lies between the curves labelled (6.1) and (6.2); the intersection with curve (4.7) defines a point whose coordinates are suitable as upper bounds. Referring to Fig. 2, it is seen that x_3 at point A is an upper bound to the amplitude and H_0 at point B is an upper bound to the initial energy. Thus

$$1.73 < x_3 < 2.50$$

and the average 2.11 of the bounds is in error by less than 23%.

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QUENCHING STRESSES IN TRANSPARENT ISOTROPIC MEDIA AND THE PHOTOELASTIC METHOD*

By R. C. O'ROURKE and A. W. SAENZ (*University of Michigan*)

Introduction. It is well known that when a transparent non-crystalline solid, such as glass, is heated to a uniform temperature T_1 and then rapidly quenched in a bath at temperature $T_0 (T_1 > T_0)$ there results a non-uniform stress distribution. Depending on T_1, one can divide these stresses into two distinct classes. If T_1 is sufficiently lower than the softening temperature of the material, then the stresses are transient, but if T_1 is sufficiently high (say 550° C for lime glass) then the quenching stresses are not transient but, on the contrary, remain permanently set into the glass. The latter are referred to in the literature as quenching or residual stresses. As is well known, the existence of such residual stresses is made manifest when the object is examined in polarized light, and by violent explosive characteristics of quenched objects when cut.† Each distribution of such stresses is characterized by a definite double refraction pattern. In 1841, F. E. Neumann[1] developed a general mathematical theory of the double refraction of light in non-uniformly heated isotropic solids. In turn the problem was studied theoretically by such men as Maxwell,[2] and Lord Rayleigh.[3]

The purpose of this paper is three-fold:

(1) To develop a mathematical theory of residual stresses based on a simple model

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†For details on the background of the subject see E. G. Coker and L. N. G. Filon, *Treatise on photo-elasticity*, Cambridge, 1931, §§ 332 and 333.

[1]F. E. Neumann, *Die Gesetze der Doppelbrechnung des Lichtes in comprimierten oder ungleichförmig unkrystallinischen Körpern*, Abh. Königl. Acad. Wiss. Berlin, Part II, pp. 1-254 (1841).

[2]J. C. Maxwell, *On the equilibrium of elastic solids*, Roy. Soc. Edin. Trans. **20**, 87-120 (1853).

[3]Lord Rayleigh, *On the stresses in solid bodies due to unequal heating and on the double refraction resulting therefrom*, VI-1, pp. 169-178, 1901; see also Arch. Neerl. (II) **5**, 32-42, (1900) and *Collected Papers*, vol. 4, p. 502.

of the quenching process. This theory is essentially a careful reinterpretation of the classical theory of thermal stresses, and is implicit in the work of Lord Rayleigh.[3]

(2) To justify, at least in a preliminary way, the use of the photoelastic method in the study of quenching stresses with particular emphasis on glass.

(3) To recognize the problem of determining the distribution of stresses photoelastically in symmetrically quenched spheres and cylinders as a simple problem in integral equations.

1. Photoelastic preliminaries. In this section the necessary photoelastic formulas used in the study of residual stresses in spheres and cylinders will be developed. The Maxwell-Neumann stress-optic law[4] states that the secondary principal stresses in the plane of the wave front are related to the retardation dR, suffered in traveling a distance dy, as follows:

$$dR = C(P - Q)\, dy \tag{1.01}$$

provided the direction of the principal stresses in the plane of the wavefront does not vary along the light path (Fig. 1). It is desired to calculate the integrated retardation suffered by a plane monochromatic wave in passing through symmetrically quenched spheres and cylinders. The wavefronts are parallel to the x-z plane (Fig. 1). For the integrated retardation one can write:

$$R(\eta) = C \int_{-\eta}^{+\eta} (P - Q)\, dy \tag{1.02}$$

(C is the stress-optic coefficient in Brewsters; y is measured in mm; $(P - Q)$ is the difference of the two principal stresses in the plane of the wavefront).

