar one, proved in many books:[4]

$$\lim_{t\to\infty} \int_a^b f(x) \sin tx\, dx = 0.$$

In (15), for example, the left-hand member can be written, because of the symmetry of $(1 - \cos tx)$

$$\lim_{t\to\infty} \frac{1}{2} \int_{-a}^a \frac{f(x) - f(-x)}{x} (1 - \cos tx)\, dx.$$

[4]Whittaker and Watson, *Modern Analysis*, Cambridge University Press, 3rd ed., Sec. 9.41.

With mild restrictions upon $f(x)$, the first fraction here will be finite at $x = 0$, and the $\cos tx$ term will give zero in the limit; for the remainder, note that

$$\int_{-a}^{a} f(x) \frac{dx}{x} = - \int_{-a}^{a} f(-x) \frac{dx}{x}\,.$$

The integral in formula (14) when $a < 0 < b$ can be reduced in an analogous manner to the integral,

$$\int_{-\infty}^{\infty} \sin tx\, dx/x = \pi \qquad \text{for} \qquad t > 0.$$

Using these formulas, the cosine terms in ψ_1 and the $(\sigma + \omega)$ term in ψ_2 are zero, whereas the $(\sigma - \omega)$ term in ψ_2 can be evaluated after changing from dk to $d(\sigma - \omega)$; since $\sigma^2 = gk$, $dk = 2\sigma\, d\sigma/g$. Thus (11) becomes, using (4) and (10),

$$\begin{aligned} \phi = 2\omega S\biggl[\frac{1}{4\pi} \int_0^{\infty} f(\beta) \log \frac{x^2 + (\beta + y)^2}{x^2 + (\beta - y)^2}\, d\beta \\ + \frac{1}{\pi} \int_0^{\infty} \frac{dk}{k - k_0} e^{-ky} \cos kx\, K(k)\, dk\biggr] \cos \omega t \\ + 2\omega S \exp(-k_0 y) \cos k_0 x\, K(k_0) \sin \omega t, \end{aligned} \tag{16}$$

where

$$k_0 = \omega^2/g. \tag{17}$$

Nearly the same integrals occur in (12), which becomes

$$\eta = \frac{2\omega^2 S}{g}\left[\cos k_0 x\, K(k_0) \cos \omega t - \frac{1}{\pi} \sin \omega t \int_0^{\infty} \frac{dk}{k - k_0} \cos kx\, K(k)\, dk\right]. \tag{18}$$

The boundary conditions (2) and (3b) are again easily verified. In $\partial\phi/\partial y$, $-k/(k - k_0) = -1 - k_0/(k - k_0) = -1 - \omega^2/g(k - k_0)$; the first term serves to cancel the contribution from the first integral when $y = 0$ because of (4) and (6).

Equations (16) and (18) represent exactly harmonic motion, but they may also be used at a given point as approximate expressions for the motion after a start from rest, provided sufficient time has elapsed. It is necessary that, as σ varies past the value ω in the integrations in (11) and (12), $\sin \sigma t$ or $\cos \sigma t$ shall execute many periods while $\cos kx$ varies little from $\cos k_0 x$. Consequently, a good approximation to the limits as $t \to \infty$ will be obtained. This requires that

$$\omega t \gg k_0 x = \omega^2 x/g = 2\pi x/\lambda$$

where $\lambda = 2\pi g/\omega^2$ represents the wave length of small traveling waves of frequency $\omega/2\pi$. In other words, the total number of waves emitted from the boundary since the start, as represented by $\omega t/2\pi$, must greatly exceed x/λ or the number of waves included between the boundary and the point x.

On the other hand, as x increases beyond the first few wave lengths from the boundary, the exactly harmonic motion approximates more and more closely simple traveling waves having the frequency of the boundary. We can show this by transforming the remaining integrals in k.

