A point of some interest is the fact that, as time goes on, the slope $\partial T(x, t)/\partial x$ of the curve of Eq. (5) does not increase without limit, but instead approaches an asymptotic value α/v, shown in Fig. 1. This asymptotic behavior may be seen by differentiating Eq. (5) with respect to x, and then inserting the conditions $t \gg x/v$ and $t \gg K/v^2$.

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DEGENERATE TWO-DIMENSIONAL NON-STEADY IRROTATIONAL FLOWS OF A COMPRESSIBLE GAS*

By N. COBURN (*University of Michigan*)

1. Introduction. A class of non-steady, two-dimensional, irrotational, compressible flows which are very similar to steady, two-dimensional, irrotational, compressible flows will be studied. In order to do this, we introduce the well-known potential equation and Bernoulli relation for general non-steady flows. Our degenerate flows are defined by requiring that two families of cylindrical characteristic surfaces (with generators parallel to the time axis) exist in space-time. These flows have the following properties: (1) the wave fronts are stationary; (2) each of the velocity components and the speed of sound depends upon a single function of time multiplied by appropriate functions, which we shall call "reduced" velocities, of the space variables; (3) the single function of time is such that the motion decays as time increases. A canonical characteristic system, consisting of five equations with five dependent variables (the reduced velocities and the rectangular coordinates of the plane) and two independent variables, is obtained. It is shown that simple waves do not exist. Finally, it is shown that a degenerate non-steady flow, whose stream lines are logarithmic spirals, exists.

2. The system of flow equations and the potential equation. Let $x^\lambda(\lambda = 1, 2)$ denote a rectangular Euclidean coordinate system in the physical plane, and let t denote the time variable. If v^λ $(\lambda = 1, 2)$ denotes the components of the velocity vector in the x^λ-coordinate system, and ρ and c denote the density and local speed of sound, respectively, then the equations of motion and the equation of continuity may be written as

$$\frac{\partial v_\lambda}{\partial t} + v^\mu \frac{\partial v_\lambda}{\partial x^\mu} + \frac{c^2}{\rho} \frac{\partial \rho}{\partial x^\lambda} = 0, \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v^\lambda}{\partial x^\lambda} + v^\lambda \frac{\partial \rho}{\partial x^\lambda} = 0. \tag{2.2}$$

In the above equations, the contravariant and covariant components of a vector are equal since the coordinate system is Euclidean orthogonal. However, we have introduced the notation of tensor analysis in order to use the summation convention.

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For irrotational motion, a velocity potential $\phi(t, x^\lambda)$ exists such that

$$v_\lambda = \frac{\partial \phi}{\partial x^\lambda}. \tag{2.3}$$

Further, we shall define the time component of the generalized velocity to be

$$v_0 = \frac{\partial \phi}{\partial t}. \tag{2.4}$$

The integrability conditions for (2.3) and (2.4) are the irrotationality conditions

$$\frac{\partial v_\lambda}{\partial x^\mu} = \frac{\partial v_\mu}{\partial x^\lambda}, \qquad \frac{\partial v_\lambda}{\partial t} = \frac{\partial v_0}{\partial x^\lambda}. \tag{2.5}$$

By substituting (2.5) into (2.1), we obtain the Bernoulli equation

$$2v_0 + q^2 + 2P = 0, \tag{2.6}$$

where $q^2 = v^\lambda v_\lambda$, and ϕ has been chosen so that the right-hand side of (2.6) vanishes. For adiabatic flows P is defined by

$$P = \int^\rho \frac{c^2}{\rho}\, d\rho = \frac{c^2}{\gamma - 1}, \qquad \gamma = 1.4. \tag{2.7}$$

By differentiating (2.6) with respect to t, x^λ (where P is defined by 2.7) and substituting into (2.2), (2.1), we obtain the desired potential equation

$$\frac{\partial v_0}{\partial t} + 2v^\mu \frac{\partial v_\mu}{\partial t} + a^{\lambda\mu} \frac{\partial v_\lambda}{\partial x^\mu} = 0, \tag{2.8}$$

where $a^{\lambda\mu} = v^\lambda v^\mu - c^2 g^{\lambda\mu}$ and $g^{\lambda\mu}$ is the metric tensor.

3. Degenerate flows. Let $\omega(t, x^\lambda) =$ constant denote the equation of a family of ∞^1 characteristic surfaces. These loci satisfy the first order equation[1]

$$\left(\frac{\partial \omega}{\partial t}\right)^2 + 2v^\lambda \frac{\partial \omega}{\partial x^\lambda}\frac{\partial \omega}{\partial t} + a^{\lambda\mu} \frac{\partial \omega}{\partial x^\lambda}\frac{\partial \omega}{\partial x^\mu} = 0. \tag{3.1}$$

If a flow possesses two families of characteristic surfaces which are cylinders with generators parallel to the time axis (that is, ω is not a function of t), then the flow will be called degenerate.