In the case of a long cylinder, whose surface is kept at T_0, the stress tensor in cylindrical coordinates assumes the following diagonal form:

$$\| \sigma_{ij} \| = \begin{Vmatrix} \sigma_\rho(\rho) & 0 & 0 \\ 0 & \sigma_\phi(\rho) & 0 \\ 0 & 0 & \sigma_z(\rho) \end{Vmatrix}, \tag{1.03}$$

where σ_ρ, σ_ϕ, and σ_z are functions of ρ only. In order to obtain the principal stresses in the plane of the wavefront, one must transform the stress tensor (1.03) to appropriate cartesian axes (See Fig. 1) by means of a rotation about the z axis through an angle φ, given by

$$R_z(\phi) = \begin{Vmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{Vmatrix} \tag{1.04}$$

which transforms the stress tensor to:

[4]For the basic photoelastic concepts, see Coker and Filon, *loc. cit.*, Chapter III, especially paragraph 3 and 10.

$$\left\| \begin{matrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{matrix} \right\| = \left\| \begin{matrix} \cos^2 \phi\sigma_\rho + \sin^2 \phi\sigma_\phi & \sin \phi \cos \phi(\sigma_\rho - \sigma_\phi) & 0 \\ \sin \phi \cos \phi(\sigma_\rho - \sigma_\phi) & \sin^2 \phi\sigma_\rho + \cos^2 \phi\sigma_\phi & 0 \\ 0 & 0 & \sigma_z \end{matrix} \right\| \tag{1.05}$$

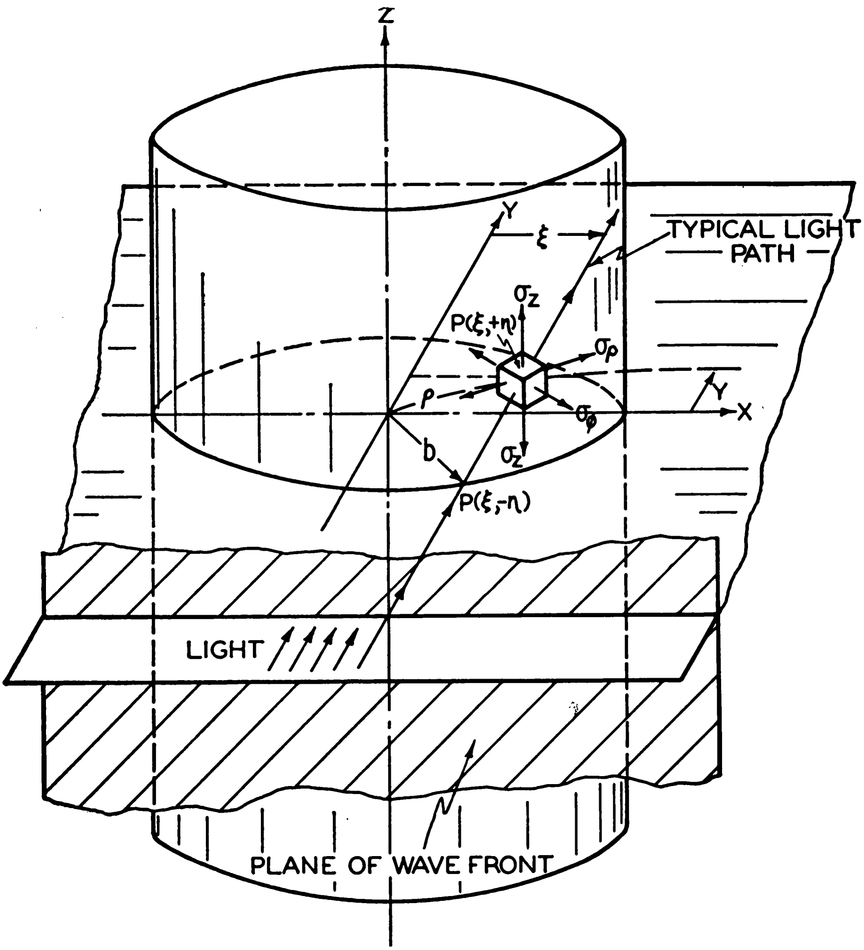

FIG. 1. Normal incidence on cylinder.