Putting $\zeta = k + im$ and integrating around the first quadrant of the ζ-plane, indented at $x = k_0$, we find that

$$\oint \frac{\exp(-\gamma\zeta + i\alpha\zeta)}{\zeta - k_0}\,d\zeta = \int_0^\infty \frac{\exp(-\gamma k + i\alpha k)}{k - k_0}\,dk$$

$$- i\pi \exp(-\gamma k_0 + i\alpha k_0) - i\int_0^\infty \frac{\exp(-i\gamma m - \alpha m)}{im - k_0}\,dm = 0$$

for $\gamma > 0$ and $\alpha > 0$; the principal value of the k integral is intended. The last integral can also be written

$$-\int_0^\infty \frac{im + k_0}{m^2 + k_0^2}\exp(-i\gamma m - \alpha m)\,dm.$$

Hence, taking real parts, we obtain the formula,

$$\int_0^\infty \frac{dk}{k - k_0} e^{-\gamma k}\cos\alpha k\,dk = -\pi\exp(-\gamma k_0)\sin\alpha k_0$$

$$+ \int_0^\infty \frac{m\cos\gamma m - k_0\sin\gamma m}{m^2 + k_0^2} e^{-\alpha m}\,dm.$$

For our present purpose, change m to k, set $\alpha = x$ and $\gamma = \beta + y$ in ϕ or $\gamma = \beta$ in η; the factor $e^{-\beta k}$ is to be found in the integral for K, as given in (13). It is then found from (16), (18), (13), that

$$\begin{aligned}\phi = {} & 2\omega S\,K(k_0)\exp(-k_0 y)\sin(\omega t - k_0 x) \\ & + \frac{2}{\pi}\omega S\cos\omega t\int_0^\infty f(\beta)\,d\beta\left[\frac{1}{4}\log\frac{x^2 + (\beta + y)^2}{x^2 + (\beta - y)^2}\right. \\ & \left. + \int_0^\infty \frac{k\cos k(\beta + y) - k_0\sin k(\beta + y)}{k^2 + k_0^2} e^{-kx}\,dk\right],\end{aligned} \tag{19}$$

$$\begin{aligned}\eta = \frac{2\omega^2 S}{g}\Bigg[& K(k_0)\cos(\omega t - k_0 x) \\ & - \frac{1}{\pi}\sin\omega t\int_0^\infty f(\beta)\,d\beta\int_0^\infty \frac{k\cos\beta k - k_0\sin\beta k}{k^2 + k_0^2} e^{-kx}\,dk\Bigg].\end{aligned} \tag{20}$$

The first term in (19) or (20) represents a train of harmonic waves traveling outward from the moving vertical boundary. In terms of the wave length, $\lambda = 2\pi/k_0 = 2\pi g/\omega^2$, the surface amplitude of these waves is, from (20) and (13),

$$A = \frac{4\pi S}{\lambda}\int_0^\infty \exp(-2\pi y/\lambda)\,f(y)\,dy. \tag{21}$$

The second term in (19) or (20) represents a local disturbance near the wave-making boundary, superposed upon the waves in time quadrature with them. The resultant disturbance near the boundary in the exactly harmonic motion will thus have an amplitude in general larger than that of the distant waves.

At the boundary itself, the expression for the surface amplitude in the exactly harmonic motion becomes simpler. With $x = 0$, in terms of $k' = k - k_0$, the integral in (18) becomes, using (13),

$$\int_0^\infty f(\beta)\, d\beta \int_{-k_0}^\infty \frac{dk'}{k'} \exp\left[-\beta(k_0 + k')\right] = -\int_0^\infty \exp\left(-\beta k_0\right) Ei(\beta k_0) f(\beta)\, d\beta,$$

$$Ei(x) = \int_{-\infty}^x \frac{e^t}{t}\, dt = -\int_{-x}^\infty \frac{e^{-t}}{t}\, dt.$$

Hence at $x = 0$, in terms of $\lambda = 2\pi g/\omega^2$, from (18) and (13),

$$\eta = \frac{4\pi S}{\lambda} \int_0^\infty \left[\cos \omega t + \frac{1}{\pi} Ei(k_0 y) \sin \omega t\right] \exp\left(-k_0 y\right) f(y)\, dy. \tag{22}$$

Here, as $y \to 0$, $Ei(k_0 y)$ becomes logarithmically infinite; nevertheless, if $f(0+)$ is absolutely bounded and integrable, the integral will converge.

