QUARTERLY

OF

APPLIED MATHEMATICS

VOLUME VIII — APRIL • 1950 — NUMBER 1

QUARTERLY
OF
APPLIED MATHEMATICS

This periodical is published quarterly under the sponsorship of Brown University, Providence, R. I. For its support, an operational fund is being set up to which industrial organizations may contribute. To date, contributions of the following industrial companies are gratefully acknowledged:

BELL TELEPHONE LABORATORIES, INC.; NEW YORK, N. Y.,
THE BRISTOL COMPANY; WATERBURY, CONN.,
CURTISS WRIGHT CORPORATION; AIRPLANE DIVISION; BUFFALO, N. Y.,
EASTMAN KODAK COMPANY; ROCHESTER, N. Y.,
GENERAL ELECTRIC COMPANY; SCHENECTADY, N. Y.,
GULF RESEARCH AND DEVELOPMENT COMPANY; PITTSBURGH, PA.,
LEEDS & NORTHRUP COMPANY; PHILADELPHIA, PA.,
PRATT & WHITNEY, DIVISION NILES-BEMENT-POND COMPANY; WEST HARTFORD, CONN.,
REPUBLIC AVIATION CORPORATION; FARMINGDALE, LONG ISLAND, N. Y.,
UNITED AIRCRAFT CORPORATION; EAST HARTFORD, CONN.,
WESTINGHOUSE ELECTRIC AND MANUFACTURING COMPANY; PITTSBURGH, PA.

The QUARTERLY prints original papers in applied mathematics which have an intimate connection with application in industry or practical science. It is expected that each paper will be of a high scientific standard; that the presentation will be of such character that the paper can be easily read by those to whom it would be of interest; and that the mathematical argument, judged by the standard of the field of application, will be of an advanced character.

Manuscripts submitted for publication in the QUARTERLY OF APPLIED MATHEMATICS should be sent to the Managing Editor, Professor W. Prager, Quarterly of Applied Mathematics, Brown University, Providence 12, R. I., either directly or through any one of the Editors or Collaborators. In accordance with their general policy, the Editors welcome particularly contributions which will be of interest both to mathematicians and to engineers. Authors will receive galley proofs only. Seventy-five reprints without covers will be furnished free; additional reprints and covers will be supplied at cost.

The subscription price for the QUARTERLY is $6.00 per volume (April-January), single copies $2.00. Subscriptions and orders for single copies may be addressed to: Quarterly of Applied Mathematics, Brown University, Providence 12, R. I., or to Box 2-W, Richmond 5, Va.

Entered as second class matter March 14, 1944, at the post office at Providence, Rhode Island, under the act of March 3, 1879. Additional entry at Richmond, Virginia.

WILLIAM BYRD PRESS, INC., RICHMOND, VIRGINIA

CONTENTS

BOOK REVIEWS

On the theory of stochastic processes and their application to the theory of cosmic radiation. By Niels Arley. John Wiley & Sons, Inc., New York, 1948. 240 pp. $5.00.

This book aims at extending the general theory of stochastic processes, as developed by Kolmogoroff, Feller, Lundberg and others, and applying the resulting generalization to the theory of cosmic ray showers. The work is thus naturally divided into two parts, which as the author says in the Preface, may be read independently of each other.

The mathematical section contains an elegant and rigorous, albeit somewhat repetitious, discussion of stochastic processes,—processes involving one or more random or statistical variables, the probability distribution of which depends on one continuously varying parameter. Although reference is made for the sake of generality to events labeled by Borel point sets in Euclidean space with a finite number of dimensions, the theory is immediately specialized to so-called *Markoff chains*, in which the stochastic variables take on only an enumerable manifold of values.

The central problem in the theory is the existence of solutions of the diffusion equations for the required probability distributions, given the elementary transition probabilities. Arley discusses this problem in a quite general way, using a convenient matrix representation, in which the row and column labeling represents respectively the final and initial states of the stochastic variables. He introduces the concept of *absolute exponentiability*, namely the existence of a matrix

$$\sum_{m=0}^{\infty} \mathbf{K}^m (t - s)^m / m!$$

where **K** is a bounding matrix of the fundamental transition probability operator. It is then shown, except for normalization, that solutions of the diffusion equations exist for absolutely exponentiable types of operators. One of the contributions of Arley is to introduce a new set of sufficient conditions for absolute exponentiability, in addition to conditions established by Feller, Kolmogoroff, and Lundberg, whose results are included in detail in this monograph. Arley's conditions are, indeed, just suitable for the cosmic ray problem.

