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SEPARATION OF LAPLACE'S EQUATION*

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ABSTRACT

The following results are established in this paper:

(I)** For the Laplace equation $\Delta\theta = 0$ in curvilinear co-ordinates (u, v, w) in Euclidean space to be directly separable† into two equations, one for S and one for Z, when the solution is $\theta = R(u, v, w)S(u, v)Z(w)$ with fixed R, it is necessary and sufficient that the surfaces $w =$ constant (1) be orthogonal to the surfaces $u =$ constant, $v =$ constant and (2) be parallel planes, planes with a common axis, concentric spheres, spheres tangent at a common point, or one of the two sets of spheres generated by the co-ordinate lines when bicircular co-ordinates are rotated about the line joining the poles or about its perpendicular bisector.

(II) We have $R = 1$ always and only in the first three cases, namely, when the surfaces $w =$ constant are parallel planes, planes with a common axis, or concentric spheres.

(III) In these three cases, but only these, the wave equation separates in the sense RSZ, and hence, for the wave equation, $R = 1$ automatically.

(IV) For further separation of the equation found above for S, when $S = X(u)Y(v)$ so that the solution is now $RXYZ$, it is necessary and sufficient that the co-ordinates be toroidal, or such that the wave equation so separates, or any inversions of these.

(V) The co-ordinates where the wave equation so separates, that is, admits solutions $RX(u)Y(v)Z(w)$, are only the well-known cases where this happens with $R = 1$, namely, degenerate ellipsoidal or paraboloidal co-ordinates (but see Sec. 8.2).

(VI). In these cases, but only these, $R = 1$ for the Laplace equation too.

(VII) Co-ordinates for RSZ or $RXYZ$ separation of the Laplace equation have the group property under inversion.

(VIII) In all cases R can be found by inspection of the linear element.

INTRODUCTION

0.1. The problem. Separation of variables is one of the simplest and most frequently used methods of solving partial differential equations subject to given boundary conditions. In this method, with which we assume the reader to be familiar, the surface over which the boundary values are specified must coincide with one of the co-ordinate

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†See Sec. 0.2.

surfaces, and hence any restriction on the co-ordinates which can be used is also a restriction on the physical situations to which the method can be applied. It is natural to inquire how we shall determine all co-ordinate systems in which a given equation can be solved by separation of variables. Such is the type of problem with which the present paper is concerned. In a later paper we give a brief history of the subject and a more general discussion of separation methods.

Partly because it includes the wave equation in problems of this type (cf. Sec. 4.3), and partly because the results are still unknown, the emphasis here is on separation of variables in Laplace's equation. We confine the discussion to three dimensions as the case of greatest practical interest. Also this loss of generality, which is believed to be inconsequential, enables us to put the derivation in a form readily followed by the general reader whose interests are not primarily mathematical. Such considerations have weight because the problems treated are of interest to physicists as well as mathematicians.

0.2. The manner of separation. Turning now to the question of the mode of separation, we observe that a more general form than the XYZ generally considered will achieve the objectives desired. Specifically we may assume the form $RXYZ$, where R is a single fixed function[1] of the three co-ordinates: $R = R(u, v, w)$. If X, Y and Z form a complete set [2] and R is known, the operations can be carried out with no increase in complexity. It might be thought that a partial differential equation would have to be solved to get R, but we shall see that this is not the case.

Assuming the form $RXYZ$ rather than XYZ, we have a slightly milder restriction on the solutions and we might expect, therefore, to find a larger class of permissible co-ordinates. Such is indeed the case; with toroidal co-ordinates, for example, the equation is known to separate in the extended sense but not in the restricted sense [11], and the same is true for the so-called Dupin cyclides [7].

A discussion of separation must consider the method of obtaining the solution, as well as its form. In most circumstances the solution is obtained by separating the equation in the literal sense; that is, we multiply through by a fixed function of the co-ordinates to obtain two sets of terms, one set involving w only, for example, and the other involving u and v only. The same procedure is then used for the u, v terms. In the present work we assume not only that the solution has the form $XYZR$, but also that the equation can be separated in this way. Such an assumption is restrictive; in particular we do not obtain the Dupin cyclides even though the solutions have the prescribed form. One of the categories of co-ordinates in [3] is also omitted. In a subsequent paper we consider the general case, with its relation to this and other special modes of obtaining the solutions.

I. SOLUTIONS $R(u, v, w)S(u, v)Z(w)$

SEPARATION OF THE EQUATION

1.0. Laplace's equation, and the second derivative terms. Consider the Laplace operator under a transformation of co-ordinates from (x, y, z) in Euclidean three-

[1]Feshbach has raised the question of co-ordinate systems for which the wave equation separates in this sense.

[2]This means that a sufficiently well-behaved, but otherwise arbitrary, function $f(x, y)$ can be expanded in the form $\Sigma A_{ab}X_aY_b$; similarly for other pairs (X, Z), (Y, Z).

dimensional space to curvilinear co-ordinates (u, v, w). Since the appearance of cross derivatives involving w makes separability impossible, we consider at first the case of orthogonal co-ordinates. It will be found later that the general case of oblique co-ordinates can be easily reduced to this one (Sec. 4.1). In the curvilinear co-ordinates (u, v, w), then, let the linear element be

$$ds^2 = f^2\,du^2 + g^2\,dv^2 + h^2\,dw^2 \tag{1.0}$$

where f, g, and h are functions of u, v, and w. Then [11]

$$\Delta\theta = \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\,\theta_u\right) + \frac{\partial}{\partial v}\left(\frac{hf}{g}\,\theta_v\right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\,\theta_w\right)\right]$$

which becomes, if $\theta = R(u, v, w)S(u, v)Z(w)$,

$$\begin{aligned}\Delta\theta = SZR\Bigg[&\frac{1}{f^2}\frac{S_{uu}}{S} + \frac{1}{g^2}\frac{S_{vv}}{S} + \frac{1}{h^2}\frac{Z''}{Z}\\ &+ F_1\frac{S_u}{S} + G_1\frac{S_v}{S} + H_1\frac{Z'}{Z} + F_2 + G_2 + H_2\Bigg],\end{aligned} \tag{1.1}$$

where

$$F_1 = \frac{2}{f^2}\frac{R_u}{R} + \frac{1}{fgh}\left(\frac{gh}{f}\right)_u \tag{1.2}$$

and similarly for G_1 and H_1. Also

$$F_2 = \frac{1}{f^2}\frac{R_{uu}}{R} + \frac{1}{fgh}\left(\frac{gh}{f}\right)_u\frac{R_u}{R} \tag{1.3}$$

and similarly for G_2 and H_2.

For separability of $\Delta^2\theta = 0$ in the form (1.1) we require that the function Z separate off into an ordinary differential equation. That is, there must exist a function $A^2(u, v, w)$ such that when $\Delta\theta$ is multiplied by A^2, only the variable w appears in the coefficient of Z''/Z and Z'/Z, and w does not appear in the coefficient of any other differentiated terms. Moreover $A^2(F_2 + G_2 + H_2)$ must break up into the sum of a function of w only and a function of u, v only.

Using the condition on the coefficients of the terms involving second derivatives of S and Z, and bearing in mind the definition (1.0) of f, g and h, we see that the linear element must have the form

$$ds^2 = A^2[F^2\,du^2 + G^2\,dv^2 + H^2\,dw^2], \tag{1.4}$$

where F, G are functions of u, v only and H is a function of w only.

1.1. Permissible changes of variable. If we replace w by a function of w, and the u, v co-ordinates by new co-ordinates which still do not involve w, we may put (1.4) in the form

$$ds^2 = A^2[F^2\,(du^2 + dv^2) + dw^2] \tag{1.5}$$

where F is a function of u and v alone. Here we have written A for the new value of A and u, v, w for the new values of u, v and w. In addition, we have used the fact that any element $F^2\,du^2 + G^2\,dv^2$ can be mapped conformally on a plane.

