Obviously we can combine all cases and write

$$\text{Max}\,\xi = \Re\left\{\frac{a_1}{3} - \frac{1}{(3)^{1/2}}(-p)^{1/2}\right\}.$$

This is obtained for

$$a_3 = \Re\left\{\frac{a_1 a_2}{3} - \frac{2}{27}a_1^3 + \frac{2}{9}(3)^{1/2}(-p)^{3/2}\right\}.$$

PERIODIC MOTIONS OF A NON-LINEAR DYNAMIC SYSTEM*

By H. SERBIN (*Purdue University*)

1. Introduction. It is a matter of common knowledge that many problems of dynamics are non-linear in character and that the linearization of such problems is an expediency adopted in the face of mathematical difficulties attending the non-linearity. Fortunately, the linearized treatment gives useful results in a number of problems. On the other hand, there are cases where the essential characteristics of the phenomenon are altered by the assumption of linearity.

In the absence of general methods for handling non-linear problems, it is desirable to investigate typical non-linear phenomena to as great detail as possible with the intention of arriving at general methods of analysis.

One such problem is van der Pol's equation

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0, \qquad \left(\dot{x} = \frac{dx}{dt},\ \ddot{x} = \frac{d^2x}{dt^2}\right). \tag{1.1}$$

This defines the one-dimensional motion of a particle of unit mass suspended on a spring of unit stiffness and subjected to negative damping at low amplitudes and positive damping at high amplitudes.

The existence of periodic solutions of an equation of more general form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{1.2}$$

was considered by Liénard, Ref. 1, who proved, for $g(x) = x$ and under fairly general conditions on $f(x)$ (see Ref. 2), that there exists a periodic solution. The proof used in Ref. 1 was reproduced in Ref. 3 for $g(x) \neq x$.

In the present paper, the problem is considered from a different point of view. The periodicity of the solution is shown to exist for a wider class of differential equations (1.2) which can best be described and compared as in the following table:

	$F(x) = \int_0^x f(x)\,dx$	$G(x) = \int_0^x g(x)\,dx$	$f(x),\ g(x)$
Lefschetz	$F(\infty) = \infty$	$G(\infty) > 0$	Symmetric
Present paper	$F(\infty) > 0$	$G(\infty) = \infty$	No symmetry required
	$F(\infty) = \infty$	$G(\infty) > 0$	No symmetry required

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In addition, the case in which $F(\infty)$ and $G(\infty)$ are positive and both finite is discussed. It is shown that an example can be constructed in which a periodic solution does not exist.

A number of inequalities relating to the initial energy and amplitude of motion are derived on the basis of which one can, in some cases, calculate upper and lower bounds of the amplitude of the periodic motion without solving the differential equation. This should be useful in application to practical problems. It is conceivable, in some instances, that an average of the upper and lower bounds will define the amplitude with sufficient accuracy for the purpose required.

Since the problem is non-linear, one must expect such inequalities to be non-linear, for instance, algebraic. The solution of these inequalities is carried out conveniently by graphical methods. This must not be compared with graphical methods of solution of the differential equation. It is conceivable that for a given degree of accuracy, examples may be constructed of the type (1.2) in which the graphical method of solution by isoclines would give results incorrect both qualitatively and quantitatively, showing a periodic solution where none exists.

It should be pointed out that the assumption of symmetry in $f(x)$ and $g(x)$ made heretofore is not consistent with the nature of the problem. If these functions are determined empirically, then one will not have symmetry in the mathematical sense if x and $-x$ represent two different positions of the system. One is accustomed, in small deflections, to replacing an empirical curve which is nearly parabolic at the origin by an exact parabola. However, for large deflections, this is not permissible. Any lack of symmetry vitiates the validity of the theorem of periodicity as known heretofore.

