A GEOMETRICAL INTERPRETATION OF THE RELAXATION METHOD*

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Let a_{ij} , B_i $(i, j = 1, 2, \cdots, n)$ be given constants and consider the equations

$$\sum_{j=1}^{n} a_{ij}x_j - B_i = 0, \qquad (i = 1, 2, \cdots, n). \tag{1}$$

As n increases, the solution by means of determinants becomes burdensome. Then the *relaxation method* provides a set of easy steps by which the solution is approached.

*Synge*** has given a geometrical description which illustrates the relaxation process. He confines himself, however, to the special case where the coefficients are such that $a_{ij} = a_{ji}$ and where, moreover, the quadratic form $\sum a_{ij}x_ix_j$ is positive definite. In his interpretation a set of homothetic ellipsoids in n-space plays a fundamental part.

The purpose of the present note is to give a simple geometrical interpretation for the general case in which the matrix a_{ij} is not necessarily symmetric.

If we substitute in the left member of (1) the arbitrary set of values y_i , we get

$$\sum_{j=1}^{n} a_{ij}y_j - B_i = R_i^{(0)}, \tag{2}$$

where R_i are the so-called *residues*. It is the task of the relaxation process to construct a sequence of successively "better" sets of values so that the residues tend to zero. Following Southwell, we do this by correcting one of the quantities y, say y_k , while the others are left unaltered. If y_k is replaced by $y_k + d_1$, we get the new residues

$$R_i^{(1)} = R_i^{(0)} + a_{ik}d_1 \ . \tag{3}$$

Now let us regard R_i as the rectangular Cartesian coordinates of a point P in Euclidean n-space. The purpose of the process is to approach the origin O of the coordinate system. By means of (3) the arbitrarily chosen point P_0 is replaced by P_1 . It is obvious that the direction of the line $l = P_0P_1$ is determined by the given coefficients; this direction is specified by a_{ik} $(i = 1, 2, \cdots, n)$.

The procedure which we have to follow is not completely defined, but it seems reasonable to move P from P_0 in the given direction in such a way that its distance to the origin becomes as small as possible. In this case d_1 has to be chosen so that the quadratic function

$$d_1^2 \sum_i a_{ik}^2 + 2d_1 \sum_i a_{ik}R_i^{(0)} + \sum_i (R_i^{(0)})^2$$

is a minimum. Hence $d_1 = (-\sum a_{ik}R_i^{(0)})/(\sum a_{ik}^2)$, and this is indeed the method frequently used; it is in accordance with the principal of least squares. But it is clear that P_1 is the orthogonal projection of O on the line l; in other words, P_1 *is the projection of* P_0 *on the* $(n - 1)$*-dimensional space through* O *which is perpendicular to the line* l. Since y_k has now been corrected, we take another of the unknown quantities and proceed in the same way.

It is thus obvious what kind of construction in our geometrical representation is

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the analogue of the procedure. We consider the $(n-1)$-dimensional spaces V_{ki} $(i = 1, 2, \cdots, n)$ through O, whose equations are $\sum a_{ik}x_i = 0$. When we take arbitrary values for the unknown quantities and correct them in the order $x_{k_1}, x_{k_2}, \cdots$, the construction runs as follows. An arbitrary point P_0 is chosen in n-space; P_1 is the orthogonal projection of P_0 on V_{k_1}; P_2 is the projection of P_1 on V_{k_2}, etc. And so the procedure is illustrated in a simple geometrical way. We add some supplementary remarks. The point P_m $(m > 0)$ lies always in one of the spaces V_k. These spaces are linearly independent if $|a_{ik}| \neq 0$. At each step (after the first) one of the V_{ki} is projected by parallel projection onto the following one. In this way, an affine correspondence is established between the two successive V_{ki}, the *modulus* of the affinity being $\cos \alpha$, where α is the angle between them. Thus the convergence of the procedure can easily be proved, provided that the corrections take place in a fixed cyclic order $x_{k_1}, x_{k_2}, \cdots, x_{k_n}$ where $k_1, k_2, \cdots, k_n$ is a permutation of $1, 2, \cdots, n$.

If two successive V_{ki} are perpendicular to each other, the projection of the first onto the second coincides with their $(n-2)$-dimensional space of intersection. It follows, therefore, that if all the V_{ki} are mutually perpendicular (that is, if the matrix $||a_{ik}||$ is orthogonal), the point P_1 lies on V_{k_1}, P_2 on the intersection of V_{k_1} and V_{k_2}, P_3 on $(V_{k_1}, V_{k_2}, V_{k_3})$ and so on; hence P_n coincides with O. In this case the procedure ends automatically after n steps.

A SIMPLIFIED METHOD OF DIFFERENTIATING AND EVALUATING FUNCTIONS REPRESENTED BY FOURIER SERIES*

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1. Introduction. This paper shows how to eliminate the difficulties caused by discontinuities of Fourier sine series at the ends of the interval of periodicity.

Applications of Fourier series to exact solutions of problems in mathematical physics involve the following essential considerations. In an interval $-a \leqq x \leqq a$, it is assumed that a function $f(x)$ and its successive derivatives up to some finite order $f^{(m)}(x)$ all comply with sufficient conditions of continuity, bounded variation, differentiability and integrability. They thus permit representation by Fourier series, which can be differentiated to give the derivative of next higher order, and integrated to give an expression for the derivative of next lower order.

When $f(x)$ is an x-odd function, and $f(a) \neq f(-a) \neq 0$, particular difficulties are encountered. The corresponding Fourier sine series is discontinuous at $x = \pm a$ and does not conveniently represent the values of $f(a)$ and $f(-a)$. In addition, the derivative $f'(x)$ is represented by a complicated Fourier series which is not readily evaluated at $x = \pm a$. Thus

$$f(x) = \sum_1^\infty b'_n \sin \beta_n x, \tag{1}$$

in which $\beta_n = n\pi/a$. The expression for the derivative is

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