The w transformation is certainly permissible, as it amounts merely to a change of scale for the w co-ordinate. Hence it does not alter the geometrical configuration of the co-ordinate surfaces. When we say that a change of variable is *permissible,* in this connection, we mean that the geometric properties relevant to separation of Laplace's equation are essentially unchanged. Hence proof of the existence of a certain geometric property in the transformed system shows that the same property was present in the original system.

That the change of u, v co-ordinates is permissible in this sense follows from the fact that the new linear element will have the form (1.4). Hence the condition that the equation should separate is still satisfied (note that Eq. (1.8) will also persist in form). Moreover, since every linear element of the form (1.4) corresponds to a triply orthogonal system, we see that the equations used later, (2.0) and (2.1), will continue to be valid in the new system. From these and similar remarks concerning changes of co-ordinates we conclude finally that it suffices to specify the surfaces w = constant. If the equation separates in our sense for a particular system of u and v surfaces orthogonal to these it will separate for every other such system. Conversely, if a change of u and v co-ordinates leads to a certain set of u and v surfaces, and it is then found that the w surfaces must have certain special properties, these properties will in fact persist for every choice of the u or v surfaces. To simplify the analysis these two operations, changing the w scale and changing the u, v co-ordinates, will be repeatedly used in the ensuing discussion.

1.2. The first derivative terms. Turning now to the coefficients of terms involving first derivatives, we use the fact that AH_1 must be a function of w only to find, by virtue of (1.2) and (1.5), that

$$A^2\left[\frac{2}{A^2}\frac{R_w}{R} + \frac{1}{F^2A^3}(F^2A)_w\right] = h_1(w),$$

which simplifies to

$$\frac{2R_w}{R} + \frac{A_w}{A} = h_1(w).$$

We integrate with respect to w, noting that the constant of integration may be an arbitrary function of u and v, and we take the exponential of each side, to find finally

$$R^2A = h_2(w)F_3(u, v). \tag{1.6}$$

Since it is permitted that R involve u, v, and w we may always modify R in such a way that

$$R^2A = 1. \tag{1.7}$$

Thus, the term h_2 in (1.6) may be absorbed in Z and F_3 in S. Equation (1.7) leads to the result: *one may determine R explicitly by putting the linear element in the form* (1.5), *and then taking $R = 1/A^{1/2}$.*

It may be verified that the coefficients of S_u/S and S_v/S are already functions of u and v only, without any new condition (actually they are zero). From separation of the equation, therefore, we have only (1.5), (1.7), and an extra condition,

$$A^2(F_2 + G_2 + H_2) = h_4(w) + F_4(u, v). \tag{1.8}$$

It will be found that (1.8) is a consequence of the Euclidean character of the space.

USE OF PROPERTIES OF EUCLIDEAN SPACE

2.0. The functional form of A. If the space is Euclidean then the linear element (1.0) satisfies the relations [2], [6],

$$f_{uv} - \frac{g_w f_v}{g} - \frac{h_v f_w}{h} = 0, \tag{2.0}$$

$$\left[\frac{g_u}{f}\right]_u + \left[\frac{f_v}{g}\right]_v + \frac{f_w g_w}{h^2} = 0. \tag{2.1}$$

We also have those relations obtained by simultaneous cyclic permutation of (fgh) and (uvw). These equations, which state that the Riemannian curvature of the space is zero, give a necessary and sufficient condition that the co-ordinate system be imbedded in Euclidean space.

In terms of (1.5), the relation (2.0) containing g_{uw} becomes

$$\frac{A_{uw}}{A_u} = 2\frac{A_w}{A} \tag{2.2}$$

after simplification. Integrating with respect to w, as in the derivation of (1.6), we find that $(1/A)_u$ is a function of u and v alone. The same is true of $(1/A)_v$. Thus it follows that $1/A = t(u, v) + H(w)$, or

$$A = 1/[t(u, v) + H(w)]. \tag{2.3}$$

The two relations (2.0) used in the derivation are now satisfied, and in terms of the new linear element the third gives

$$t_{uv} = F_v t_u/F + F_u t_v/F. \tag{2.4}$$

Turning now to the relations (2.1), we find that (2.1) as it stands leads, without detailed calculation, to an equation of the form[3]

$$H'^2 + c_1 H^2 + 2c_2 H + c_3 = 0, \tag{2.5}$$

where the coefficients c are functions of u and v only. Differentiating with respect to w we find that

$$H'(H'' + c_1 H + c_2) = 0. \tag{2.6}$$

Since the solution depends on w only, while the coefficients depend on u and v only, we may give to u and v any constant values to get a linear differential equation for H with constant coefficients. This equation may be solved to give

$$H = 0,\ w,\ w^2,\ e^w,\ \sinh w,\ \cosh w,\ \sin w \tag{2.7}$$

as the only possible values of H that are essentially distinct. Here the case $H' = 0$ has been reduced to $H = 0$ by absorbing the constant value of H in the function t. If H is not zero we substitute back into (2.6) and find that the coefficients, which we already knew were independent of w, are actually constant. This result will be needed later.

In summary, we have found that the linear element must be of the form

$$ds^2 = \frac{F^2(du^2 + dv^2) + dw^2}{(t + H)^2} \tag{2.8}$$

[3]The equation is written explicitly in (2.9).

with t and F functions of u and v only, while H is one of the functions (2.7). The only additional conditions are (1.8), (2.1), and (2.4).

To use the relations (2.1) we substitute for f, g and h their values as given by (2.8). The equation (2.1) as it stands gives

$$\frac{t_{uu} + t_{vv}}{t + H} = \frac{t_u^2 + t_v^2 + F^2H'^2}{(t + H)^2} + \left(\frac{F_u}{F}\right)_u + \left(\frac{F_v}{F}\right)_v \tag{2.9}$$

while the sum of the two others found by permutation leads to

$$t_{uu} + t_{vv} = 2\,\frac{t_u^2 + t_v^2 + F^2H'^2}{t + H} - 2F^2H'' \tag{2.10}$$

and their difference gives

$$t_{uu} - t_{vv} = 2t_u\,\frac{F_u}{F} - 2t_v\,\frac{F_v}{F}\,. \tag{2.11}$$

The system (2.9)-(2.11), slightly simpler than (2.1), is equivalent to it. These three equations and (2.4) give the necessary and sufficient condition that the metric (2.8) be for orthogonal co-ordinates in Euclidean space. The fact that ds^2 has the form (2.8), plus the relation (1.8), on the other hand, gives the necessary and sufficient condition that this co-ordinate system be one in which Laplace's equation separates. Systems satisfying both sets of conditions, and only those, are the ones we are seeking.

2.1. The second fundamental form. It is known from differential geometry [2], [6] that a triply orthogonal system is completely determined, except for its orientation in space, by the linear element ds^2. Hence the second fundamental form (as well as the first) for each of our co-ordinate surfaces is specified by f, g and h of Eq. (1.0) and we might expect to find specific relations from which they could be computed. Such relations actually exist [2], [7]; on the surface w = constant we have

$$-d\bar{x}\cdot d\bar{N} = l\,du^2 + m\,du\,dv + n\,dw^2,$$

where

$$l = -ff_w/h, \qquad m = 0, \qquad n = -gg_w/h; \tag{2.12}$$

the results in other cases are obtained by cyclic permutation of u, v, w and f, g, h.

In particular for the linear element (2.8) we obtain that

$$-d\bar{x}\cdot d\bar{N} = \frac{F^2H'}{(t + H)^2}\,(du^2 + dv^2). \tag{2.13}$$

On the other hand setting $dw^2 = 0$ in (2.8) gives

$$d\bar{x}\cdot d\bar{x} = \frac{F^2}{(t + H)^2}\,(du^2 + dv^2) \tag{2.14}$$

for the first fundamental form. Since the two forms (2.13) and (2.14) are proportional, we know that the surfaces w = constant must be spheres or planes [13]. Computing the Gaussian curvature as the ratio of the two discriminants d^2/D^2 (Ref. [13]) we find the radius of the sphere corresponding to a given value of w:

$$\text{radius} = 1/H'(w). \tag{2.15}$$

This result will frequently be used in the ensuing investigation.