The pressure on the boundary is also obtainable in closed form, for the case of exactly harmonic motion. It is $p = \rho(\partial\phi/\partial t)_{x=0}$. For the evaluation of the integral in (19), the following formula is easily inferred from formulas given in Bierens de Haan's table of integrals:

$$\int_0^\infty \frac{k \cos pk - q \sin pk}{k^2 + q^2}\, dk = -e^{-pq}\, Ei(pq),$$

where p and q are constants. For the present case, take $q = k_0$, $p = \beta + y$. The pressure at depth y is thus found to be, using (13),

$$\begin{aligned} p = {} & 2\omega^2 \rho S \cos \omega t \exp\left(-k_0 y\right) \int_0^\infty \exp\left(-k_0 \beta\right) f(\beta)\, d\beta \\ & - \frac{2\omega^2 \rho S}{\pi} \sin \omega t \int_0^\infty f(\beta) \left\{ \frac{1}{4} \log \frac{(\beta + y)^2}{(\beta - y)^2} \right. \\ & \left. - \exp\left[-k_0(\beta + y)\right] Ei\left[k_0(\beta + y)\right] \right\} d\beta. \end{aligned} \tag{23}$$

The second term in p is a quadrature component that, on the whole, does no work on the boundary; the first term provides the energy carried away by the waves.

Any motion consistent with the assumed harmonic boundary conditions can be resolved into the motion already described and a complementary component satisfying the condition of zero horizontal velocity over the vertical plane $x = 0$.

2. The pure-wave case. If $f(y) = \exp\left(-k_0 y\right)$, the boundary moves as does a vertical layer of fluid particles during the passage of harmonic waves; the entire motion must then reduce to that of the wave train.

Now, if $f(y) = \exp\left\{-2\pi y/\lambda\right\}$, (21) gives at once the amplitude $A = S$, which is correct, since in deep waves the vertical and horizontal amplitudes are equal. That the local disturbance vanishes completely in this case is most easily seen if the corresponding term in ϕ is put into Havelock's form.

The first of the following equations is easily obtained, and integration of it with respect to α yields the second:

$$\int_0^\infty e^{-\alpha x} (\cos px - \cos qx)\, dx = \frac{\alpha}{\alpha^2 + p^2} - \frac{\alpha}{\alpha^2 + q^2}\,;$$

$$\int_0^\infty \frac{\cos px - \cos qx}{x} e^{-\alpha x}\, dx = \frac{1}{2} \log \frac{\alpha^2 + q^2}{\alpha^2 + p^2}\,.$$

The constant of integration in the latter formula must be zero since both members of the equation vanish as $\alpha \to \infty$.

Using this last formula with $x = k$, $\alpha = x$, $p = \beta - y$ and $q = \beta + y$ to transform the logarithm in (19), we obtain as the coefficient of $f(\beta)d\beta$ in (19),

$$\int_0^\infty \left[\frac{1}{2} \frac{\cos (\beta - y)k - \cos (\beta + y)k}{k} + \frac{k \cos k(\beta + y) - k_0 \sin k(\beta + y)}{k^2 + k_0^2}\right] e^{-kx}\, dk,$$

and, after consolidating, the integral in β in (19) becomes

$$\int_0^\infty f(\beta)\, d\beta \int_0^\infty (k \cos \beta k - k_0 \sin \beta k) \frac{k \cos ky - k_0 \sin ky}{k(k^2 + k_0^2)} e^{-kx}\, dk.$$

The entire expression for ϕ in (19) then agrees with Havelock's expression except for a change of notation. Furthermore, it is easily verified, by carrying out the β integration, that if $f(y) = \exp(-k_0 y)$, the expression just written vanishes. The right-hand member of (19) thus reduces in this case to the first term, representing the waves. In the same way the $\sin \omega t$ term in (20) disappears.

II. Case of Finite Depth

Let it now be assumed that the liquid has only a finite depth h. Nothing essentially new is introduced by this limitation. The principal formulas will accordingly be written down with a minimum of explanation.