The problem of proving the normalization of the fundamental solution, (which was obtained using Peano-type product-integrals) is a difficult one, and Arley succeeds only in proving it for his "generalized elementary processes", namely, processes for which in each *upward* change, the stochastic variable cannot change by more than a certain number l of units.

This section of the book will be valuable to statisticians and mathematicians interested in extending the general existence theory for stochastic processes. It is not of much practical use for applications, since as the author himself says, the existence of solutions is generally assured in physical problems, and the equations may be written down from physical considerations. Although, as stated on p. 46, the product-integral is the ideal mathematical tool for the theory of discontinuous processes, it is rarely of help in numerical work.

The author introduces the second part of the work by saying (p. 15): *"It is, namely, a fundamental, empirical law of nature that the region of validity of the simplifying approximations of which any theory must necessarily make use, is always far wider than might be justified by theoretical arguments."* (Author's italics). The simplifying approximations made here are indeed extensive, and while the agreement with experiment is good, and perhaps more than would be expected at first glance, careful discussion would reveal sufficiently valid reasons for the agreement to make such a bold assertion unnecessary. In any case, it could stand further discussion.

The second part of the work deals with cosmic ray showers, which are produced by the combination of the radiation of high-energy photons by electrons, and the production of electron pairs by photons, in their passage through matter, with the atomic collisions by the electrons accounting for the loss of energy and ultimate disappearance of the shower. A brief but adequate description of this process is given in the text. The most serious simplification made in the theory is the disregard of the energy-dependence of the various probabilities involved. The inclusion of this dependence would throw the problem outside the realm of application of Arley's theory, and require a further generalization.

The author is aware of this difficulty, and proposes to overcome it by making a model in which particles duplicate themselves with a constant probability, but "die" with a probability proportional to their depth in the shower-evoking material. It is hard to see the extent of the error produced by using this model instead of one in which particles "die" when their energy is reduced to zero.

Another simplifying assumption is to make the two duplication processes symmetrical—electrons each producing two photons only, to correspond with the reverse process—in order to make the equations amenable. Even then it is only possible to find the first and second moments of the desired distribution, which is the probability of observing n electrons at depth t. So a "reasonable" distribution with two adjustable parameters is used, the Polyà distribution

$$p(n) = \left(\frac{-b\bar{n}}{1 + b\bar{n}}\right)^n \binom{-1/b}{n} (1 + b\bar{n})^{-1/b}$$

and for each value of t, b and $\bar{n}$ are chosen to yield the calculated first and second moments, $\bar{n}$ and $\bar{n}^2$.

The reasonableness of this procedure is shown, at least for large $\bar{n}$ and small t, by comparison with a solution obtained by numerical integration of the basic diffusion equations. The theory is applied to the actual cosmic-ray case, by using the fairly well-established calculations of Bhabha and Heitler for the latter case, of the first-moments problem (the average numbers), adjusting the "death-rate" of the model to yield the same value of the number of particles at the maximum, then using the model to obtain the second moments, and finally using the Polyà distribution as indicated.

The results are first compared with Poisson distributions, which clearly are not valid owing to the correlated nature of the statistics, and are applied to the Geiger-counter shower measurements according to Rossi. The probabilities have to be integrated over an assumed initial spectrum of photons and electrons, which further complicates the calculation and makes difficult an evaluation of the validity of the procedure. The comparison with experiment is, however, fairly good.

It should be stated, parenthetically, that much better average-number calculations for shower theory are now available (H. S. Snyder, Phys. Rev. **76**, 1563 (1949)), and that in particular the low energy calculations of Arley and Ericksen may be off by a large factor.

The general result is the demonstration of a practical way of predicting, albeit roughly, the shape of the Rossi curves and the results of similar statistical experiments. Little light, however, seems to have been thrown on the general and difficult analytical problem of the fluctuations in cosmic ray showers. It has been shown (Scott and Uhlenbeck, Phys. Rev. **62**, 497, (1942)) that the second moments and the fluctuation can be calculated directly in the actual case, and an approach of this sort may prove more helpful in the general theory.

The typography of the book is excellent, and there seem to be no essential misprints. The English is good, giving one but few reminders that the author is Danish. The photo-offset reproduction has reduced the original about 10% with little loss of clarity.

W. T. Scott

Mathematics dictionary. By A. A. Alchian, E. F. Beckenbach, C. Bell, H. V. Craig, G. James, R. C. James, A. D. Michal, I. S. Sokolnikoff. Edited by Glenn James and Robert C. James. D. Van Nostrand Company, Inc., New York, 1949. v + 432 pp. $7.50.