SEPARATE EXAMINATION OF CASES

3.0. The case $H = 0$. For each value of H in (2.7) we obtain a new set of relations for F. These equations are not easy to solve as they stand, and our procedure will be to seek a change of u, v variables that will reduce them to simpler form. By the discussion of Sec. 1.1 we know that any restrictions on the w surfaces obtained after the change must have been valid before it as well.

In case $H = 0$, as assumed here, we see by (2.13) that $l = m = n = 0$ and hence the surfaces $w =$ constant must be planes [13]. On one of these planes let us pick the u, v co-ordinates so that the linear element takes the Cartesian form $du^2 + dv^2$. This is possible, since the surface is a plane. By comparison with (2.8) when $dw^2 = 0$ we see that the present u and v co-ordinates make $F/t = 1$. Also since both t and F are independent of w, we now have $F/t = 1$ for all values of w, not merely for the constant value first selected.

With this procedure Eqs. (2.10) and (2.11) become

$$t_{uu} + t_{vv} = 2(t_u^2 + t_v^2)/t,$$

$$t_{uu} - t_{vv} = 2(t_u^2 - t_v^2)/t,$$

whence, after adding and dividing by t_u,

$$t_{uu}/t_u = 2t_u/t$$

with a similar result for t_{vv}. Integrating twice we find that

$$1/t = uf_1(v) + f_2(v),$$

and a corresponding result with u and v interchanged. The two together show that $1/t$ must have the form $a + bu + cv + duv$, with a, b, c, d constant; Eq. (2.4) tells us that $d = 0$. The remaining conditions (2.9) and (1.8) are now satisfied, so that there is no other restriction.

If $b = c = 0$ the surfaces $w =$ constant represent parallel planes, as we see by comparison with the linear element for Euclidean co-ordinates. If b or c is not zero, however, we have $ds^2 = du^2 + dv^2 + u^2\,dw^2$ or, renaming the variables, $ds^2 = dr^2 + dz^2 + r^2\,d\theta^2$. This shows that the $w =$ constant surfaces are planes with a common axis.

3.1. The case $H = w$. Turning now to the case $H = w$ we find from (2.10) that

$$(t_{uu} + t_{vv})(t + w) = 2(t_u^2 + t_v^2 + F^2).$$

Since this is an identity in w the coefficient of w must vanish. Consequently $t_{uu} + t_{vv} = 0$, and hence also $t_u^2 + t_v^2 + F^2 = 0$. This is impossible since it makes $F = 0$.

3.2. The case $H = w^2$. Next if $H = w^2$ we find that

$$(F_u/F)_u + (F_v/F)_v = 0, \tag{3.1}$$

$$t_{uu} + t_{vv} = 4F^2, \tag{3.2}$$

$$t_{uu} + t_{vv} = (t_u^2 + t_v^2)/t, \tag{3.3}$$

by equating to zero the coefficients of w^4, w^2 and 1 in (2.9). Nothing new is obtained from (2.10), and hence these with (2.4) and (2.11) are the only conditions. Equations (3.1) and (3.3) are equivalent to the statements that log F and log t are harmonic functions.

By simply writing the Gauss equation for curvature in terms of the first fundamental form, we find that for any F satisfying (3.1), this curvature vanishes for the surfaces with $ds^2 = F(du^2 + dv^2)$. These surfaces therefore are developable, and we may introduce a new set of u, v co-ordinates which will make $F = 1$. When this is the case, Eq. (2.4) tells us that $t_{uv} = 0$, so that $t = \alpha(u) + \beta(v)$. Equation (3.2) now reduces to $\alpha'' + \beta'' = 4$, which gives $\alpha'' = 2 + c$, $\beta'' = 2 - c$ with c a constant. It follows that $\alpha = u^2$, $\beta = v^2$, after making a change of variable, if necessary, to eliminate the arbitrary constants. Our linear element now takes the form

$$ds^2 = \frac{du^2 + dv^2 + dw^2}{(u^2 + v^2 + w^2)^2} \tag{3.4}$$

after t has been given its value as determined above. We observe from (7.1) and (7.2) that the linear element (3.4) is the one which would be obtained by inversion of Cartesian co-ordinates in the origin. Because the linear element is sufficient to determine the co-ordinate system completely, as noted in Sec. 2.1, it follows that the co-ordinate surfaces, as well as the linear element, will be the same as those which would be obtained by the inversion described. In particular the surfaces $w =$ constant must consist of a plane and a set of spheres all tangent to it at one point. This result, incidentally, can be obtained directly from (3.4), as in the examples considered below.

3.3. The case $H = e^w$. Turning now to the case $H = e^w$ we substitute in (2.10) and equate to zero the coefficients of 1 and e^w, respectively, to find:

$$t(t_{uu} + t_{vv}) = 2(t_u^2 + t_v^2),$$

$$t_{uu} + t_{vv} = -2F^2 t,$$

which gives

$$t_u^2 + t_v^2 + F^2 t^2 = 0. \tag{3.5}$$

Equation (3.5) implies that t is zero. We have then from (2.8) that

$$ds^2 = \frac{F^2(du^2 + dv^2) + dw^2}{e^{2w}} . \tag{3.6}$$

The other relations are now all satisfied, if F is suitably restricted.

By the second fundamental form the radius of the sphere $w =$ constant is e^{-w}. The distance from $w = \infty$ to the sphere along $u = a$, $v = b$ is also e^{-w}, by Eq. (3.6). Thus the spheres are concentric with $w = \infty$ as the common center.

3.4. The case $H = \sinh w$. When $H = \sinh w$, we substitute into (2.10) as usual, then put everything in terms of $\sinh w$ by using $\cosh^2 w = 1 + \sinh^2 w$, and finally equate to zero the coefficients of 1 and $\sinh w$. We thereby obtain

$$t(t_{uu} + t_{vv}) = 2(t_u^2 + t_v^2 + F^2), \tag{3.7}$$

$$t_{uu} + t_{vv} = -2F^2 t. \tag{3.8}$$

Together these relations show that

$$t_u^2 + t_v^2 + F^2 + F^2 t^2 = 0 \tag{3.9}$$

which is impossible, since it makes $F = 0$.

3.5. The case $H = \cosh w$. If $H = \cosh w$ we proceed as above to obtain, from (2.9) and (2.10) after slight simplification,

$$t_{uu} + t_{vv} + 2tF^2 = 0, \tag{3.10}$$

$$t_u^2 + t_v^2F^2(t^2 - 1) = 0, \tag{3.11}$$

$$F^2 + (F_u/F)_u + (F_v/F)_v = 0. \tag{3.12}$$

Let us notice by (2.15) that the surface $w = 0$ is a plane, so that reasoning as before we may assume $F = t + 1$.