A mechanical system illustrating the non-symmetric case is afforded by a pendulum in which friction acts proportional to velocity. If one adds a device which acts through a limited range of motion of the pendulum at the bottom of the swing in such a way as to sense the velocity and then to force the motion proportional to the velocity over the friction, then one would have the features characteristic of Eq. (1.2). If the constant of proportionality is different for x and $-x$, the asymmetric case will exist.

2. Qualitative character of the solution. Consider the differential equation (1.2) and define

$$
\begin{aligned}
F(x) &= \int_0^x f(x)\,dx, \\
G(x) &= \int_0^x g(x)\,dx, \qquad\qquad (2a) \\
H &= \tfrac{1}{2}\dot{x}^2 + G(x).
\end{aligned}
$$

Make the assumptions:

(A) $f(x)$ and $g(x)$ are continuous for $x \geqq 0$,
(B) $f(x) < 0$ for $0 < x < x_1$,
(C) $f(x) > 0$ for $x > x_1$,
(D) $F(\infty) > 0$,
(E) $g(x) > 0$ for $x > 0$,
(F) $F(\infty)G(\infty) = \infty$.

Then $F(x)$ has only one positive zero, $x = x_2$.

Since

$$\frac{dH}{dt} = \dot{x}[\ddot{x} + g(x)],$$

it follows from Eq. (1.2) that

$$\frac{dH}{dt} = -f(x)\dot{x}^2, \tag{2.1}$$

which shows that H increases with time for $x < x_1$ and decreases for $x > x_1$. Dividing both sides of Eq. (2.1) by $\dot{x}$ and simplifying,

$$\frac{dH}{dx} = \mp f(x)[2(H - G)]^{1/2}, \tag{2.2}$$

where the sign is minus or plus according as the velocity is plus or minus.

Suppose H plotted versus x in an (x, H)-plane (Fig. 1). It is seen that $H \geqq G$, the equality implying that $dH/dx = 0$. At a point where $H > G$, the right side of Eq. (2.2) satisfies the Lipschitz condition in H. Since $f(x)$ and $G(x)$ are continuous, it follows from the fundamental existence theorem on the solution of ordinary differential equations that there is one and only one solution passing through (x, H) for $H > G$. Referring to Fig. 1, there are four distinct parts of the trajectory:

Part	$\dot{x}$	f	$\frac{dH}{dx}$
01	+	−	+
13	+	+	−
35	−	+	+
56	−	−	−

The subscripts 0, 1, etc., will indicate the value of a variable at the corresponding point of the trajectory.

It is clear that $H_1, H_2, \cdots, H_6, x_3$ are monotone, continuous, functions of H_0. The proof given in Ref. 3 shows, in addition, that $H_6 - H_0$ is a monotone increasing function of x_3.

The analysis in the following pages is directed toward showing that $H_6 - H_0$, regarded as a function of H_0 on the interval $0 \leqq H_0 < \infty$, changes sign. Because of the monotone character of $H_6 - H_0$, it will follow that there is one and only one trajectory for which $H_6 = H_0$.

3. Behaviour of the solution for $0 < x \leqq x_2$. Consider only those trajectories for which $x_3 \geqq x_2$. Define

$$I_{ij}(G) = \int_{ij} [2(H - G)]^{1/2}\, dH,$$

where the subscript ij indicates integration along the trajectory in the sense $i \rightarrow j$. If $G(x) < G'(x)$ on the part ij of the trajectory, then $I_{ij}(G) < I_{ij}(G')$. Since

$$I_{ij} = \mp \int_{ij} f(x)\, dx = \mp(F_j - F_i),$$

it follows that[1]

$$\begin{aligned} I_{01}(0) &= (2H_1)^{1/2} - (2H_0)^{1/2} < I_{01}(G) \\ &= -F_1 < I_{01}(G_1) = [2(H_1 - G_1)]^{1/2} - [2(H_0 - G_1)]^{1/2}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} I_{21}(G_1) &= [2(H_1 - G_1)]^{1/2} - [2(H_2 - G_1)]^{1/2} < I_{21}(G) \\ &= -F_1 < I_{21}(G_2) = [2(H_1 - G_2)]^{1/2} - [2(H_2 - G_2)]^{1/2}. \end{aligned} \tag{3.2}$$

