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THE DIRICHLET PROBLEM: BOUNDS AT A POINT FOR THE SOLUTION AND ITS DERIVATIVES*

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1. Introduction. Bounds in the mean square sense for elastostatic boundary value problems have been found by Prager and Synge [1]† by a method in which they split the original boundary value problem into two relaxed problems such that the common solution of the two relaxed problems is the solution of the original problem. They associate a set of six stress components with a point or vector in function space, and define the metric of the function space by a suitable choice of the scalar product. They show that the solution of the original boundary value problem, interpreted as a vector in function space, must lie on a hypercircle. Since the metric of the function space is positive definite, the hypercircle is bounded, so that the location of the solution of the original boundary value problem on the hypercircle bounds it in the mean square sense. They further show that these bounds may be improved by introducing additional solutions of the relaxed problems.

In a later paper by Synge [2], the method of the hypercircle is applied to boundary value problems other than the elastostatic problems; these include the Dirichlet and Neumann problems. In that paper bounds are derived for the mean value of the solution in the neighborhood of a point. The method used differs from that of the present paper, but there are certain underlying ideas in common.

In the present paper, we consider only the Dirichlet problem; we assume that the solution has already been located on a hypercircle in function space, using the general approach of the two preceding papers; we then proceed to determine bounds for the solution at a point.

We consider the problem in Euclidean N-space E_N. Apart from any further significance of this generalization, it enables us to discuss simultaneously the two most interesting cases, $N = 2, N = 3$.

In addition to determining bounds for the solution at a point, the same general method makes it possible to obtain bounds for the various derivatives of the solution. Bounds for the derivatives are considered separately in Sec. 4.

The work up to this point is concerned specifically with the problem of determining bounds for the solution and its derivatives at a point interior to the domain of definition

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†Numbers in brackets refer to the bibliography at the end of the paper.

of our problem. However, for a certain type of problem (e.g. the torsion problem), it is useful to have bounds for the derivatives at points on the boundary. In Sec. 5, bounds are obtained for the normal derivative at a point on a plane face.

In Sec. 6, a numerical example is given to illustrate the practical application of the method. At a specified point, two bounds are obtained for the numerical value of the solution of the Dirichlet problem in a plane region.

An alternative method, suggested by the work of Greenberg, is discussed in Sec. 7. This method is discussed for the two dimensional problem, but the reader may extend it to N-dimensions without difficulty.

2. Notation and auxiliary theorem. Latin suffixes take the range 1, 2, $\cdots$, N, and the summation convention holds for repeated suffixes. The summation convention does not hold for Greek suffixes. The coordinates are rectangular cartesian, and differentiation with respect to a coordinate is indicated by a subscript following a comma ($u_{,i} = \partial u/\partial x_i$).

We shall denote by V an open domain in Euclidean N-space E_N, bounded by a closed surface B of $(N - 1)$ dimensions: V may be simply or multiply connected. We shall have occasion to make use of an N-dimensional sphere with center at a general point P (contained in V) and with radius a. The N-dimensional interior of any such sphere will be denoted by v and its bounding $(N - 1)$-surface by b (Fig. 1). The unit normal

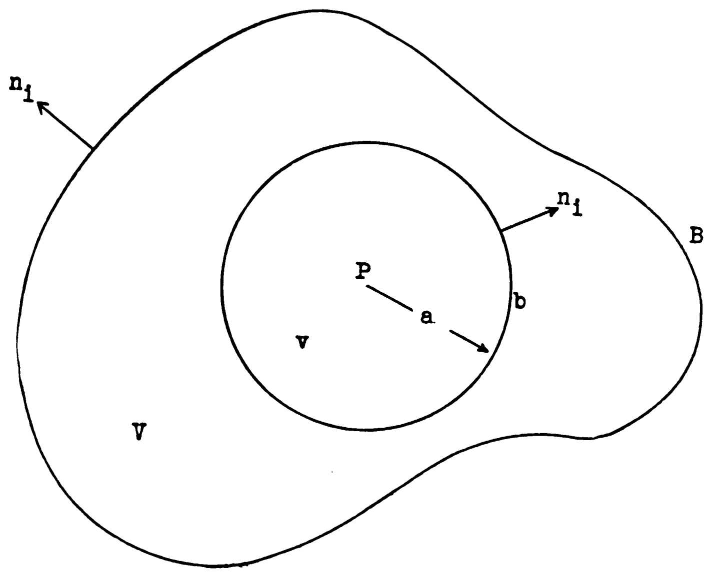

FIG. 1. In Euclidean N-space, V is an arbitrary open domain bounded by an $(N - 1)$-dimensional surface B. The point P is surrounded by an N-dimensional sphere b of radius a and center at P: v is the volume enclosed by b. The unit normal n_i is directed outward from V on B and outward from v on b.