This work is based on the "James Mathematics Dictionary" which was devoted to the terms of elementary mathematics. To those have now been added the basic terms in "metric differential geometry, theory of functions of real and complex variables, advanced calculus, differential equations, theory of groups, theory of metrices, theory of summability, point-set topology, general analysis (referring in particular to integral equations and the calculus of variations, analytic mechanics, theory of potential, and statistics."

W. Prager

Description of a relay calculator. By The Staff of the Computation Laboratory. Harvard University Press, Cambridge, Massachusetts, 1949. xvi + 366 pp. $8.00.

This book gives a complete description of the well-known Mark II calculator which was built for installation at the Naval Proving Ground, Dahlgren, Virginia by the Harvard Computation Laboratory under the direction of Professor H. N. Aiken.

The machine, now in operation at Dahlgren, is a general purpose digital computer with an inner relay memory of 96 registers together with 24 sets of switches for the storage of constants, and 2 function tables for evaluating certain elementary functions, together with 4 interpolator mechanisms; it has 2 adding units and 4 multiplying units and it receives its orders from a punched tape. It has been so designed that it is possible to split it into two machines and operate each independently of the other.

The code and control are so arranged that in each second the calculator can form 2 multiplications, 4 additions and 6 numeric transfers.

The body of the book is devoted to a careful and detailed description of these various units and of their modes of operation. It is clear that anyone who is interested in carrying out a calculation on the machine will find this work indispensable. It should also prove interesting to all those engaged in machine design since some of the points of view expressed are of importance in all branches of the design problem.

The last chapter of the book is devoted to a discussion of problem preparation. After the statement of some general principles a number of examples are considered in detail; these serve to give the reader a grasp of the system of coding used.

HERMAN H. GOLDSTINE

Extrapolation, interpolation, and smoothing of stationary time series with engineering applications. By Norbert Wiener. The Technology Press of The Massachsuetts Institute of Technology, John Wiley & Sons, Inc., New York and Chapman & Hall, Ltd., London. ix + 163 pp. $4.00.

This monograph is the second of a series. The first, *Cybernetics* (Wiley, N.Y., 1948; Math. Rev. 9, 598), presented the philosophy of "control and communication in the animal and the machine". This book presents in detail some of the methods and techniques for cybernetics. In particular, the book considers problems of statistical prediction and filtering with applications to the design of communication systems. It was first published as a classified NDRC report and has been supplemented with two appendices by N. Levinson. The author was the first to formulate problems in communication theory on a statistical basis and his methods are the alloying of methods from the two fields: communication engineering and the statistical study of time series.

After introducing general concepts and summarizing some basic mathematical notions, the author continues in a readable and orderly fashion to explain the method. He introduces numerous examples to illustrate its application. The first problem studied is that of linear prediction for single stationary time series. The past of a message up to and including time t and certain statistical information about the message are assumed known; the message is a time series and the statistical parameter is its autocorrelation coefficient. He then shows how to select from a class of physically realizabie operations on the message that operation which gives the best prediction of the message at time $t + \alpha$ in the future. The allowable operators are linear, invariant with respect to the origin of time, and the result of the operation depends only upon the past of the message. The best prediction is in the sense of least mean (over all t) square error. The best operator depends only upon the autocorrelation coefficient; it is best for the whole class of messages which are statistically alike in that they have the same autocorrelation coefficient. Noise is then assumed to be superimposed on the message. In a similar manner a method is given for finding the linear operator which gives the best approximation of the message at time $t + \alpha$. Here the best operator depends only upon the auto-correlation and cross-correlation coefficients of the noise and the message. The degree of lag (α negative) which is advantageous in filtering is discussed; α positive corresponds to the combined problem of filtering and predicting. The methods have the virtue that they do not ignore phase distortion and are not limited to steady-state behavior. The extension of the method to multiple stationary time series is indicated. Application of the method to approximate differentiation, interpolation and extrapolation are discussed in the last chapter. Appendix A contains a table of Laguerre functions. Appendices A and B by N. Levinson discuss a simpler computational procedure and give a heuristic account of the author's theory.

Except for certain practical difficulties in computation and in the construction of circuits, the most obvious obstacle to the use of the theory is in making a choice of correlation coefficients suitable to the problem at hand. In this regard the engineer will need to gain more insight into the significance of these statistical quantities. Already this book has influenced the work of many people. Its public publication should widen the scope of its influence and should hasten the advancements which the author foresees as possibilities.

J. P. LaSalle