From (2.4) we have that

$$t_{uv} = 2t_ut_v/(t + 1)$$

which may be integrated, divided by $(f + 1)^2$ and integrated again, to give finally

$$1/(t + 1) = \alpha(u) + \beta(v). \tag{3.13}$$

After division by $(t + 1)^4$ Eq. (3.11) becomes

$$\alpha'^2 + \beta'^2 + 1 = 2(\alpha + \beta). \tag{3.14}$$

When differentiated with respect to u, v being constant, this leads to the linear equation

$$\alpha'(\alpha'' - 1) = 0.$$

We have a similar relation for β; whence we conclude that

$$\begin{aligned} \alpha &= a \qquad \text{or} \qquad (1/2)u^2 + au + b, \\ \beta &= A \qquad \text{or} \qquad (1/2)v^2 + Av + B. \end{aligned} \tag{3.15}$$

That both α and β cannot be constant is seen by (3.13), (3.10), and (3.11); the equations require $F = 0$, which is not permissible. If α alone is constant, moreover, the relation (3.14) tells us that $A^2 + 1 = 2(\alpha + \beta)$. The value of F thus obtained does not satisfy (3.12). Hence neither α nor β is constant. After substituting (3.15) into (3.14) to find $b + B = 1/2$, making a change of variable to get $a = A = 0$, and using (3.13), we find that

$$t = 2/(u^2 + v^2 + 1) - 1.$$

This value of t, which satisfies all relations, leads to

$$ds^2 = \frac{4(du^2 + dv^2) + (1 + u^2 + v^2)^2\, dw^2}{[2 + (\cosh w - 1)(1 + u^2 + v^2)]^2}$$

as the linear element when $H = \cosh w$. Using $\cosh w - 1 = 2 \sinh^2 (w/2)$, writing w for $w/2$, and taking $\cosh^2 w = 1 + \sinh^2 w$ in the denominator, we find that

$$ds^2 = \frac{du^2 + dv^2 + (1 + u^2 + v^2)^2\, dw^2}{[\cosh^2 w + (u^2 + v^2) \sinh^2 w]^2} \tag{3.16}$$

which assumes the proper form, we note, when $w = 0$.

The change of variable $u = r \cos \theta$, $v = r \sin \theta$ leads to

$$ds^2 = \frac{dr^2 + r^2\, d\theta^2 + (1 + r^2)^2\, dw^2}{[1 + (1 + r^2) \sinh^2 w]^2}$$

and in this case the surfaces $\theta =$ constant are all planes, as we see by computing the second fundamental form. On these surfaces $\theta =$ constant we have

$$ds^2 = \frac{dt^2 + dw^2}{[\sin^2 t + \sinh^2 w]^2}$$

after making the change of variable $r = \cot t$. The linear element last obtained coincides with that for bipolar co-ordinates, whence we conclude that the spheres $w =$ constant must be one of the sets of surfaces generated when bipolar co-ordinates are revolved about the line joining the two poles. The fact that $w = 0$ is plane shows that our spheres must be those which have their centers in the line joining the two poles.

3.6. The case $H = \sin w$. The final case is $H = \sin w$, which gives

$$t_u^2 + t_v^2 = F^2(t^2 - 1), \tag{3.17}$$

$$t_{uu} + t_{vv} = 2F^2 t, \tag{3.18}$$

$$(F_u/F)_u + (F_v/F)_v = F^2 \tag{3.19}$$

when we substitute in (2.9) and (2.10), replace $\cos^2 w$ by $1 - \sin^2 w$, and equate to zero the coefficients of 1, $\sin w$ and $\sin^2 w$. By (2.15) the surface $w = \pi/2$ is a plane. Hence we may assume $F = t + 1$ to obtain the result (3.13), as before. Equation (2.4) is now satisfied; the only ones remaining are (2.11) and those just obtained.

By (3.13) and (3.17)

$$\alpha'^2 + \beta'^2 + 2(\alpha + \beta) = 1.$$

Proceeding as in the discussion of (3.15) we find that neither α' nor β' is zero, and that

$$t = 2/(1 - u^2 - v^2) - 1$$

so that the linear element becomes

$$ds^2 = \frac{4(du^2 + dv^2) + (1 - u^2 - v^2)\, dw^2}{[2 + (\sin w - 1)(1 - u^2 - v^2)]^2}.$$

Continuing as for (3.16) we find that the surfaces $w =$ constant are the other spheres generated by the co-ordinate lines when bipolar co-ordinates are rotated, that is, a set of spheres passing through a single fixed circle. This result can also be reached directly by inspection of ds^2.

CONCLUDING REMARKS

4.0. Dismissal of an auxiliary condition. To complete the discussion we must show that (1.8) is satisfied in each of the cases considered. With this end in view we use (1.7), (1.5), and (1.3) to obtain, for the left member of (1.8),

$$\frac{1}{4F^2}\left[\frac{A_u^2 + A_v^2 + F^2 A_w^2}{A^2} - 2\,\frac{A_{uu} + A_{vv} + F^2 A_{ww}}{A}\right]. \tag{4.1}$$

Replacing A by its proper value (2.3) and simplifying we find

$$\frac{1}{4F^2}\left[2\,\frac{t_{uu}+t_{vv}+H''F^2}{t+H}-3\,\frac{t_u^2+t_v^2+F^2H'^2}{(t+H)^2}\right] \tag{4.2}$$

which becomes

$$(t_{uu}+t_{vv})/8F^2(t+H)-H''/4(t+H) \tag{4.3}$$

if we use (2.10).

When $H = 0$, the expression in (4.3) is a function of u and v alone, so that (1.8) is certainly satisfied. Similarly, if $H = w^2$ we may replace $t_{uu} + t_{vv}$ by its value (3.2), whence it is seen again that (4.3) has the proper form as prescribed by (1.8). If $H = e^w$ the result is again true, since t is zero; for $H = \cosh w$ it is a consequence of (3.10); for $H = \sin w$ it follows from (3.18). Thus Eq. (1.8) is satisfied automatically in all cases, and the co-ordinate systems hitherto obtained will actually lead to separation. What we have shown is that (1.8) is a consequence of the fact that the space is Euclidean.

4.1. Non-orthogonal co-ordinates. As noted, the w-surfaces are orthogonal to the others, since there must be no cross derivative terms involving the variables to be separated. It is really not necessary, however, that the u and v surfaces be orthogonal to each other, although up to now we have assumed this to be the case. Thus, orthogonality of the u and v surfaces is essential to our derivations, because the relations of Sec. 2 presuppose that the w = constant surfaces are imbedded, or at any rate can be imbedded, in a triply orthogonal system. Our aim now is to discard this condition of orthogonality as an initial assumption, and to show that the equations dependent thereon will be satisfied anyway as a consequence of separation.

To this end we assume that

$$ds^2 = edu^2 + 2f\,du\,dv + gdv^2 + hdw^2 \tag{4.4}$$

rather than (1.0) and obtain [12] in place of (1.1),

$$\frac{\partial}{\partial u}\left(\frac{gh^{1/2}}{d}\,\theta_u-\frac{fh^{1/2}}{d}\,\theta_v\right)+\frac{\partial}{\partial v}\left(\frac{eh^{1/2}}{d}\,\theta_v-\frac{fh^{1/2}}{d}\,\theta_u\right)+\frac{\partial}{\partial w}\left(\frac{eg}{h^{1/2}d}\,\theta_w\right)=0. \tag{4.5}$$

Upon assuming a solution RSZ as in Sec. 1.0 we find that

$$\frac{g}{d^2}\frac{S_{uu}}{S}-2\,\frac{f}{d^2}\frac{S_{uv}}{S}+\frac{e}{d^2}\frac{S_{vv}}{S}+\cdots=0, \tag{4.6}$$

where the terms not written involve first derivatives of the unknown functions only, besides e, f, g, h and R. In (4.5) and (4.6) d^2 is the discriminant of the $u - v$ quadratic form,

$$d^2 = eg - f^2. \tag{4.7}$$

Since the equation separates, there exists a function $A(u, v, w)$ such that when the equation is multiplied by A, the coefficients of terms involving S are functions of u and v only, while the coefficients of terms with Z are functions of w only (cf. Sec. 1). Also the term free of unknowns must break up into a function of u, v plus a function of w. For our present purposes we need only the coefficients of S_{uu}, S_{uv} and S_{vv}, which tell us that

$$Ag/d^2,\ Af/d^2,\ Ae/d^2 \tag{4.8}$$

are functions of u and v alone. It follows that any combination of these expressions is also a function of u and v alone. Hence, in particular,

$$\left(\frac{Ag}{d^2}\right)\left(\frac{Ae}{d^2}\right) - \left(\frac{Af}{d^2}\right)^2 = \frac{A^2}{d^2} \tag{4.9}$$

has this property. Multiplying (4.8) by A and using (4.9) we find that e/A, f/A and g/A are functions of u and v alone, so that the linear element (4.4) has the form

$$ds^2 = A[e_1(u, v)\, du^2 + f_1(u, v)\, du\, dv + g_1(u, v)\, dv^2] + h dw^2. \tag{4.10}$$

Confining our attention to the terms in brackets, we see that it is always possible to make a change of variables, replacing u by $\bar{u}(u, v)$ and v by $\bar{v}(u, v)$, so that in the new variables we will have $f_1 = 0$. This is an analytic expression of the well-known geometrical fact that every surface admits a set of parametric curves which are orthogonal. When such a change of variables is made, the linear element (4.10) reduces to the form (1.0).