$n_i (i = 1, 2, \cdots , N)$ will always be directed outward from V on B and outward from v on b.

In all cases, integration will be denoted by a single sign and the range of integration indicated by the element of integration unless otherwise stated.

We introduce the idea of function space, which we shall denote by F, taking care to distinguish between a vector in F and a vector in the Euclidean N-space, E_N. The former will be denoted by heavy type, $\mathbf{S}$; the latter will carry a subscript, u_i.

We define a vector $\mathbf{S}$ in F to be any vector field p_i in $V + B$. We require p_i to satisfy certain differentiability conditions, namely, that V can be divided into a finite number of subregions such that p_i is continuous and has continuous first derivatives in each such subregion. We further require that the normal component $(p_i n_i)$ of p_i be continuous across the $(N - 1)$-dimensional surface separating two such subregions.

The scalar product of two vectors $\mathbf{S}$ and $\mathbf{S}'$ will be denoted by $\mathbf{S}\cdot\mathbf{S}'$ and defined by

$$\mathbf{S}\cdot\mathbf{S}' = \int p_i p_i'\, dV, \tag{2.1}$$

where the volume element dV indicates that the integration extends throughout V. On putting $\mathbf{S}' = \mathbf{S}$, we get

$$|\,\mathbf{S}\,|^2 = \mathbf{S}^2 = \mathbf{S}\cdot\mathbf{S} = \int p_i p_i\, dV. \tag{2.2}$$

This gives the metric in F and $\mathbf{S} = 0$ implies $p_i = 0$ throughout V.

Synge [2] has established the fact that the extremity of the solution vector $\mathbf{S}$ lies on a hypercircle Γ in F and has expressed this fact by writing

$$\mathbf{S} = \mathbf{C} + R\mathbf{J}, \tag{2.3}$$

where the extremity of $\mathbf{C}$ is the center of Γ and R its radius: $\mathbf{C}$ and R are considered to be known quantities for present purposes and are given explicitly by the following formulas:

$$\begin{aligned} R^2 &= \frac{1}{4}\left[\mathbf{S}^{*2} - \sum_{\mu=1}^{m_1} (\mathbf{S}^*\cdot\mathbf{I}_\mu')^2 - \sum_{\nu=1}^{m_2} (\mathbf{S}^*\cdot\mathbf{I}_\nu'')^2\right], \\ \mathbf{C} &= \frac{1}{2}\left[\mathbf{S}^* - \sum_{\mu=1}^{m_1} \mathbf{I}_\mu'(\mathbf{S}^*\cdot\mathbf{I}_\mu') + \sum_{\nu=1}^{m_2} \mathbf{I}_\nu''(\mathbf{S}^*\cdot\mathbf{I}_\nu'')\right]. \end{aligned} \tag{2.4}$$

The vector $\mathbf{S}^*$ is a known vector, and the vectors $\mathbf{I}_\mu'$ and $\mathbf{I}_\nu''$ are two orthonormal sets of unit vectors which together form an orthonormal set of $(m_1 + m_2)$ unit vectors, $\mathbf{I}_\rho$:

$$\mathbf{I}_\rho\cdot\mathbf{I}_\sigma = \delta_{\rho\sigma}\,, \qquad (\rho, \sigma = 1, 2, \cdots, m = m_1 + m_2), \tag{2.5}$$

where $\delta_{\rho\sigma}$ is the Kronecker delta. The vector $\mathbf{J}$ is arbitrary except for the conditions expressing that $\mathbf{J}$ is a unit vector, and that it is orthogonal to the known orthonormal set $\mathbf{I}_\rho$:

$$\mathbf{J}\cdot\mathbf{J} = 1, \qquad \mathbf{J}\cdot\mathbf{I}_\rho = 0, \qquad (\rho = 1, 2, \cdots, m). \tag{2.6}$$

Let $\mathbf{G}$ be any vector in F and consider the scalar product $\mathbf{S}\cdot\mathbf{G}$ of $\mathbf{G}$ and the solution vector $\mathbf{S}$. We determine maximum and minimum values for $\mathbf{S}\cdot\mathbf{G}$ as $\mathbf{S}$ ranges over the hypercircle Γ.