From (3.1), it follows that

$$\{[2H_1]^{1/2} + F_1\}^2 < 2H_0 < \{[2(H_1 - G_1)]^{1/2} + F_1\}^2 + 2G_1 ,$$

and from (3.2) that

$$\{[2(H_1 - G_1)]^{1/2} + F_1\}^2 + 2G_1 < 2H_2 < \{[2(H_1 - G_2)]^{1/2} + F_1\}^2 + 2G_2 .$$

The latter two inequalities yield

$$\begin{aligned} 0 < 2(H_2 - H_0) &< -2F_1\{[2H_1]^{1/2} - [2(H_1 - G_2)]^{1/2}\} \\ &= \frac{-4F_1G_2}{\{[2H_1]^{1/2} + [2(H_1 - G_2)]^{1/2}\}} < \frac{-4F_1G_2}{(2H_0)^{1/2}} . \end{aligned} \tag{3.3}$$

The smallest value of x_3 for which the derivation of (3.3) is applicable is $x_3 = x_2$; for this choice of x_3 , it follows from the first part of (3.3) that $H_2 = G_2 > H_0$. Hence when H_0 exceeds G_2 , x_3 will be larger than x_2 and (3.3) is valid.

A similar argument applied to the trajectory 456 yields

$$\begin{aligned} 0 < 2(H_6 - H_4) &< -2F_1\{[2H_5]^{1/2} - [2(H_5 - G_2)]^{1/2}\} \\ &< \frac{-4F_1G_2}{\{[2H_5]^{1/2} + [2(H_5 - G_2)]^{1/2}\}} < \frac{-4F_1G_2}{\{(2H_6)^{1/2} + F_1\}} , \end{aligned} \tag{3.4}$$

where the last inequality in (3.4) is based on the assumptions $H_5 - G_2 > 0$ and $[2H_6]^{1/2} + F_1 > 0$. However, when $H_6 \geqq H_0$, these are both satisfied if H_0 is chosen so large that

$$(2H_0)^{1/2} \geqq (2G_2)^{1/2} - F_1 . \tag{3.5}$$

For,

$$(2H_5)^{1/2} = (2H_6)^{1/2} - I_{56}(0) > (2H_6)^{1/2} + F_1 \geqq (2H_0)^{1/2} + F_1 \geqq (2G_2)^{1/2} > 0.$$

Therefore, assuming $H_6 \geqq H_0$ and (3.5), one has from Eqs. (3.3) and (3.4)

$$\begin{aligned} H_2 - H_4 < (H_2 - H_0) + (H_6 - H_4) &< -\frac{2F_1G_2}{(2H_0)^{1/2}} - \frac{2F_1G_2}{(2H_0)^{1/2} + F_1} \\ &= -2F_1G_2\left\{\frac{1}{(2H_0)^{1/2}} + \frac{1}{[(2H_0)^{1/2} + F_1]}\right\}. \end{aligned} \tag{3.6}$$

[1]The following analysis, which leads to inequality (3.3), sharper than the author's original result, is due to the reviewer.

4. Behaviour of the solution for $x \geqq x_2$. Since $I_{32}(G) = I_{43}(G) = F_3$ and $(H - G)_{32} > (H - G)_{43}$,

$$H_2 - H_3 = \int_{32} dH > \int_{43} dH = H_3 - H_4 ,$$
$$H_2 - H_4 = (H_2 - H_3) + (H_3 - H_4) < 2(H_2 - H_3). \tag{4.1}$$

Also,

$$I_{32}(G_2) = [2(H_2 - G_2)]^{1/2} - [2(H_3 - G_2)]^{1/2} < I_{32}(G)$$
$$= F_3 < I_{32}(G_3) = [2(H_2 - G_3)]^{1/2} < [2(H_2 - H_4)]^{1/2}. \tag{4.2}$$