From (1.05) one readily identifies the principal stresses in the plane $x - z$ of the wave-front as

$$P = \sigma_z ,$$
$$Q = \cos^2 \phi\sigma_\rho + \sin^2 \phi\sigma_\phi . \tag{1.06}$$

Reversing roles of P and Q merely changes the sign of the retardation. This is equivalent to inverting the Babinet Compensator.[5] Thus the total retardation becomes:

$$R(\eta) = C \int_{-\eta}^{+\eta} \{\sigma_z - \cos^2 \phi\sigma_\rho - \sin^2 \phi\sigma_\phi\} \, dy. \tag{1.07}$$

Now, this integral can be simplified because of the inherent symmetry involved in the study of spheres and cylinders. One expresses $\cos^2 \varphi$, $\sin^2 \varphi$, and y in terms of ξ (Fig. 1)

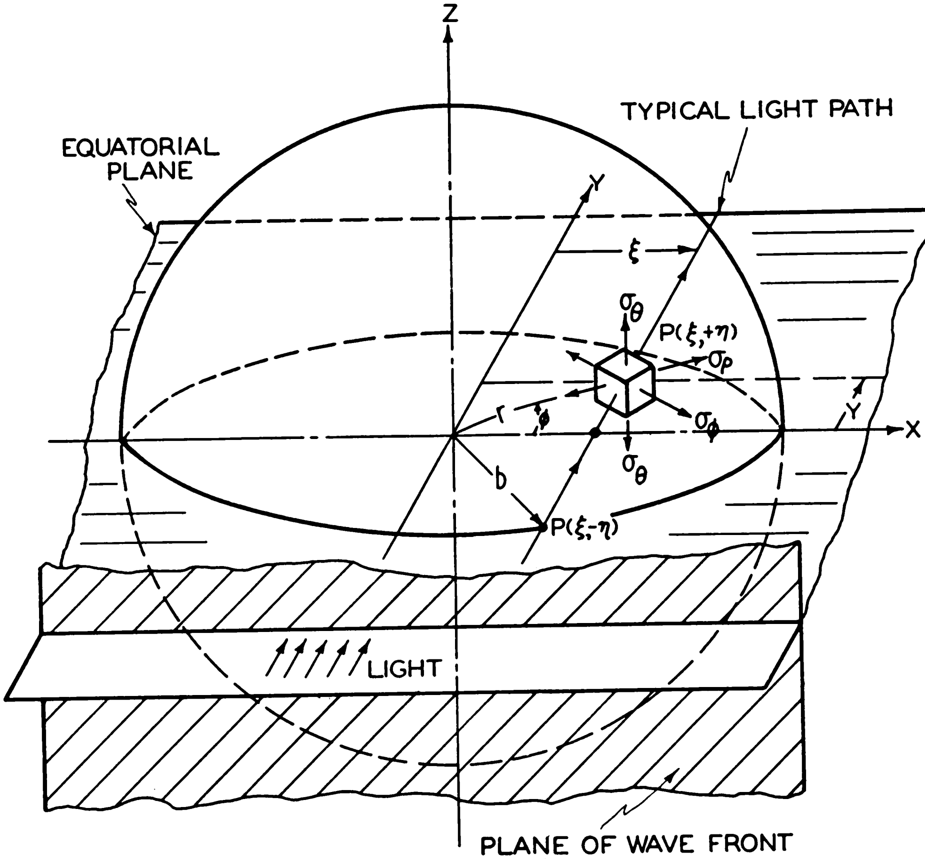

FIG. 2. Normal incidence on sphere.

and ρ, and notes the fact that σ_ρ , σ_ϕ and σ_z are functions of ρ only. Then equation (1.07) becomes:

$$R(\xi) = 2C \int_\xi^b \left\{(\sigma_z - \sigma_\phi) + (\sigma_\phi - \sigma_\rho) \frac{\xi^2}{\rho^2}\right\} \frac{\rho \, d\rho}{(\rho^2 - \xi^2)^{1/2}} \cdot \tag{1.08}$$