For degenerate flows, (3.1) reduces to

$$a^{\lambda\mu} \frac{\partial \omega}{\partial x^\lambda}\frac{\partial \omega}{\partial x^\mu} = 0. \tag{3.2}$$

From this fact, it is easily verified that these flows have the following properties:

(1) they are always supersonic;

(2) the normal vectors to the two families of characteristic surfaces and the corresponding bicharacteristic vectors are independent of time;

(3) the direction of the velocity vector and the Mach number are independent of time; and

[1] R. Courant and D. Hilbert *Methoden der mathematischen Physik*, vol. II, Julius Springer, Berlin, 1937, p. 375.

(4) if η^λ, $'\eta^\lambda$ denote the unit normal vectors to the two families of characteristic surfaces and t^λ, $'t^\lambda$ denote the unit vectors of the corresponding bicharacteristic curves, then

$$v^\lambda = c\eta^\lambda + (q^2 - c^2)^{1/2} t^\lambda, \tag{3.3}$$

$$v^\lambda = c'\eta^\lambda - (q^2 - c^2)^{1/2} {'t^\lambda}. \tag{3.4}$$

From property (2), we see that the wave fronts are stationary. In view of property (3), we may write

$$v^\lambda = f(t)\bar{v}^\lambda, \qquad c = f(t)\bar{c}, \tag{3.5}$$

where the barred quantities are the "reduced" velocities, etc., and depend only on the variables x^λ. From (2.6), (2.7), it follows that

$$v_0 = [f(t)]^2 \bar{v}_0 \, . \tag{3.6}$$

Substituting (3.5), (3.6) into (2.5), (2.8), we find that $f(t)$ must be

$$f(t) = -\frac{1}{kt + b}. \tag{3.7}$$

where k, b are constants. Further (2.5), (2.8), (2.6), reduce to

$$\frac{\partial \bar{v}_\lambda}{\partial x^\mu} = \frac{\partial \bar{v}_\mu}{\partial x^\lambda}, \qquad \frac{\partial \bar{v}_0}{\partial x^\lambda} = k\bar{v}_\lambda \, , \tag{3.8}$$

$$\bar{a}^{\lambda\mu} \frac{\partial \bar{v}_\lambda}{\partial x^\mu} + 2k(\bar{v}_0 + \bar{q}^2) = 0, \tag{3.9}$$

$$2\bar{v}_0 + \bar{q}^2 + \frac{2\bar{c}^2}{\gamma - 1} = 0. \tag{3.10}$$

In our future work, we shall drop the bars over the "reduced velocities" etc, but shall limit our considerations to the system (3.8) through (3.10).

4. The characteristic system for degenerate non-steady flows. In view of (3.8), Eq. (3.9) is a quasi-linear partial differential equation of the second order. The characteristic system for such an equation can be obtained immediately from H. Lewy's work.[2] Let α = constant, β = constant denote the traces of the two families of cylindrical characteristic surfaces on the x^λ-plane. In addition, in order to compare our results with those of the steady case, we introduce the notation

$$x = x^1, \qquad y = x^2, \qquad u = v_1 = v^1, \qquad v = v_2 = v^2. \tag{4.1}$$

An application of Lewy's method (and a little algebra) furnishes the characteristic system

[2]R. Courant and D. Hilbert, loc. cit., pp. 326-332.

$$(u(q^2 - c^2)^{1/2} + cv)\frac{\partial y}{\partial \beta} - (v(q^2 - c^2)^{1/2} - cu)\frac{\partial x}{\partial \beta} = 0, \tag{4.2}$$

$$(u(q^2 - c^2)^{1/2} - cv)\frac{\partial y}{\partial \alpha} - (v(q^2 - c^2)^{1/2} + cu)\frac{\partial x}{\partial \alpha} = 0, \tag{4.3}$$

$$(v(q^2 - c^2)^{1/2} + cu)\frac{\partial v}{\partial \beta} + (u(q^2 - c^2)^{1/2} - cv)\frac{\partial u}{\partial \beta} + \frac{2kq^2(v_0 + q^2)}{u(q^2 - c^2)^{1/2} + cv}\frac{\partial x}{\partial \beta} = 0, \tag{4.4}$$

$$(v(q^2 - c^2)^{1/2} - cu)\frac{\partial v}{\partial \alpha} + (u(q^2 - c^2)^{1/2} + cv)\frac{\partial u}{\partial \alpha} + \frac{2kq^2(v_0 + q^2)}{u(q^2 - c^2)^{1/2} - cv}\frac{\partial x}{\partial \alpha} = 0, \tag{4.5}$$

$$\frac{\partial v_0}{\partial \alpha} - ku\frac{\partial x}{\partial \alpha} - kv\frac{\partial y}{\partial \alpha} = 0. \tag{4.6}$$

The algebraic relation (3.10) must also be added to the system (4.2) through (4.6). Somewhat different forms of (4.4), (4.5) can be obtained by solving (4.3), (4.6) for $\partial x/\partial \alpha$, and (4.2) and the equation in β corresponding to (4.6) for $\partial x/\partial \beta$. We find that

$$\frac{\partial x}{\partial \alpha} = \frac{(u(q^2 - c^2)^{1/2} - cv)}{kq^2(q^2 - c^2)^{1/2}}\frac{\partial v_0}{\partial \alpha}, \qquad \frac{\partial x}{\partial \beta} = \frac{(u(q^2 - c^2)^{1/2} + cv)}{kq^2(q^2 - c^2)^{1/2}}\frac{\partial v_0}{\partial \beta}. \tag{4.7}$$