In using the source method, infinite trains of line images are now required in order to preserve the boundary condition both on the bottom and on the free surface. The sources associated with an element dy of the boundary at a depth $y = \beta$ fall into two series, beginning respectively at $y = \pm\beta$; in each series the spacing is $2h$ and the signs alternate, since reflection in the free surface reverses the sign whereas reflection in the bottom does not. This alternation of sign suggests use of the complex potential $C \log \tan az$, whose real part, if C and a are real, is $(C/2)$ [$\log (\cosh 2ay - \cos 2ax) - \log (\cosh 2ay + \cos 2ax)$]. The proper periodicity is obtained if $a = \pi/4h$; and for small x this expression reduces, except for an added constant, to $(C/2) \log (x^2 + y^2)$ or to the potential of a line source at the origin. For one train of sources, y is replaced by $(y - \beta)$, for the other, by $(y + \beta)$. It is thus found that, for the motion in the absence of gravity, (2) and (3) representing the potential and the corresponding upward surface velocity are replaced by

$$\phi_1' = \frac{S}{2\pi} \int_0^h F_t(\beta, t) \log \left(\frac{\cosh bx - \cos b(\beta + y)}{\cosh bx + \cos b(\beta + y)} \frac{\cosh bx + \cos b(\beta - y)}{\cosh bx - \cos b(\beta - y)}\right) d\beta, \tag{24}$$

$$v_{n1}' = \left(\frac{\partial \phi_1'}{\partial y}\right)_{y=0} = \frac{S}{h} \cosh bx \int_0^h \frac{F_t(\beta, t) \sin b\beta}{\sinh^2 bx + \sin^2 b\beta}\, d\beta, \tag{25}$$

with

$$b = \frac{\pi}{2h}.$$

The expression for v'_{n1} is then replaced by a Fourier integral:

$$v'_{n1}(x, t) = \frac{2}{\pi} \int_0^\infty \cos kx \, dk \int_0^\infty v'_{n1}(\alpha, t) \cos k\alpha \, d\alpha. \tag{26}$$

The formula needed to reduce this integral can be obtained by integrating $e^{ikz} \cosh bz(\sinh^2 bz + \sin^2 b\beta)^{-1} dz$ around the upper half plane for $z = x + iy$. Poles occur on the imaginary axis where $\sin by = \pm \sin b\beta$, or $y = \beta + n\pi/b$, $y = -\beta + (n + 1)\pi/b$, where $n = 0, 1, 2, \cdots$. The sum of the residues thus leads to a series, namely, $1 - \exp\{-k\pi/b\} + \exp\{-2k\pi/b\} \cdots = (1 - \exp\{-k\pi/b\})^{-1}$. The real part of the integral is thus found to yield the formula

$$\int_0^\infty \frac{\cos kx \cosh bx}{\sinh^2 bx + \sin^2 b\beta} dx = \frac{\pi}{2\, b \sin b\beta} \left[e^{-k\beta} + \frac{2 \sinh k\beta}{e^{\pi k/b} + 1} \right]. \tag{27}$$

By means of this formula, (26) with $v'_{n1}(\alpha, t)$ inserted from (25) reduces to

$$v'_{n1} = \frac{2S}{\pi} \int_0^\pi \cos kx \, Q'(k, t) \, dk, \qquad Q'(k, t) = \int_0^h F_t(\beta, t) \left(e^{-k\beta} + \frac{2 \sinh k\beta}{e^{2kh} + 1} \right) d\beta.$$

This equation for v'_{n1} replaces (6). With the usual modifications to fit the finite depth, the former procedure then gives for the total velocity potential ϕ' and surface elevation η' in place of (8) and (9),

$$\phi' = \phi'_1 + \frac{2gS}{\pi} \int_0^t dt' \int_0^\infty \sigma^{-1} \sin \sigma(t - t') \frac{\cosh k(h - y)}{\cosh kh} \cos kx \, Q'(k, t') \, dk, \tag{28}$$

$$\eta' = \frac{2S}{\pi} \int_0^t dt' \int_0^\infty \cos \sigma(t - t') \cos kx \, Q'(k, t') \, dk, \tag{29}$$

where now

$$\sigma^2 = gk \tanh kh.$$

The boundary conditions are easily verified, including the new one that $\partial\phi/\partial y = 0$ when $y = h$.