From the first part of (4.2), it follows that

$$2(H_2 - G_2) < [2(H_3 - G_2) + F_3]^2,$$
$$2(H_2 - H_3) < F_3^2 + 2F_3[2(H_3 - G_2)]^{1/2} = F_3^2 + 2F_3[2(G_3 - G_2)]^{1/2}. \tag{4.3}$$

Combining with (4.1), we obtain

$$H_2 - H_4 < F_3^2 + 2F_3[2(G_3 - G_2)]^{1/2}. \tag{4.4}$$

Using the second part of (4.2), we have

$$\tfrac{1}{2}F_3^2 < H_2 - H_4 < F_3^2 + 2F_3[2(G_3 - G_2)]^{1/2}. \tag{4.5}$$

Using (3.6) and (4.5), we can now state:

Theorem 1. *When conditions* (A)-(E) *are satisfied and H_0 is so large that*

$$(2H_0)^{1/2} \geqq (2G_2)^{1/2} - F_1 , \tag{4.6}$$

$$F_3^2 \geqq -4F_1G_2\left\{\frac{1}{(2H_0)^{1/2}} + \frac{1}{[(2H_0)^{1/2} + F_1]}\right\}, \tag{4.7}$$

then $x_3 > x_2$ and $H_6 < H_0$.

As H_0 increases, H_2 increases. If $H_2 - H_4 \to \infty$, then, from (4.5), $F(\infty)\cdot G(\infty) = \infty$. On the other hand, $F(\infty) = \infty$ implies $H_2 - H_4 \to \infty$, by (4.5). If $G(\infty) = \infty$, then $H_2(> G_3)$ approaches infinity as $x_3 \to \infty$. Hence by (3.3), $H_0 \to \infty$. Therefore condition (F) is equivalent to:

$$H_0 \to \infty \qquad \text{as} \qquad x_3 \to \infty .$$

Now suppose that the differential equation (1.2) is defined for $x < 0$ and that conditions corresponding to (A)-(F) are satisfied for $x \leqq 0$; specifically, assume that:

(A′) $f(x)$, $g(x)$ are continuous for $x \leqq 0$,
(B′) $f(x) < 0$ for $x_1' < x < 0$,
(C′) $f(x) > 0$ for $x < x_1'$,
(D′) $-F(-\infty) > 0$,
(E′) $-g(x) > 0$ for $x < 0$,
(F′) $-F(-\infty)G(-\infty) = \infty$.

If one replaces x by $-\xi$, eq. (1.2) becomes

$$\ddot{\xi} + f_1(\xi)\dot{\xi} + g_1(\xi) = 0,$$

where

$$f_1(\xi) = f(-\xi),$$

$$g_1(\xi) = -g(-\xi),$$

Then

$$f_1(\xi),\ g_1(\xi),\ F_1(\xi) = \int_0^\xi f_1(\xi)\,d\xi, \qquad G_1(\xi) = \int_0^\xi g_1(\xi)\,d\xi$$

satisfy conditions (A)-(F), although with x_1 replaced by $-x_1'$.

Now consider in the region $x \geqq 0$ a trajectory starting with initial energy H_0, returning to $x = 0$ with energy H_6, continuing into the region $x < 0$ (with initial energy H_6), and returning to $x = 0$ with energy H_{12}. Because of assumption (F), H_6 and H_{12} are monotone increasing functions of H_0 in the interval $0 \leqq H_0 < H_{12}$. According to Theorem 1, $H_6 < H_0$ for sufficiently large H_0. If H_6 were unbounded as $H_0 \to \infty$, then, applying Theorem 1 to the region $\xi = -x \geqq 0$, H_6 may be chosen so large that $H_{12} < H_6$. Hence for large H_0, $H_{12} < H_0$.

Now consider motions of small amplitude. When $x_3 \leqq x_2$, $H_0 < H_6$. A similar result holds for negative x. Therefore for small H_0, $H_{12} > H_0$.