We may represent the vector $\mathbf{G}$ as the sum of its projections on each of the unit vectors $\mathbf{I}_\rho$ $(\rho = 1, 2, \cdots, m)$ and on the subspace of the hypercircle: that is

$$\mathbf{G} = M\mathbf{J}_0 + \sum_{\rho=1}^{m} N_\rho \mathbf{I}_\rho\,, \tag{2.7}$$

where $M\mathbf{J}_0$ ($\mathbf{J}_0$ is a unit vector) is the vector projection of $\mathbf{G}$ on the plane of the hypercircle and

$$N_\rho = \mathbf{G}\cdot\mathbf{I}_\rho\,. \tag{2.8}$$

To obtain M we square (2.7) and again use the orthogonality conditions; we obtain

$$M^2 = \mathbf{G}^2 - \sum_{\rho=1}^{m} (\mathbf{G} \cdot \mathbf{I}_\rho)^2. \tag{2.9}$$

This value is always positive, so that M is real, and we shall understand by M the positive square root of (2.9). Making use of (2.3), we may write

$$\mathbf{S}\cdot\mathbf{G} - \mathbf{C}\cdot\mathbf{G} = R\ \mathbf{J}\cdot\mathbf{G} = R\ \mathbf{J}\cdot\left[M\mathbf{J}_0 + \sum_{\rho=1}^{m} N_\rho \mathbf{I}_\rho\right] = RM\ \mathbf{J}\cdot\mathbf{J}_0\,. \tag{2.10}$$

Hence

$$|\,\mathbf{S}\cdot\mathbf{G} - \mathbf{C}\cdot\mathbf{G}\,| = RM\ |\,\mathbf{J}\cdot\mathbf{J}_0\,| \leq RM. \tag{2.11}$$

This may be summed up in the following theorem.

Theorem I. Let $\mathbf{G}$ *be an arbitrary vector in* F *and* $\mathbf{S}$ *the solution vector, the extremity of which is confined to lie on a hypercircle in* F*. Then bounds for the scalar product* $\mathbf{S}\cdot\mathbf{G}$ *are given by*

$$|\,\mathbf{S}\cdot\mathbf{G} - \mathbf{C}\cdot\mathbf{G}\,| \leq RM, \tag{2.12}$$

Where R *is the radius of the hypercircle and* M *is the positive square root of* (2.9).

3. Bounds for the solution at an interior point. The Dirichlet problem requires the determination of a function ϕ which is harmonic in a volume V and assumes assigned values on its surface B. These conditions may be written

$$\Delta\phi = 0, \qquad (\phi)_B = f, \tag{3.1}$$

where Δ is the Laplacian operator in E_N and f is a function assigned on the $(N - 1)$-dimensional bounding surface B; it will be assumed that f is piecewise continuous. The normal n_i $(i = 1, 2, \cdots, N)$ to B will be assumed to be piecewise continuous and such that a solution ϕ exists which makes the various integrals occurring converge.

It is well known that the fundamental solution of the Laplace equation is

$$\begin{aligned} G &= \ln r\ , && \text{for } N = 2, \\ &= \frac{r^{2-N}}{(2 - N)}\,, && \text{for } N \geq 3. \end{aligned} \tag{3.2}$$

As a first step toward obtaining bounds for the solution ϕ at a point P with coordinates $y_i = c_i$, we define in V a free Green's vector, obtained by differentiation of the fundamental solution:

$$\begin{aligned} G_i &= x_i r^{-N} && \text{for } r \geq a, \qquad (i = 1, 2, \cdots, N) \\ &= 0 && \text{for } r < a. \end{aligned} \tag{3.3}$$

This formula applies for $N = 2, 3, \cdots$. The coordinates x_i are relative to $P(x_i = y_i - c_i)$, r is the distance measured from $P(r^2 = x_i x_i)$, and a does not exceed the distance from P to the nearest point of B, so that the sphere $r = a$ with center at P does not cut B. It should be noted that G_i defined by (3.3) is divergence-free, that is,

$$G_{i,i} = Nr^{-N} - Nx_i x_i r^{-N-2} = 0, \tag{3.4}$$

since $x_i x_i = r^2$. This property is also evident from the fact that G_i is the gradient of the fundamental harmonic function.