3. The harmonic case. The treatment of this case then proceeds as before. It is found that, in the exactly harmonic case where the displacement of the boundary is at all times $s = Sf(y) \cos \omega t$, instead of (16) and (18),

$$\begin{aligned} \phi' = \phi'_1 + 2\omega S \Bigg\{ & \frac{\cosh k_1(h - y)}{\cosh k_1 h} \frac{K'(k_1)}{\tanh k_1 h + k_1 h \operatorname{sech}^2 k_1 h} \sin \omega t \cos k_1 x \\ & + \frac{\cos \omega t}{\pi} \int_0^\infty \frac{\cosh k(h - y)}{\cosh kh} \cos kx \frac{K'(k) \, dk}{k \tanh kh - k_1 \tanh k_1 h} \Bigg\} \end{aligned} \tag{30}$$

where k_1 is the positive real root of the equation

$$\omega^2 = gk_1 \tanh k_1 h \tag{31}$$

and

$$K'(k) = \int_0^h f(\beta)\left[e^{-k\beta} + \frac{2\sinh k\beta}{e^{2kh}+1}\right] d\beta, \tag{32}$$

$$\begin{aligned}\eta' = \frac{2\omega^2 S}{g}\Bigg\{&\cos\omega t\,\cos k_1 x\,\frac{K'(k_1)}{\tanh k_1 h + k_1 h\,\mathrm{sech}^2\, k_1 h}\\ &- \frac{\sin\omega t}{\pi}\int_0^\infty \cos kx\,\frac{K'(k)\,dk}{k\tanh kh - k_1\tanh k_1 h}\Bigg\}.\end{aligned} \tag{33}$$

In deducing (30) from (28), an equation is first obtained that is identical with (11) except that ϕ and ϕ_1 are replaced by ϕ' and ϕ_1', $K(k)$ by $K'(k)$ as given in (32) above, and e^{-ky} by $\cosh k(h - y)/\cosh kh$. With the same changes, the expressions given under Eq. (11) for $\psi_1(x, y)$ and $\psi_2(x, y)$ become the amplitude functions ψ_1' and ψ_2' for ϕ', such that $\phi' = \phi_1' + \psi_1'(x, y)\cos\omega t + \psi_2'(x, y)\sin\omega t$. In the further reduction, however, k_1 replaces k_0, and the relation $\sigma^2 - \omega^2 = g(k - k_0)$ is replaced by $\sigma^2 - \omega^2 = g(k\tanh kh - k_1\tanh k_1 h)$, so that the latter expression in parentheses replaces $(k - k_0)$; furthermore, the relation $dk = 2\sigma\, d\sigma/g$ is replaced by $dk = 2\sigma\, d\sigma/g(\tanh kh + kh\,\mathrm{sech}^2\, kh)$, so that at $k = k_1$ an additional factor $(\tanh k_1 h + k_1 h\,\mathrm{sech}^2\, k_1 h)$ is introduced. The differences between (30) and (16) are thus explained. The expression for η' is then easily obtained from (3a).

The waves at large x can be discovered as before by transforming the remaining integrals by contour integration. Here, the expression $k\tanh kh - k_1\tanh k_1 h$ appears in (30) and (33) where $(k - k_0)$ appears in (16) and (18); hence at the pole, which occurs here at $k = k_1$, the following additional factor is obtained in the denominator:

$$\left[\frac{d}{dk}(k\tanh kh - k_1\tanh k_1 h)\right]_{k=k_1} = \tanh k_1 h + k_1 h\,\mathrm{sech}^2\, k_1 h.$$

The terms in $\sin\omega t$ thus combine again with the $\cos\omega t$ terms to represent traveling waves at large x. The remaining expressions are complicated, but they involve x only in a factor of the form $\exp(-k'x)$ where k' is real and positive, and so represent again a local disturbance near the boundary.

The surface amplitude A of the waves can be inferred from (33) with (31) and (32):

$$\begin{aligned}A = \frac{4\pi S}{\lambda_1}&(1 + k_1 h\,\mathrm{sech}\, k_1 h\,\mathrm{csch}\, k_1 h)^{-1}\\ &\times\int_0^h f(y)\left[\exp(-k_1 y) + \frac{2\sinh k_1 y}{\exp(2k_1 h) + 1}\right] dy\end{aligned} \tag{34}$$

where $k_1 = 2\pi/\lambda_1$ in terms of the wave length λ_1. As $h \to \infty$, this expression reverts to that for deep water as given in (21).

If, on the other hand, h/λ_1 is small, so is $k_1 y$ throughout the range of integration, and $k_1 h\,\mathrm{sech}\, k_1 h\,\mathrm{csch}\, k_1 h = 1$, nearly; thus, approximately,

$$A = \frac{2\pi S}{\lambda_1}\int_0^h f(y)\,dy. \tag{35}$$

This is easily seen to be the correct expression for canal waves.