Therefore H_{12} is a continuous, monotone-increasing function of H_0 for $0 \leqq H_0 < \infty$ such that $H_{12} - H_0$ changes sign. For some H_0, $H_{12} = H_0$, showing that there is a periodic solution of eq. (1.2). It is shown in Ref. 3 that $H_6 - H_0$ is monotone decreasing. Then $H_{12} - H_6$ and $H_6 - H_0$ are also monotone-decreasing and the periodic solution is unique.

Theorem 2. *If Eq.* (1.2) *satisfies conditions* (A)-(F) *and* (A′)-(F′), *there is a unique periodic solution.*

5. A counter-example. Let us denote a trajectory *recurrent* when $H_6 = H_0$. It is shown in Ref. 3 that if $f(x)$ is even and $g(x)$ odd relative to $x = 0$, a periodic characteristic is recurrent. It will now be shown that when condition (F) is not satisfied, there is, in general, no recurrent trajectory and therefore no periodic solution.

Consider the family of equations (1.2) defined by

$$\begin{aligned} F_\epsilon(x) &= F(x), \qquad 0 \leqq x \leqq x_2, \\ &= \epsilon F(x), \qquad x \geqq x_2, \qquad 0 < \epsilon \leqq 1 \end{aligned}$$

For every ϵ,

$$G_2 \leqq H_4 \leqq G(\infty) < \infty. \tag{5.1}$$

Now for every choice of H_4 satisfying (5.1), consider the trajectory 456 (Fig. 1) and that trajectory 012 for which $H_0 = H_6$ of the afore-mentioned part 456. Then $H_2 - H_4$ is a positive-valued, continuous function of H_4 on the closed interval (5.1); $H_2 - H_4$ therefore attains a minimum $\Delta > 0$ for some H_4.

If there were a recurrent solution for (1.2) for every ϵ, $0 < \epsilon \leqq 1$, then from (4.4),

$$0 < \Delta \leqq H_2 - H_4 < [F_\epsilon(x_3)]^2 + 2F_\epsilon(x_3)\{2[G(x_3) - G_2]\}^{1/2}$$

$$< \epsilon^2[F(\infty)]^2 + 2\epsilon F(\infty)\{2[G(\infty) - G_2]\}^{1/2}.$$

By choosing ϵ small enough, the inequality may be violated. Hence for such an ϵ, $H_0 \neq H_6$.

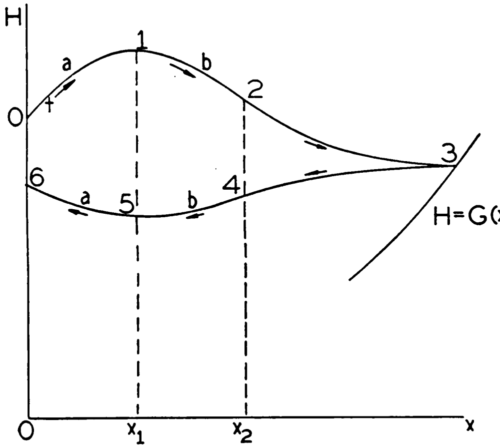

Fig. 1. Qualitative characteristics of the trajectory.

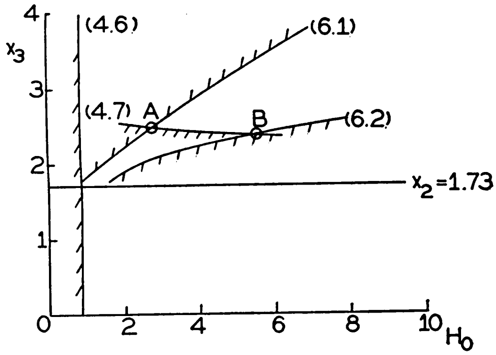

Fig. 2. Graphical solution of upper bounds.

6. Numerical bounds. It is useful, for application to specific problems, to develop numerical bounds to the amplitude x_3 of recurrent motion. Since $x_3 > x_2$, a lower bound is x_2. The following set of inequalities is useful, in some cases, for obtaining an upper bound to x_3 and H_0.