What we have shown is that there exists a change of u and v parameters which will make the new u and v surfaces orthogonal to each other, if they were not originally. Such a change is permissible in the sense of Sec. 1.1, and may therefore be carried out at the beginning of the investigation. We now have a linear element of the form (1.0), and the foregoing derivation proceeds without further change. Thus we have completed the proof of the first result:

With fixed R, if solutions $R(u, v, w)S(u, v)Z(w)$ satisfy Laplace's equation, and if separate differential equations for S and Z can be found by multiplying the equation by a suitably chosen function, then the co-ordinate surfaces w = constant must be orthogonal to the other two co-ordinate surfaces, and must consist of parallel planes, planes with a common line of intersection, spheres tangent at a common point, concentric spheres, the plane and set of spheres obtained when one set of bicircular co-ordinates is revolved about the line joining the poles, or a set of spheres all passing through a single fixed circle. Also, if the surfaces w = constant have any of these forms, and if the u and v surfaces are orthogonal to them, then Laplace's equation can always be separated in the prescribed manner.

4.2. Cases for which $R = 1$. Let us inquire when we may assume that $R = 1$. A necessary and sufficient condition is that R have the form $F(u, v)G(w)$, since in that case, but not otherwise, R may be absorbed in the solution $S(u, v)Z(w)$. From the equation $R^2A = 1$ we see that this condition, and hence the possibility of $R = 1$, is satisfied when, and only when, A also has the form $F(u, v)G(w)$. We know, however, that A must have the form (2.3), whence we conclude that t, H, or both must be constant for $R = 1$. These considerations lead to the second result:

We may assume that $R = 1$ always and only when the surfaces w = constant consist of parallel planes, planes with a common line of intersection, or concentric spheres.

4.3. The wave equation. Next let us consider the possibility of separating the wave equation. As noted, the theory is simpler than the preceding and included in it. Writing the equation in the form (*cf*. [16])

$$\Delta\theta + k\theta = 0, \tag{4.11}$$

we see that $F_2 + G_2 + H_2$ of (1.1) becomes $F_2 + G_2 + H_2 + k$, and this is the only change. All results previously obtained are valid here too, then, except for (1.8), which becomes

$$A^2(F_2 + G_2 + H_2 + k) = F_5(u, v) + H_5(w). \tag{4.12}$$

But in the course of the foregoing investigation it was shown that (1.8), originally postulated independently as a result of the separation, is actually not an independent relation, but follows automatically from the others (Sec. 4.0). Though not required in itself for separation of the wave equation, therefore, this relation must nevertheless hold, in view of the other conditions. Combining it with (4.12) we see that

$$A^2 = F_6(u, v) + H_6(w). \tag{4.13}$$

If t of (2.3) is not constant it depends on u, say, and the same is true of F_6 in view of (1.8) and (4.13). Equating the two expressions for A, we differentiate with respect to u, solve for $(t + H)^3$, and differentiate the result with respect to w to get

$$3(t + H)^2H' = 0$$

which shows that H is constant. Thus either H or t must be constant. We therefore have $R = 1$, immediately, for the case of partial separation. In this way we obtain the third result:

The wave equation separates in the sense RSZ for the three cases having $R = 1$, and for these only.

II. Solutions $R(u, v, w)X(u)Y(v)Z(w)$

FURTHER SEPARATION OF THE EQUATION

5.0. General. It has been supposed hitherto that the solution is only partially separated, being of the form $R(u, v, w)S(u, v)Z(w)$ with a fixed function R. If we assume that the solution separates further to give the form $R(u, v, w)X(u)Y(v)Z(w)$, so that $S(u, v) = X(u)Y(v)$ for each function of the family, then we obtain new conditions on the co-ordinates. Because the cross derivative terms make separability impossible, for example, it is known at the outset that all three co-ordinate surfaces must now be orthogonal.

The initial stage of the separation led to an ordinary differential equation for $Z(w)$ and, with m constant, to

$$\frac{A^2}{f^2}\frac{S_{uu}}{S} + \frac{A^2}{g^2}\frac{S_{vv}}{S} + A^2F_1\frac{S_u}{S} + A^2G_1\frac{S_v}{S} + F_4(u, v) = m \tag{5.1}$$

for $S(u, v)$, as we see by using (1.8) and noting that each of the separated groups of terms must be constant. This is true because the sum must be zero, and one expression involves w only while the other involves u and v only. When S has the assumed form XY, (5.1) becomes

$$\frac{A^2}{f^2}\frac{X''}{X} + \frac{A^2}{g^2}\frac{Y''}{Y} + A^2F_1\frac{X'}{X} + A^2G_1\frac{Y'}{Y} + F_4 = m. \tag{5.2}$$

For separation there must be a function $J(u, v)$ such that (5.2) separates when multiplied by it. In particular the term mJ must separate, since the only other term it can combine with, F_4J, is independent of m. If separation is to occur for as few as two distinct values of m, it already implies that F_4J and mJ must separate into the sum of a function of u only and a function of v only. In this connection it should be noted that if we try to pick a new J for each m, the ratio of the J values will have to be a function of u only and also a function of v only, in view of the condition (see below) on the coefficients of X'' and Y''. Thus the ratio J_1/J could depend on m alone, and the above considerations apply.

5.1. The differentiated terms. It has been seen that our multiplier $J(u, v)$ is of the form $\alpha(u) + \beta(v)$. When (5.2) is multiplied by this, separation must occur, and hence the coefficients of X''/X and X'/X must depend on u alone, while those of Y''/Y and Y'/Y depend on v alone. The condition for the second derivatives tells us that

$$ds^2 = A^2\{[\alpha(u) + \beta(v)]\,[du^2 + dv^2] + dw^2\}$$

after a change of scale in the u and v co-ordinates. The condition on coefficients of the first derivatives allows us to assume that $R^2A = 1$, as we see by following the derivation of (1.7). Also we know from the first separation that A has the form (2.3), so that we obtain finally

$$ds^2 = \frac{[\alpha(u) + \beta(v)](du^2 + dv^2) + dw^2}{[t(u, v) + H(w)]^2}\,. \tag{5.3}$$

From the derivation it is clear that (5.3) is sufficient as well as necessary for separation.

In connection with (5.3) we note that du^2 and dv^2 have the same coefficients, just as in (1.5). The former result was obtained merely by introducing a change of parameters; it was not a consequence of separability. In the present case, on the contrary, a change of parameters is not permissible. The most we can do is make a change of scale, such as replacing u by a function of u. That the linear element has the same coefficients for du^2 and dv^2 is a result which had to be proved from separability of the equation. Also since the u and v co-ordinates cannot be changed at will, the methods formerly used to simplify the equations are not available here; but there is some compensation in that the form of F is now known.

5.2. The term free of unknowns. The foregoing results follow from consideration of the coefficients of the differentiated terms. From the other terms we obtain an equation analogous to (1.8), namely

$$(\alpha + \beta)F_4 = \alpha_1(u) + \beta_1(v). \tag{5.4}$$

In combination with (1.8), which we have seen is always satisfied, (5.4) gives

$$(\alpha + \beta)A^2(F_2 + G_2 + H_2) = (\alpha + \beta)h_4(w) + \alpha_1(u) + \beta_1(v). \tag{5.5}$$

Using (4.3) in place of $A^2(F_2 + G_2 + H_2)$ and taking $F^2 = \alpha + \beta$, we find from (5.5) that

$$-\frac{t_{uu} + t_{vv}}{8(t + H)} + \frac{H''(\alpha + \beta)}{4(t + H)} = -(\alpha + \beta)h_4 - \alpha_1 - \beta_1\,. \tag{5.6}$$

It may be shown that this relation, like (1.8), is a consequence of the others; we omit the details. The proof is closely analogous to that presented in full in Sec. 4.0.