This equation is of fundamental importance for the study of residual stresses in cylinders and spheres. It can be studied in two different ways, both of which are treated in detail below. The function $R^*(\xi)$, (an asterisk will signify that the function is obtained

[5]Coker and Filon, *loc. cit.* paragraphs 1 and 36.

experimentally), is easily obtained with a suitable Babinet compensator. (1.08) reduces to an Abel integral equation with the aid of appropriate substitutions, thereby enabling one to give an exact solution for each of the three stresses, σ_ρ, σ_ϕ, σ_z. (See Section III). The second way of treating (1.08) is to employ a model of the quenching process (Sect. II), and calculate σ_ρ, σ_ϕ, and σ_z, thereby finding a theoretical retardation pattern $R(\xi)$ which can then be compared with experiment. The case of the sphere lends itself to an analogous treatment. One measures $R^*(\xi)$ along the equatorial plane (See Fig. 2) where no rotation in the plane of the wavefront occurs. The stress tensor is diagonal in spherical coordinates r, ϑ, φ, for a sphere quenched in the same way as the cylinder above. It assumes the form:

$$|| \sigma_{ij} || = \left\| \begin{matrix} \sigma_r(r) & 0 & 0 \\ 0 & \sigma_\vartheta(r) & 0 \\ 0 & 0 & \sigma_\varphi(r) \end{matrix} \right\| ; \qquad \sigma_\vartheta = \sigma_\phi . \tag{1.09}$$

The retardation integral for the sphere becomes:

$$R(\xi) = 2C\xi^2 \int_\xi^b (\sigma_\phi - \sigma_r) \frac{dr}{r(r^2 - \xi^2)^{1/2}} \cdot \tag{1.10}$$

2. Applicability of photoelastic method in the study of quenching stresses. In Sec. 3 a method will be developed which reduces the calculation of residual stresses in symmetrically quenched spheres and cylinders to the inversion of Abel's integral equation. For the cylinder, this method will depend upon the assumption that residual stresses are elastic stresses; this assumption is implied in the "sum rule." The "sum rule" is the relation $\sigma_z(\rho) = \sigma_\rho(\rho) + \sigma_\phi(\rho)$ which is characteristic of thermoelastic stresses in symmetrically quenched circular cylinders (hollow and solid). The necessity for making this assumption lies in the fact that the method being proposed uses *relative* retardations $R^*(\xi)$ (Babinet Compensator) instead of *absolute* retardations (Mach-Zehnder interferometer). If *absolute* retardations are measured, then one can invert the appropriate Abel integral equations of Sec. 3 without the "sum rule" ansatz. (This will be reported on elsewhere.) However, for practical purposes, it is much more expedient to measure *relative* retardations since, as will be shown in Sec. 3, one obtains the complete distribution of residual stresses in symmetrically quenched cylinders from a single photograph of $R^*(\xi)$.

It is therefore necessary to give some justification for the "sum rule" ansatz in the case of residual stresses of the type being considered. There are at least four exact methods for showing the elastic nature of these residual stresses, two of which will be explained now. First of all, Eq. (1.08) can be written in the following form:

$$R^*(\xi) = 2C \int_\xi^b \sigma_z \frac{\rho \, d\rho}{(\rho^2 - \xi^2)^{1/2}} , \tag{2.01}$$

since equilibrium demands that

$$\int_0^\eta Q \, dy = 0.$$

Now if one uses the "sum rule" and the equilibrium equation

$$\frac{d}{d\rho}(\rho\sigma_\rho) - \sigma_\phi = 0$$

Eq. (2.01) becomes

$$R^*(\xi) = 2C\int_\xi^b \frac{d}{d\rho}(\rho^2\sigma_\rho)\frac{d\rho}{(\rho^2 - \xi^2)^{1/2}} \cdot \tag{2.02}$$