For $x_3 > x_2$, one has, using (3.3) and (4.3)

$$G_3 < H_0 + (H_2 - H_0) < H_0 - \frac{2F_1G_2}{(2H_0)^{1/2}}, \tag{6.1}$$

$$H_0 < (H_2 - H_3) + H_3 < \frac{F_3^2}{2} + F_3[2(G_3 - G_2)]^{1/2} + G_3. \tag{6.2}$$

A trajectory satisfying inequalities (4.6), (4.7), (6.1), and (6.2) has the property $H_0 > H_6$ and its maximum amplitude x_3 and initial ordinate H_0 furnish upper bounds to the corresponding characteristics of the periodic motion.

Example. Take $f(x) = -\frac{1}{2}(1 - x^2)$, $g(x) = x$.

Then,

$$F(x) = \frac{1}{2}\left(-x + \frac{x^3}{3}\right), \qquad G(x) = \frac{x^2}{2}, \qquad x_1 = 1, \qquad x_2 = 3^{1/2},$$

$$F_1 = -\frac{1}{3}, \qquad G_2 = \frac{1}{2}, \qquad G_2 = \frac{3}{2}.$$

The graphical solution of the inequalities is shown in Fig. 2. The curve of x_3 vs. H_0

lies between the curves labelled (6.1) and (6.2); the intersection with curve (4.7) defines a point whose coordinates are suitable as upper bounds. Referring to Fig. 2, it is seen that x_3 at point A is an upper bound to the amplitude and H_0 at point B is an upper bound to the initial energy. Thus

$$1.73 < x_3 < 2.50$$

and the average 2.11 of the bounds is in error by less than 23%.

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QUENCHING STRESSES IN TRANSPARENT ISOTROPIC MEDIA AND THE PHOTOELASTIC METHOD*

By R. C. O'ROURKE and A. W. SAENZ (*University of Michigan*)

Introduction. It is well known that when a transparent non-crystalline solid, such as glass, is heated to a uniform temperature T_1 and then rapidly quenched in a bath at temperature $T_0 (T_1 > T_0)$ there results a non-uniform stress distribution. Depending on T_1, one can divide these stresses into two distinct classes. If T_1 is sufficiently lower than the softening temperature of the material, then the stresses are transient, but if T_1 is sufficiently high (say 550° C for lime glass) then the quenching stresses are not transient but, on the contrary, remain permanently set into the glass. The latter are referred to in the literature as quenching or residual stresses. As is well known, the existence of such residual stresses is made manifest when the object is examined in polarized light, and by violent explosive characteristics of quenched objects when cut.† Each distribution of such stresses is characterized by a definite double refraction pattern. In 1841, F. E. Neumann[1] developed a general mathematical theory of the double refraction of light in non-uniformly heated isotropic solids. In turn the problem was studied theoretically by such men as Maxwell,[2] and Lord Rayleigh.[3]

The purpose of this paper is three-fold:

(1) To develop a mathematical theory of residual stresses based on a simple model

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†For details on the background of the subject see E. G. Coker and L. N. G. Filon, *Treatise on photo-elasticity*, Cambridge, 1931, §§ 332 and 333.

[1]F. E. Neumann, *Die Gesetze der Doppelbrechnung des Lichtes in comprimierten oder ungleichförmig unkrystallinischen Körpern*, Abh. Königl. Acad. Wiss. Berlin, Part II, pp. 1-254 (1841).

[2]J. C. Maxwell, *On the equilibrium of elastic solids*, Roy. Soc. Edin. Trans. **20**, 87-120 (1853).

[3]Lord Rayleigh, *On the stresses in solid bodies due to unequal heating and on the double refraction resulting therefrom*, VI-1, pp. 169-178, 1901; see also Arch. Neerl. (II) **5**, 32-42, (1900) and *Collected Papers*, vol. 4, p. 502.