We denote by $\mathbf{G}$ the vector in F corresponding to G_i in V and associate the vector $\mathbf{S}$ in F with the vector field $\phi_{,i}$ in V, where ϕ is the solution of (3.1). We seek to bound ϕ at the point P. Noting that G_i has no singularity in V, the scalar product $\mathbf{S}\cdot\mathbf{G}$ is evaluated by use of Green's theorem:

$$\begin{aligned} \mathbf{S}\cdot\mathbf{G} &= \int \phi_{,i}G_i\, dV = \int_{V-v} \phi_{,i}G_i\, dV \\ &= \int \phi G_i n_i\, dB - \int \phi G_i n_i\, db - \int_{V-v} \phi G_{i,i}\, dV, \end{aligned} \tag{3.5}$$

where db is an element of the sphere ($r = a$). The first surface integral is calculable since ϕ is known on B. The normal n_i points outward from v on b and is equal to x_i/r. Then by use of a mean value theorem for harmonic functions[1], the integral over b is expressible in terms of the value of the function ϕ at the point P:

$$\begin{aligned} \int \phi G_i n_i\, db &= \int \phi n_i x_i r^{-N}\, db = \int \phi r^{1-N}\, db \\ &= a^{1-N} \int \phi\, db = K_N \phi(P), \end{aligned} \tag{3.6}$$

where K_N is a constant depending only on the dimensionality of our space and is given explicitly by (3.9).

We may now write (3.5) in the form:

$$K_N\phi(P) = \int \phi G_i n_i\, dB - \mathbf{S}\cdot\mathbf{G}. \tag{3.7}$$

This relation expresses the value of the solution at the point P in terms of a calculable quantity and the scalar product of the solution vector and the Green's vector in the function space F.

Theorem I of Section 2 provides bounds for the scalar product; so combining these results, we have bounds for the solution at the point P which are given by the following theorem.

Theorem II. Let ϕ be a function which is harmonic in V and assumes assigned values on the boundary B of V. Let P be any interior point in V. Then bounds for $\phi(P)$ are given by the inequality

$$\left| \int fG_i n_i\, dB - \mathbf{C}\cdot\mathbf{G} - K_N\phi(P) \right| \leq MR, \tag{3.8}$$

where n_i is the unit normal outward from V on B.

In (3.8), $\mathbf{C}$, R are given by (2.4), M is the positive square root of (2.9), G_i is given by (3.3), and

$$\begin{aligned} K_N &= \frac{2\pi^{N/2}}{\Gamma(N/2)}, \qquad (K_2 = 2\pi,\ K_3 = 4\pi), \\ f &= (\phi)_B, \\ \mathbf{C}\cdot\mathbf{G} &= \int_{V-v} C_i x_i r^{-N}\, dV, \end{aligned} \tag{3.9}$$

[1] O. D. Kellogg, *Foundations of potential theory*, Berlin, 1929, pp. 223-224.

where C_i denotes the vector field in V corresponding to the vector $\mathbf{C}$ in F whose extremity lies at the center of the hypercircle.

4. Bounds for the derivatives of the solution at an interior point. To obtain bounds for the first derivatives, we define a new Green's vector $G_i^{(p)}$ which is derived from the Green's vector of Section 3 by differentiation of equation (3.3) with respect to x_p :

$$\begin{aligned} G_i^{(p)} &= \delta_{ip}r^{-N} - N\, x_i x_p r^{-N-2}, \qquad \text{for } r \geq a, \\ &= 0, \qquad \text{for } r < a, \qquad (i, p = 1, 2, \cdots, N). \end{aligned} \tag{4.1}$$

It is of course obvious that this vector is divergence-free since it is the second derivative of the fundamental harmonic function.