From (2.02), using Dirichlet's principle for inverting orders of integration and the fact that $\sigma_\rho(a) = \sigma_\rho(b) = 0$, one easily derives the interesting relation

$$\int_a^b R^*(\xi)\, d\xi = 0 \tag{2.03}$$

For those residual stress patterns which have been studied, (2.03) is very accurately satisfied. Secondly, it is of further interest to see if one can construct a model of the quenching stresses from which a theoretical expression for $R(\xi)$ can be computed and compared with the experimental pattern $R^*(\xi)$. This is indeed possible and will be briefly discussed here. First of all, one assumes the existence of a "state of ease" at a temperature T^*, such that for $T > T^*$ all stresses in the body are negligibly small compared to the stresses for $T \ll T^*$ (of order $10^{kg/}$ mm^2 for lime glass). Secondly, one neglects the temperature variation of the elastic and thermal constants. The striking agreement obtained so far between theory and experiment seems to indicate that the model is essentially correct. Many experimental investigations have revealed the fact that the properties of amorphous materials, such as glass, change rather abruptly at a certain critical temperature.[5] The proposed model merely idealizes this fact by making the elastic constants change discontinuously at the critical temperature T^*.

These assumptions readily lead to the following rule:

To calculate the residual stresses in a body (A) quenched from the uniform temperature T_1 to the temperature T_0 of a bath ($T_1 > T_0$), one calculates the thermal stresses at time $t = t^*$ ($T = T^*$) in an elastic body (B), with the same elastic and thermal constants as (A), whose initial temperature is T_1 and whose surface beginning at $t = 0$ is maintained at T_1 . The stresses in (B) at $t = t^*$ are the negative of the residual stresses at $t \to \infty$ in (A). (2.04)

Since the problems of thermoelastic stresses in spheres and cylinders have been dealt with thoroughly in the literature, it is not necessary to discuss them here. Essentially, one takes the pertinent solutions (say from Timoshenko[6]), replaces t by t^*, changes the signs of the stresses, according to (2.04), and substitutes into (1.08) and (1.10) to obtain the theoretical retardation patterns. The theoretical retardation patterns for cylinders and spheres have been calculated in this fashion and are given below:

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[5] R. Houwink, *Elasticity, plasticity, and structure of matter*, Cambridge, 1940, Chap. 3 and 6.

[6] S. Timoshenko, *Theory of elasticity*, New York, 1934, Chap. 11.

Solid Cylinder (radius b):

$$R(\lambda) = -\frac{4C\alpha E(T_1 - T_0)}{1-\nu} \cdot b \cdot \sum_{n=1}^{\infty} \frac{\exp(-\chi\xi_n^2 t^*)}{\xi_n^2 b^2 J_1(\xi_n b)} R(n;\lambda)$$

$$R(n;\lambda) = -J_1(\xi_n b)(1-\lambda^2)^{1/2} + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-)^{k+l}(2l+1)\cdot(\xi_n b)^{2k+2l}}{4^{k+l}\cdot k!(k+2l+1)!} \qquad (2.05)$$

where

$$\cdot(1-\lambda^2)^{k+l-1/2} \cdot \frac{J_{2l+1}(\lambda\xi_n b)}{\lambda}\left\{\frac{1-\lambda^2}{2(k+l)+1} + \frac{2(k+l)\lambda^2}{2(k+l)-1}\right\}$$

ξ_n are positive roots of $J_0(\xi b) = 0$ and $\lambda = \xi/b$ (Fig. 1)

In practice t^* is sufficiently large so that only the "shape factor" $R(1, \lambda)$ need be calculated (see Fig. 3):