5.3. Transformations leaving equations invariant. It is convenient to note the changes in functions or variables which leave the equations essentially unaltered. As before, use of such properties permits simplification of the methods used. By inspection of the linear element (5.3), or of the equations themselves, we see that the following transformations will not lead to any essential change, if the k_i are constant:

$$t \to k_0 t, \tag{i}$$

$$\alpha \to k_1\alpha, \qquad \beta \to k_1\beta, \tag{ii}$$

$$\alpha \to \alpha + k_2\,, \qquad \beta \to \beta - k_2\,, \tag{iii}$$

$$u \to f(u), \qquad v \to g(v), \tag{iv}$$

$$\alpha \rightleftarrows \beta, \qquad u \rightleftarrows v, \tag{v}$$

DETERMINATION OF α AND β

6.0. An ordinary differential equation when $H' \neq 0$. Starting from (2.9), following through the derivation of (2.6) in detail, and using $F = (\alpha + \beta)^{1/2}$, we find that

$$c_1 = \frac{\alpha'' + \beta''}{2(\alpha + \beta)^2} - \frac{\alpha'^2 + \beta'^2}{2(\alpha + \beta)^3}\,, \tag{6.1}$$

where c_1 is constant. This result assumes $H' \neq 0$. In (6.1) we hold v constant and write α for $\alpha + \beta$ to obtain

$$2c_1\alpha^3 = \alpha(\alpha'' + c_2) - (\alpha'^2 + c_3),$$

which becomes

$$2c_1\alpha^3 = \alpha\left(p\,\frac{dp}{d\alpha} + c_2\right) - (p^2 + c_3)$$

when we take α instead of u as the independent variable, $p = \alpha'$ instead of α as unknown, and use $\alpha'' = p\,dp/d\alpha$. Upon introduction of a new variable $S = p^2 = \alpha'^2$, this in turn gives

$$\frac{dS}{d\alpha} - \frac{2S}{\alpha} - 4c_1\alpha^2 + 2c_2 - \frac{2c_3}{\alpha} = 0. \tag{6.2}$$

The integrating factor is $1/\alpha^2$ and leads to

$$S = \alpha'^2 = 4c_1\alpha^3 + 2c_2\alpha - c_3\,. \tag{6.3}$$

The symmetry of the equations noted in item (v) of Sec. 5.3 shows that we have a relation of the same type for β. Equation (6.1) becomes a polynomial in α and β, as we see by computing α'' and β'' from (6.3). Multiplying (6.1) by $(\alpha + \beta)^3$ we find that $\alpha'^2 + \beta'^2$ must be zero when $\alpha = -\beta$. It follows, then, that the equation corresponding to (6.3) for β is

$$\beta'^2 = 4c_1\beta^3 + 2c_2\beta + c_3\,. \tag{6.4}$$

Substitution into (6.1) leads to an identity, and hence there is no additional condition. Because the differential equation (6.3) also arises in the case $H' = 0$, as we shall see below, discussion of its solutions has been deferred to Sec. 6.4.

6.1. A partial differential equation when $H' = 0$. Turning now to the case in which H is constant, we absorb it into the function t, to obtain a simplified form of Eqs. (2.10) and (2.9), Eqs. (2.4) and (2.11) remaining the same. If we take new independent variables $\phi = \log t$ and $\theta = \log F^2 = \log (\alpha + \beta)$ these relations assume a form involving derivatives only, not the unknown functions. Specifically from (2.4), (2.10), (2.11), (2.9) in that order we find the following:

$$2\phi_{uv} + 2\phi_u\phi_v = \theta_u\phi_v + \phi_u\theta_v , \tag{6.5}$$

$$\phi_u^2 + \phi_v^2 = \phi_{uu} + \phi_{vv} , \tag{6.6}$$

$$\phi_{uu} - \phi_{vv} + \phi_u^2 - \phi_v^2 = \theta_u\phi_u - \theta_v\phi_v , \tag{6.7}$$

$$\theta_{uu} + \theta_{vv} = 2(\phi_{uu} + \phi_{vv}). \tag{6.8}$$

Because of the form of θ we also have that

$$\theta_{uv} + \theta_u\theta_v = 0. \tag{6.9}$$

Let us differentiate the relation

$$2(\phi_u^2 + \phi_v^2) = \theta_{uu} + \theta_{vv} , \tag{6.10}$$

obtained from (6.6) and (6.8), with respect to u to find that

$$4(\phi_u\phi_{uu} + \phi_v\phi_{uv}) = \theta_{uuu} + \theta_{uvv} . \tag{6.11}$$

Upon eliminating the second derivatives ϕ_{uu} and ϕ_{uv} by use of the other relations, one finds that the terms involving ϕ all cancel, leaving the equation in $\theta[= \log (\alpha + \beta)]$

$$\theta_u(\theta_{uu} + \theta_{vv}) = \theta_{uuu} + \theta_{uvv} . \tag{6.12}$$

It will be seen that α and β can be determined from this.

6.2. Complete solution of the case $H' = \beta' = 0$. If β does not depend on v we may replace α by $\alpha + \beta$, as suggested in Sec. 5.3 item (iii), and β by $\beta - \beta = 0$. We have then

$$\theta_u\theta_{uu} = \theta_{uuu} \tag{6.13}$$

from (6.12), with θ equal to $\log \alpha$. Letting $\psi = \theta_u = \alpha'/\alpha$, substituting and integrating we get

$$\psi^2/2 = \psi' - c^2/2$$

with c constant. If $c \neq 0$ we have $\psi = c \tan (u/2 + c')$, but if $c = 0$ then $\psi = -2/(u + c')$. As the general solutions we thus find that

$$\alpha = \frac{c''}{\cos^2 (uc + c')} \qquad \text{or} \qquad \alpha = \frac{c'}{(u + c)^2} . \tag{6.14}$$

In view of the equivalence (iv) of Sec. 5.3 the only essentially distinct cases are $\alpha = csch^2 u$, $csc^2 u$, or $1/u^2$.

6.3. An ordinary differential equation for the case $H' = 0$ but $\alpha'\beta' \neq 0$. We shall suppose now that $\alpha'\beta' \neq 0$, but retain the assumption that $H' = 0$. With θ replaced by its value $\log (\alpha + \beta)$, Eq. (6.12) now gives

$$\alpha'''(\alpha + \beta) - 4\alpha''\alpha' - 2\alpha'\beta'' + 3\alpha' \frac{\alpha'^2 + \beta'^2}{\alpha + \beta} = 0. \tag{6.15}$$

Proceeding as in the derivation of (6.2) we let u be constant to find that

$$\frac{dS}{d\beta} - \frac{3S}{\beta} + c_1\beta + c_2 + \frac{c_3}{\beta} = 0, \tag{6.16}$$

where $s = \beta'^2$ and β has been written for $\alpha + \beta$. The integrating factor is $1/\beta^3$ and it gives, after integration and multiplication by β^3,

$$S = \beta'^2 = A + B\beta + C\beta^2 + D\beta^3. \tag{6.17}$$

By the symmetry noted in (v), Sec. 5.3, we have also that

$$\alpha'^2 = a + b\alpha + c\alpha^2 + d\alpha^3, \tag{6.18}$$

where in both cases the coefficients are constant.