When (4.7) are substituted into (4.4), (4.5), we obtain equations involving derivatives of u, v, v_0. Further, if we introduce θ, the slope of the velocity vector

$$u = q\cos\theta, \qquad v = q\sin\theta, \tag{4.8}$$

then (4.4), (4.5) become

$$q(q^2 - c^2)^{1/2}\frac{\partial q}{\partial \beta} + cq^2\frac{\partial \theta}{\partial \beta} + \frac{2(v_0 + q^2)}{(q^2 - c^2)^{1/2}}\frac{\partial v_0}{\partial \beta} = 0, \tag{4.9}$$

$$q(q^2 - c^2)^{1/2}\frac{\partial q}{\partial \alpha} - cq^2\frac{\partial \theta}{\partial \alpha} + \frac{2(v_0 + q^2)}{(q^2 - c^2)^{1/2}}\frac{\partial v_0}{\partial \alpha} = 0. \tag{4.10}$$

Simple waves, (that is, a family of straight line bicharacteristics, say β = constant, along which u, v are constant) *do not exist for degenerate non-steady flows.* For if we require u, v to be constant along a curve β = constant and the slope of (4.3) to be independent of α, then a differentiation shows that c must be constant along β = constant. From (3.10), it follows that v_0 is also constant along β = constant. Use of (4.5), shows that $q = 0$ or $v_0 + q^2 = 0$ along curves of this family. Finally, equating the slopes obtained from (4.6) with v_0 constant and (4.3), we see that $q = 0$ or $q = c$. Hence $q = 0$ is the only common solution of our equations.

The case $k = 0$, v_0 = constant, *furnishes steady two-dimensional flow. Another case of interest is when* $v_0 + q^2$ *is zero throughout the flow.* From (3.10), it follows that such a flow can exist only at the Mach number, $M = 5^{1/2}$. The Eqs. (4.9), (4.10) may be integrated immediately. We obtain

$$q = ae^{-\theta\tan\omega}, \qquad \text{along } \alpha = \text{constant}, \tag{4.11}$$

$$q = be^{\theta\tan\omega}, \qquad \text{along } \beta = \text{constant}, \tag{4.12}$$

where ω is the Mach angle for $M = 5^{1/2}$ and a, b are arbitrary functions of α, β, respectively. The curves (4.11), (4.12) are logarithmic spirals in the hodograph plane. For the case $v_0 + q^2 = 0$, the (x,y)-plane map of the bicharacteristics $\alpha =$ constant (or $\beta =$ constant), is orthogonal to the (u,v)-plane map of $\beta =$ constant (or $\alpha =$ constant). *Hence, the* (x,y)*-plane map of* (4.11), (4.12) *consists of logarithmic spirals.* Specific formulas can be obtained easily. Since the bicharacteristics are inclined to the stream lines at the constant angle ω, *the stream lines are also logarithmic spirals.* Evidently, a similar method can be used to investigate those degenerate non-steady flows for which $v_0 = f(q^2), f(q^2) < 0$.

NOTE ON THE CHARACTERISTICS IN UNSTEADY ONE-DIMENSIONAL FLOWS WITH HEAT ADDITION*

By C. C. LIN (*Massachusetts Institute of Technology*)

1. In a very interesting paper, Kahane and Lees[1] studied the problem of one-dimensional wave propagation in a gas when heat is being added. They used the method of numerical integration, by taking finite differences along the characteristics. However, since the form of the characteristic equations contained more than two dependent variables, they were led to use some rather artificial approximations besides those involved in taking finite differences. As the type of work done by Kahane and Lees will probably be continued by people interested in jet propulsion, it seems desirable that the most convenient form of the characteristic equations be derived and a simple procedure of numerical integration be developed. It is the purpose of this note to show that with a proper choice of the dependent variables, the characteristic equations are much simpler, and the numerical integration can be carried out in a straightforward manner.[2]

2. The fundamental equations for one-dimensional unsteady flow are

$$u_t + uu_x + \frac{1}{\rho} p_x = 0, \tag{1}$$

$$\rho_t + u\rho_x + \rho u_x + \rho u A^{-1} A_x = 0, \tag{2}$$

$$s_t + us_x = q^*, \tag{3}$$

$$p = \text{constant} \times \rho^\gamma e^s, \tag{4}$$

where u is the velocity in the direction of the x-axis, A is the cross-sectional area of the tube, t is the time, and p, ρ, s, γ are the familiar symbols for the pressure, the density, the entropy (divided by the specific heat at constant volume c_v) and the ratio of specific

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[1]A. Kahane and L. Lees, J. Aero. Sci. 15, 665-670 (1948).

[2]For further discussions of this problem, see a note by William Swartz, which is to appear in J. Aero. Sci.