(B) Hollow Cylinder: $(a \le \rho \le b)$

$$R(\lambda) = -2C\frac{\alpha E}{1-\nu}\sum_{n=1}^{\infty}\exp(-\chi\xi_n^2 t^*)\frac{A_n}{\xi_n}\left\{\frac{2a^2 Z_1(\xi_n b) - 2abZ_1(\xi_n a)}{b^2 - a^2}\right.$$

$$\cdot(1-\lambda^2)^{1/2} + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-)^{k+l+1}(2l+1)\cdot(\xi_n b)^{2k+2l}}{4^{k+l}k!(2l+k+1)!}\frac{Z_{2l+1}(\lambda\xi_n b)}{\lambda} \qquad (2.06)$$

$$\left.\cdot(1-\lambda^2)^{k+l-1/2}\left\{\frac{1-\lambda^2}{2(k+l)+1} + \frac{2(k+l)\lambda^2}{2(k+l)-1}\right\}\right\},$$

where

$$A_n = \frac{\pi^2}{2}(T_1 - T_0)\xi_n\frac{J_0^2(\xi_n a)}{J_0^2(\xi_n b) - J_0^2(\xi_n a)}\{bZ_1(\xi_n b) - aZ_1(\xi_n a)\}$$

$$Z_0(x) = J_0(x)N_0(\xi_n b) - N_0(x)J_0(\xi_n b)$$

ξ_n are roots of $Z_0(\xi a) = 0$

(C) Sphere (radius b)

$$R(\lambda) = -\frac{4C\alpha E}{1-\nu}(T_1 - T_0)\left(\frac{\pi^3}{2}\right)^{1/2}\cdot b\cdot\sum_{n=1}^{\infty}(-)^n n\exp\left(-\frac{n^2\pi^2\chi t^*}{b^2}\right)R(n;\lambda), \qquad (2.07)$$

where

$$R(n;\lambda) = \lambda^2\sum_{k=0}^{\infty}\frac{\pi^k(2n)^{k-1/2}k!}{(2k+1)!}(1-\lambda^2)^{k+1/2}J_{k+5/2}(n\pi)$$

The leading terms $R(1;\lambda)$ for (2.05) and (2.07) are plotted in figures 3 and 4 respectively. The agreement between these "first approximations" $R(1;\lambda)$ and the corresponding experimental patterns $R^*(\lambda)$ obtained so far furnishes strong argument for the applicability of the photoelastic method in the determination of the type of quenching stresses under study here.

3. Photoelastic determination of quenching stresses in spheres and cylinders. Equation (2.02) is Abel's integral equation so that the problem of finding the distribution of residual stresses in symmetrically quenched cylinders reduces to the inversion of this

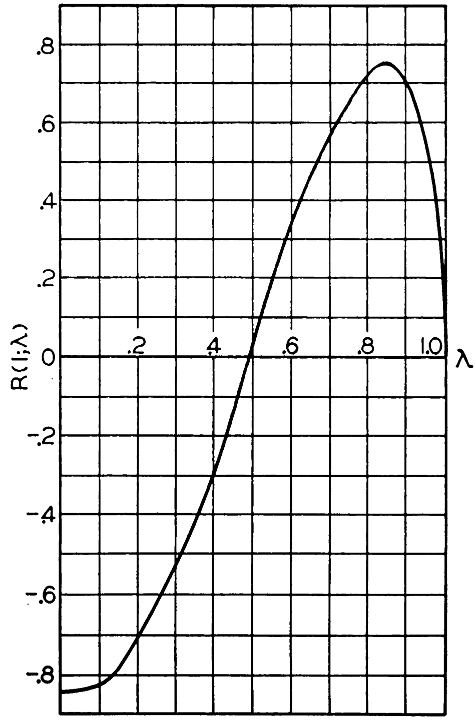

FIG. 3. Characteristic photoelastic curve for quenched solid cylinder.