Inspection of (6.15) tells us that $\alpha'^2 + \beta'^2$ must be zero whenever $\alpha = -\beta$, since α'' and β'' are finite by (6.17) and (6.18). If α'^2 is given by (6.18), then β'^2 is uniquely determined as

$$\beta'^2 = -a + b\beta - c\beta^2 + d\beta^3. \tag{6.19}$$

It may be verified that (6.15) is now satisfied, and hence that there is no additional condition. Since these equations are equivalent to (6.3) and (6.4), obtained for the case $H' = 0$ (cf. Sec. 6.4), their solution completes the determination of α and β. It is seen, incidentally, that if $\alpha(u)$ satisfies (6.18), then the function $-\alpha(iv)$ will satisfy (6.19).

6.4. Canonical forms of the equation. Let us make use of Sec. 5.3 to simplify (6.18) and (6.19). Writing $-\alpha$ for α will make $d > 0$, if originally $d < 0$; and replacing α by $A\alpha + B$, u by u/C, we get

$$A^2\alpha'^2C^2 = a + b(A\alpha + B) + c(A\alpha + B)^2 + d(A\alpha + B)^3. \tag{6.20}$$

Supposing that d has been made greater than zero, we set

$$a + bB + cB^2 + dB^3 = 0, \tag{6.21}$$

$$b + 2cB + 3dB^2 = dA^2, \tag{6.22}$$

and choose $C = (Ad)^{1/2}$ to obtain the canonical form

$$\alpha'^2 = \alpha^3 + \lambda\alpha^2 + \alpha. \tag{6.23}$$

The corresponding substitution for β must be $\beta \rightarrow A\beta - B$ to be permissible; it gives (6.23) with α replaced by β and λ by $-\lambda$.

We see that A, B, and C may be assumed real (this is important), and that $A = 0$ is necessary only when there is a triple root of the original equation. These observations follow from the facts that the left-hand side of (6.22) is the derivative of (6.21) and that $d > 0$. For a triple root the canonical form is

$$\alpha'^2 = \alpha^3, \qquad \beta'^2 = \beta^3. \tag{6.24}$$

If we suppose now that $d = 0$ but $c \neq 0$ we find, in a similar way, that

$$\alpha'^2 = \alpha^2 + \lambda\alpha + 1, \qquad \beta'^2 = -\beta^2 + \lambda\beta + 1. \tag{6.25}$$

This form includes the double root case, which can be shown, however, to be impossible anyway. One would at first expect a $\pm$ sign in front of α'^2, but this is accounted for by interchanging α and β.

Next $c = d = 0$ gives

$$\alpha'^2 = \alpha, \qquad \beta'^2 = \beta \tag{6.26}$$

and $\alpha' = \beta' = 0$ is the last possibility. In Eq. (6.23) we must distinguish the three cases $|\lambda| < 2$, $|\lambda| = 2$, $|\lambda| > 2$, whereas in (6.25) we have only the one case $|\lambda| > 2$, since α' and β' must be real simultaneously.

6.5. Solution of the ordinary differential equations for α and β. We have seen that whenever H' or $\alpha'\beta' \neq 0$, the functions α and β must be solutions of one or another of the canonical differential equations in Sec. 6.4. These equations, though non-linear, may all be solved by elementary methods. The work is rather tedious, particularly since one must be careful to keep all possible solutions. We, therefore, content ourselves with a single example. From (6.23) we find that

$$du = [\alpha(\alpha^2 + \lambda\alpha + 1)]^{-1/2}\, d\alpha$$

which leads (among other possible expressions) to [14]

$$u + c = \left(\pm\frac{4}{r}\right)^{1/2} F\left[\frac{1}{r}(r - r')^{1/2}, \phi\right], \qquad \sin^2\phi = \frac{\alpha - r}{r' - r},$$

when the roots r, r', 0 are real and unequal, that is, when $|\lambda| > 2$. Here $F(x, p)$ is the elliptic function of the first kind. Since we want α as a function of u rather than the converse, we introduce the Jacobi elliptic functions to find that

$$\frac{\alpha - r}{r' - r} = \operatorname{sn}^2\left(\tfrac{1}{2}r^{1/2}(u + c), \qquad [(r - r')/r]^{1/2}\right)$$

with a similar expression for β. For r and r' we substitute the proper values in terms of λ to obtain, with a new u, a one-parameter family. This is for the case $0 < r' < \alpha < r$. Other cases are similarly treated, and lead to two additional expressions when $|\lambda| > 2$. The cases $|\lambda| < 2$, which gives conjugate complex roots, and $\lambda = 2$, which gives equal roots, are dealt with in the same way; the latter and all others in (6.24)—(6.26) give elementary functions.

DETERMINATION OF t

7.0. The partial differential equations. Up to this point we have shown that the linear element of orthogonal co-ordinates in Euclidean space must be of the form (5.3) if Laplace's equation is to separate. In addition H is one of the values (2.7) and α and β are each one of the values obtained in Secs. 6.0—6.5. It remains only to determine t in (5.3), and we proceed to this question forthwith.

Treating first the case $H' = 0$, our aim is to compute the expression involving F on the right of (2.9). Assuming that $\alpha'\beta'$ or $H' \neq 0$ we have (6.18) and (6.19). Differentiating both sides with respect to u and dividing by $2\alpha'$ we get

$$\alpha'' = \frac{3}{2}a\alpha^2 + b\alpha + \frac{c}{2} \tag{7.1}$$

with a similar expression for β''. Since $F = (\alpha + \beta)^{1/2}$ we have, for the expression desired,

$$\left(\frac{F_u}{F}\right)_u + \left(\frac{F_v}{F}\right)_v = \frac{1}{2}\left[\frac{\alpha'' + \beta''}{\alpha + \beta} - \frac{\alpha'^2 + \beta'^2}{(\alpha + \beta)^2}\right]$$

which reduces to

$$\left(\frac{F_u}{F}\right)_u + \left(\frac{F_v}{F}\right)_v = (\alpha + \beta)\frac{a}{4} \tag{7.2}$$

in view of (6.18), (6.19) and (7.1). Here a is the coefficient of α^3 in (6.18), and hence $a = 1$ for the cases with canonical equations (6.23), (6.24) but $a = 0$ for (6.25), (6.26).

The above assumes (6.18), which is valid only when H' or $\alpha'\beta' \neq 0$. Assuming now that $H' = \beta' = 0$ and writing α for $\alpha + \beta$ we find that the expression on the left of (7.2) becomes $(\alpha'/\alpha)_u/2$. For the case $H' = \beta' = 0$ it has been shown that α must be one of the three functions mentioned at the end of Sec. 6.2. By direct calculation we find that $(\alpha'/\alpha)_u/2 = \alpha$ every time, and hence (7.2) is valid with $a = 4$. Thus the expression is known in all cases.

When $H = 0$ we find that

$$t_{uu} + t_{vv} = \frac{t_u^2 + t_v^2}{t} + \frac{ta}{4}(\alpha + \beta), \tag{7.3}$$

$$t_{uu} + t_{vv} = 2\,\frac{t_u^2 + t_v^2}{t} \tag{7.4}$$

from (2.9) and (2.10), after making use of (7.2) and replacing F^2 by $\alpha + \beta$. These relations tell us that $t_u^2 + t_v^2 = 0$ whenever $a = 0$, so that in every non-cubic case if H is constant t must be constant also. The only remaining relations are (2.4) and (2.11), which become

$$2t_{uv} = \frac{\beta' t_u + \alpha' t_v}{\alpha + \beta}, \tag{7.5}$$

$$t_{uu} - t_{vv} = \frac{\alpha' t_u - \beta' t_v}{\alpha + \beta}, \tag{7.6}$$

for the present situation $F^2 = \alpha + \beta$. Equations (7.5) and (7.6) are valid for all H.