We denote by $\mathbf{G}^{(p)}$ the vector in the function space F corresponding to $G_i^{(p)}$ in V and evaluate the scalar product $\mathbf{S}\cdot\mathbf{G}^{(p)}$, making use of Green's theorem:

$$\begin{aligned} \mathbf{S}\cdot\mathbf{G}^{(p)} &= \int \phi_{,i} G_i^{(p)}\, dV = \int_{V-v} \phi_{,i} G_i^{(p)}\, dV \\ &= \int \phi G_i^{(p)} n_i\, dB - \int \phi G_i^{(p)} n_i\, db - \int_{V-v} \phi G_{i,i}^{(p)}\, dV. \end{aligned} \tag{4.2}$$

The integral over B is calculable since ϕ is known on B, and the last integral vanishes since $G_i^{(p)}$ is divergence-free. We evaluate the remaining integral, making use of Green's theorem and another mean value theorem for harmonic functions[2]. We observe that $n_i = x_i/r$ and $n_i x_p = n_p x_i$ on b. Then

$$\begin{aligned} \int \phi G_i^{(p)} n_i\, db &= a^{-N} \int \phi n_p\, db - N a^{-N-2} \int \phi x_i x_p n_i\, db \\ &= (1 - N) a^{-N} \int \phi_{,p}\, dv = (1 - N) L_N\, \phi_{,p}(P), \end{aligned} \tag{4.3}$$

where L_N depends only on the dimensionality of the space and is given explicitly by

$$L_N = K_N/N = \frac{\pi^{N/2}}{\Gamma(N/2 + 1)}\, \cdot \tag{4.4}$$

Equation (4.2) is now in the form:

$$(1 - N) L_N \phi_{,p}(P) = \int \phi G_i^{(p)} n_i\, dB - \mathbf{S}\cdot\mathbf{G}^{(p)}. \tag{4.5}$$

This relation gives an expression for the value of the first derivative of the solution at the point P in terms of a calculable integral and the scalar product of the solution vector and a Green's vector in the function space F.

The result (4.5) when combined with Theorem I gives bounds for the first derivatives of the solution ϕ at the point P. Thus, we have established the following theorem.

Theorem III. Let ϕ be a function which is harmonic in V and assumes assigned values on the boundary B of V. Then bounds for the first derivatives of the solution at a point P contained in V are given by the inequality

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[2]*Loc. cit.* pp. 223-224.

$$\left| \int fG_i^{(p)} n_i \, dB - \mathbf{C} \cdot \mathbf{G}^{(p)} + (N - 1) L_N \phi_{,p}(P) \right| \le M^{(p)} R, \tag{4.6}$$

where n_i represents the unit normal outward from V on B.

The value of L_N is given by (4.4) and $M^{(p)}$ is obtained from (2.9) by replacing $\mathbf{G}$ by $\mathbf{G}^{(p)}$.

A process similar to the preceding gives bounds for the higher order derivatives of ϕ at a point P interior to V. The appropriate vector field in V is obtained in each case by differentiation of (3.3). The variables with respect to which (3.3) is to be differentiated are determined by the particular derivative of ϕ for which bounds are being sought. For example, if bounds are required for $\phi_{,pq}$, then the vector field is obtained for this case by differentiation of (3.3) with respect to x_p and x_q . The resulting vector field is then made to correspond to a Green's vector in the function space F and the scalar product of this Green's vector and the solution vector is computed. Combining the results of this computation with Theorem I gives bounds for the derivative under consideration. Such results may be stated in a manner analogous to Theorem III with appropriate changes. Thus, if bounds were required for $\phi_{,pq}$ the inequality (4.6) must be replaced by

$$\left| \int fG_i^{(pq)} n_i \, dB - \mathbf{C} \cdot \mathbf{G}^{(pq)} - \frac{N^2}{N+2} L_N \phi_{,pq}(P) \right| \le M^{(pq)} R, \tag{4.7}$$

with the obvious meaning given to the changed notation.