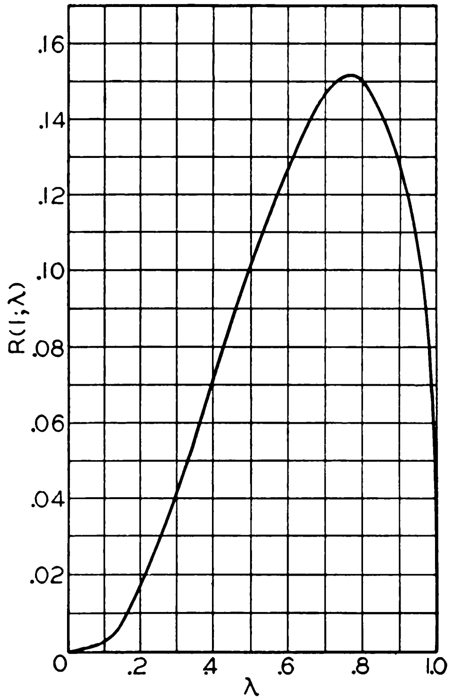

FIG. 4. Characteristic photoelestic curve for quenched glass sphere.

integral equation, which is well known. The complete solution for cylinders can be written in the form:

$$\sigma_\rho(\rho) = -\frac{1}{\pi C}\frac{1}{\rho^2}\int_\rho^b \frac{\xi R^*(\xi)\, d\xi}{(\xi^2 - \rho^2)^{1/2}},$$

$$\sigma_\phi(\rho) = \frac{d}{d\rho}(\rho\sigma_\rho(\rho)), \tag{3.01}$$

$$\sigma_z(\rho) = \sigma_\rho(\rho) + \sigma_\phi(\rho).$$

For the case of spheres, one immediately recognizes (1.10) as Abel's integral equation without the necessity of a "sum rule" (which does not exist for the corresponding thermoelastic stresses). A similar inversion yields:

$$\sigma_r(r) = -\frac{2}{\pi C}\int_r^b \frac{R^*(\xi)\, d\xi}{\xi(\xi^2 - r^2)^{1/2}},$$

$$\sigma_\vartheta(r) = \sigma_\phi(r) = \frac{r}{2}\frac{d}{dr}\sigma_r + \sigma_r. \tag{3.02}$$

Thus, the entire problem reduces to quadratures. $R^*(\xi)$ is an analytic function of the light-path variable η (or $(b^2 - \xi^2)^{1/2}$). Thus one can write

$$R^*(\xi) = \sum_{k=1}^{\infty} a_k (b^2 - \xi^2)^{k/2} \tag{3.03}$$

If (3.03) is substituted into (3.01) the quadratures can be performed and one finds for cylinders:

$$\rho^2 \sigma_\rho(\rho) = -\frac{\Gamma(1/2)}{2\pi C} \sum_{k=1}^{\infty} a_k \frac{\Gamma(k+1)}{\Gamma(k+3/2)} (b^2 - \rho^2)^{(k+1)/2} \tag{3.04}$$

Correspondingly for spheres equation (2.07) suggests the following ansatz:

$$R^*(\xi) = \sum_{k=1}^{\infty} c_k (b^2 - \xi^2)^{k/2} \xi^2 \tag{3.05}$$

and (3.02) yields

$$\sigma_r(r) = -\frac{1}{\pi C} \Gamma(1/2) \sum_{k=1}^{\infty} c_k \frac{\Gamma(k+1)}{\Gamma(k+3/2)} (b^2 - r^2)^{(k+1)/2} \tag{3.06}$$

The coefficients a_κ , c_κ are easily obtained by fitting $R^*(\xi)$ by the method of least squares.

From the above results, it is seen that the method in this section rests on the "sum rule" for the case of concentric and solid cylinders, while in that of spheres no such assumption has to be made. It is thus clear that equations (3.01) and (3.02) are independent of the detailed nature of the boundary conditions of the problem, and that they are therefore applicable when large radiation losses are present which cannot be accounted for by the simple Newton law of cooling, a situation which would preclude the use of the method of Sec. 2, from a practical standpoint.

The results obtained so far employing the procedures of Sec. 3 are of technical importance in locating exposed tension regions (i.e., unstable regions) in quenched cylinders and spheres. In such work one desires to know the sign of the boundary stresses, their absolute magnitude being of secondary importance. One could also gain rough ideas concerning residual stresses in non-transparent polycrystalline materials, such as steel, by simply changing the elastic and thermal constants appropriately and using ideas of similarity.