When H is not zero the corresponding form of (2.9) and of (2.10) is obtained by simply setting $F^2 = \alpha + \beta$ in the relations of Secs. 3.1—3.6. Specifically, if $H = w^2$ we have that

$$t_{uu} + t_{vv} = 4(\alpha + \beta), \tag{7.7}$$

$$t(t_{uu} + t_{vv}) = t_u^2 + t_v^2; \tag{7.8}$$

if $H = e^w$ then $t = 0$; if $H = \cosh w$, then

$$t_{uu} + t_{vv} + 2t(\alpha + \beta) = 0, \tag{7.9}$$

$$t_u^2 + t_v^2 + (\alpha + \beta)(t^2 - 1) = 0; \tag{7.10}$$

and if $H = \sin w$, then

$$t_u^2 + t_v^2 = (\alpha + \beta)(t^2 - 1), \tag{7.11}$$

$$t_{uu} + t_{vv} = 2(\alpha + \beta)t. \tag{7.12}$$

It may be verified that both (2.9) and (2.10) are satisfied identically when, for a given H, the function t satisfies the appropriate pair of equations from (7.3), (7.4) or (7.7)—(7.12). These relations plus (7.5) and (7.6), then, give the necessary and sufficient condition for separation in Euclidean space.

7.1. An indirect method of solving the equations for t. To obtain all possible linear elements we must have all possible values of t; in other words we must have the general solution of the above differential equations. This general solution is not easy to obtain directly and hence we use an indirect procedure, which depends on the fact that α and β are already known for every case.

Given a particular linear element ds^2 we note from (5.3) that all others using the same α and β are obtained from this one by means of a conformal transformation, with ds^2 now regarded as the metric of Riemannian V_3 . Thus, since α and β are to be the same in both cases we can change t and H only. The two elements are, consequently, proportional. By a theorem of Liouville [15], the most general conformal transformation which preserves the Euclidean character of the space—i.e., preserves the relations of Sec. 2—is a rigid motion, a reflection, or an inversion. Only the latter is of interest here. Hence given α and β, if we can somehow discover just one admissible t and H, then all others can be found from this by inversion.[4]

A single value of t is found without much trouble from the equations; in many cases it is obtained by inspection. Or one may refer to [3] and [11], which give examples of Euclidean linear elements with our α and β for each case. Knowing the single solution, we get the others by inversion as outlined above, and it remains only to see whether all inversions are permissible.

CONCLUDING REMARKS

8.0. Concerning inversion. It was found by actual trial that all inversions of permissible co-ordinates were themselves permissible, if by *permissible* we mean that the space is Euclidean and the equation separates. The new linear element had the form (5.3) and the other conditions (7.3)-(7.12) were always satisfied, independently of the center of inversion. We propose now to give a direct proof that this must necessarily be the case.

First, since the original linear element was permissible, it had the form (1.5) with $F^2 = \alpha + \beta$. Hence, because an inversion is a conformal transformation, the new linear element will also have this form with a different A, say $\bar{A}$. The equation will separate if $\bar{R}$, the new value of R, is given by $\bar{R}^2\bar{A} = 1$. This is true provided only that the relations (1.8) and (5.5) continue to hold in the new system, and we shall see that such is the case.

Since the original linear element was permissible, it was obtained from a co-ordinate system in Euclidean space. Consequently the new one, having been found by inversion, will have the same property. Now the relations (1.8) and (5.5), which are the only remaining conditions for separation of the equation, have been proved to be satisfied automatically whenever the space is Euclidean. They are therefore satisfied by the new linear element, and hence the equation separates. Moreover the fact that A has the form (2.3) was also deduced from the Euclidean relations, and hence will persist for $\bar{A}$; and the same is true of all the differential equations for t and H. Thus the linear

[4]A similar use of Liouville's theorem is made in [8].

element obtained by inversion satisfies every one of our conditions if the original linear element does. These remarks, which apply to partial as well as complete separation, complete the proof of the fourth result:

If the above procedure can be applied to the resulting differential equation for S to get further separation, so that the solutions have the form $R(u, v, w)X(u)Y(v)Z(w)$, then the co-ordinates must all be orthogonal and must be either toroidal co-ordinates, or the well-known cases (I—III *but not* IV *in* [3]) *giving separation of this type with $R = 1$, or else the co-ordinates obtained by inversion of these in a sphere.*

We have also proved that:

The set of co-ordinates giving separation RSZ or $RXYZ$, as the case may be, is closed under inversion.

In other words, if a given co-ordinate system is in the set, then the system obtained by inverting it in any sphere will likewise be in the set. This behavior of course is not found when we confine ourselves to the case $R = 1$.

It must be mentioned that these results on inversion, though interesting mathematically, are of slight practical importance. The way one would actually solve Laplace's equation in co-ordinates which are inversions of standard cases would be to solve the standard case and then invert the solution. The fact that the equation could be solved directly by separation of variables in the new co-ordinate system, though true, would not be used in practice.

8.1. Conditions for which $R = 1$. We shall have $R = 1$ when, and only when, it can be absorbed in the product XYZ, that is, when $R = p(u)q(v)r(w)$. In view of the condition $R^2A = 1$ in Eq. (1.7) we see that A too must have this form. Such a condition, when combined with (2.3), makes H constant and F of the form $r(u)s(v)$, or else it makes t constant. It is found that these conditions hold only in cases which occur in [3]. Thus,

Laplace's equation separates in the sense XYZ when, and only when, the wave equation so separates.

The "when" part of this result is well-known, but the "only when" part is believed to be new.

8.2. The wave equation. In Sec. 4.3 we considered the conditions under which the wave equation would separate partially, to give solutions of the type RSZ. Turning now to the case in which there is complete separation $RXYZ$, we encounter a difficulty when we try to show that $F^2 = \alpha(u) + \beta(v)$. The previous argument depended on the fact that one term of (5.2) involved the separation constant m while the other did not. For the wave equation, however, we may have k depending on m, and this method therefore cannot be used. Confining our attention to the case $H = 0$, which we have seen is the only case in which F need not be constant in the wave equation, we shall re-examine the separation procedure.[5]

Since Sec. 5.1 makes $J = F^2 = G^2$, the same equations are found, in general, with F^2 replacing $\alpha + \beta$. But (5.4) becomes

$$F^2\{A^2(F_2 + G_2 + H_2 + k) - m\} = \alpha_2(u) + \beta_2(v) \tag{8.1}$$

when F_4 is replaced by the F_5 of (4.12). Observe that $H = 0$ in (2.3) makes (1.3) depend

[5]This discussion was revised in proof (April, 1949); the earlier version was incorrect unless we had $k_1m_2 = k_2m_1$ in (8.2).

on u and v alone, by (1.5) and (1.7). Hence H_3 of (4.12) is constant, and may be absorbed in F_3.

If there is separation for three pairs of values (m_i, k_i) such that

$$\begin{vmatrix} 1 & m_1 & k_1 \\ 1 & m_2 & k_2 \\ 1 & m_3 & k_3 \end{vmatrix} \neq 0, \tag{8.2}$$

then the three equations (8.1) may be solved for F^2, whence we obtain the desired result $F^2 = \alpha + \beta$.

When the determinant (8.2) is zero, which can happen in practice (cf. Ref. [16]), the complete result has not been obtained. It appears that separation can occur for co-ordinates not obviously reducible for the known cases, and that $J = \alpha + \beta$ is no longer essential. Also a rather long investigation shows that $\theta = \log F^2$ must satisfy $\theta_{uu} + \theta_{vv} = cF^2$ when $F^2 \neq \alpha + \beta$, and that the constant c is zero if any minor like $m_1k_2 - m_2k_1$ vanishes. This latter condition gives $t = 0$ by (6.6) and (6.8), whence we conclude that $F^2 = \alpha + \beta$. The general problem in which the determinant (8.2) vanishes, but no minor does, is reserved for later investigation.

Now that we have obtained the proper form for F we can use all the results of the preceding sections. These results, which certainly represent a necessary condition on the co-ordinates for separation of the wave equation, tell us that we have t or H constant only in the well-known cases giving $R = 1$. Consequently,

The wave equation separates in the sense $RXYZ$ only in the known cases for which it separates in the sense XYZ, provided (8.2) *holds.*

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