5. Bounds for the normal derivative at a point on a plane boundary. The preceding work is concerned with the problem of determining bounds for the solution and its

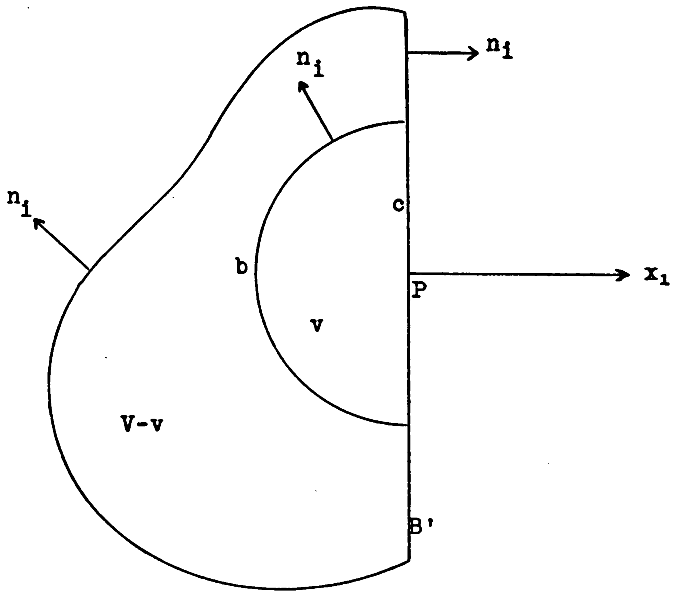

FIG. 2. $\Delta\phi = 0$ in V, $(\phi)_B = f$, a given function. The origin of coordinates is at P and the x_1-axis is perpendicular to the plane portion of the bounding surface B. The hemispherical domain surrounding P is bounded by $b + c$, where $c = B - B'$.

derivatives at a point interior to the domain V. However in certain types of problems, it is useful to have bounds for the normal derivative at a point on the boundary. In this section, we confine our attention to a domain V bounded by a closed surface B such that at least one portion of B is a plane (Fig. 2). We shall establish bounds for the normal derivative of the solution at a point P on this plane portion of the bounding surface.

A system of rectangular cartesian coordinates is introduced such that the origin is at P and the positive x_1-axis is perpendicular to B and is directed outward from V. Thus the derivative $\phi_{,1}$ is the normal derivative of ϕ at P. The point P is then surrounded by a hemispherical domain v of radius a lying entirely in V. The surface of the hemisphere is denoted by b and that portion of B which caps b is called c. We use B' to denote $B - c$.

We define a Green's vector $\mathbf{G}^{(1)}$ in the function space F corresponding to the vector field in V obtained from (4.1) by setting $p = 1$, and compute the scalar product of $\mathbf{G}^{(1)}$ and the solution vector $\mathbf{S}$. However after applying Green's theorem, the surface integrals over B and the boundary of v have the range of integration c in common so that the scalar product takes the form

$$\mathbf{S}\cdot\mathbf{G}^{(1)} = \int_{B'} \phi G_i^{(1)} n_i \, dB' - \int_b \phi G_i^{(1)} n_i \, db. \tag{5.1}$$

As in the previous instances, the first integral is calculable since $\phi = f$ on B', but we no longer have the mean value property of harmonic functions to aid in the evaluation of the last integral since it is an integral over the hemispherical surface b. However, this last integral can be evaluated by other means to obtain the result

$$\int_b \phi G_i^{(1)} n_i \, db = a^{-N} \int_0^a z^{N-1} \, dz \int_0^z F(x) \, dx + (N - 1)a^{-N} \int_c f \, dc + \frac{1}{2}(1 - N)L_N \phi_{,1}(P), \tag{5.2}$$

where

$$F(x) = (N - 1)x^{1-N} \int_c (\phi_{,22} + \phi_{,33} + \cdots + \phi_{,NN}) \, dc. \tag{5.3}$$

Since $\phi = f$ on c and the x_1-axis is perpendicular to c, the integrand of (5.3) is a known function, hence F is a known function. We note that this evaluation of the last integral of (5.1) requires that the boundary function f and its first and second derivatives be piecewise continuous and such that their integrals exist.

Upon substitution from (5.2) into (5.1), an explicit expression for $\mathbf{S}\cdot\mathbf{G}^{(1)}$ is obtained in terms of calculable integrals and the normal derivative of ϕ at the point P. Then combining this result with Theorem I, we obtain the following theorem.

Theorem IV. Let ϕ be a function which is harmonic in V and assumes assigned values on the boundary B of V. It is further required that B contains a plane portion. Then bounds for the normal derivative of ϕ at a point P on this plane portion are given by the inequality

$$\left| \int_{B'} f G_i^{(1)} n_i \, dB' - \mathbf{C}\cdot\mathbf{G}^{(1)} - a^{-N} \int_0^a z^{N-1} \, dz \int_0^z F(x) \, dx - (N - 1)a^{-N} \int_c f \, dc - \frac{1}{2}(1 - N)L_N \frac{\partial\phi(P)}{\partial n} \right| \le M^{(1)}R, \tag{5.4}$$

where n_i is the unit outward normal.

It is interesting to note that Theorem IV gives an exact value in the following case. Let ϕ be harmonic in the domain V, where V is a volume in Euclidean N-space bounded by a hemispherical surface B' of radius a and a diametral plane c. The point P is taken at the center of curvature of B' (Fig. 3). We assign ϕ the value zero on c and unity on B'

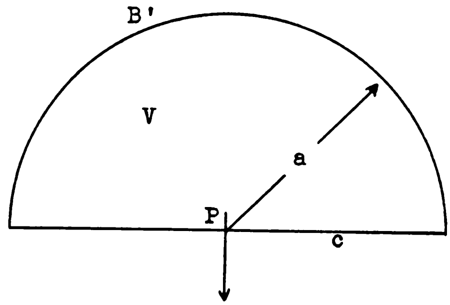

FIG. 3. $\Delta\phi = 0$ in V, $(\phi)_{B'} = 1$, $(\phi)_c = 0$. The normal derivative of ϕ at the point P is computed for the special case in which V consists of a domain bounded by a hemispherical surface and a diametral plane in Euclidean N-space.

and seek the value of the normal derivative of ϕ at the point P. We may take the volume v to coincide with V, so that obviously $M^{(1)}$ and $\mathbf{C}\cdot\mathbf{G}^{(1)}$ of (5.4) are zero. Further, since $\phi = 0$ on c, it follows that $F(a) = 0$ and that the integral of f over c is also zero. The remaining integral over B', when evaluated, gives

$$\int_{B'} fG_i^{(1)}n_i\, dB' = \frac{(1 - N)\pi^{(N-1)/2}}{a\Gamma[(N - 1)/2]} \cdot \tag{5.5}$$

Substitution of (5.5) into (5.4) gives

$$\phi_{,1}(P) = \frac{(N - 1)L_{N-1}}{a\, L_N} \cdot \tag{5.6}$$

For $N = 2$, $N = 3$, this gives respectively

$$\begin{aligned} \phi_{,1}(P) &= 2/(a\pi), \qquad (N = 2), \\ \phi_{,1}(P) &= 3/(2a), \qquad (N = 3). \end{aligned} \tag{5.7}$$

This quantity represents the intensity at the center of a circular ($N = 2$) or spherical ($N = 3$) condenser which is split symmetrically and the two halves raised to potentials $+1$ and -1.

6. A numerical example. As an illustration of the use of the method contained in the preceding sections, we consider the following problem: let the open domain V be the area in the Euclidean plane ($N = 2$) bounded by B, $x = \pm 1$, $x^2 + y^2 = 2$ (Fig. 4). We seek bounds for the function ψ which satisfies the conditions

$$\Delta\psi = 0, \qquad (\psi)_B = f = \tfrac{1}{2}(x^2 + y^2). \tag{6.1}$$

In particular, we establish numerical bounds for the function ψ at the origin (the point P in the notation of the preceding work).

We use the explicit (x, y) notation and the implicit notation interchangeably, depending on which is the more convenient.

We first find a vector field p_i^* and a scalar field ψ^* in V satisfying the conditions

$$p_i^* = \psi_{,i}^*, \qquad (\psi^*)_B = f. \tag{6.2}$$

Secondly, we find a vector field p_i'' in V satisfying

$$p_{i,i}'' = 0 \tag{6.3}$$

and no boundary conditions. The common solution of (6.2) and (6.3) is the solution of (6.1). We can obtain solutions of (6.3) by means of skew differentiation: $p_i'' = \epsilon_{ij}\chi_{,j}''$.

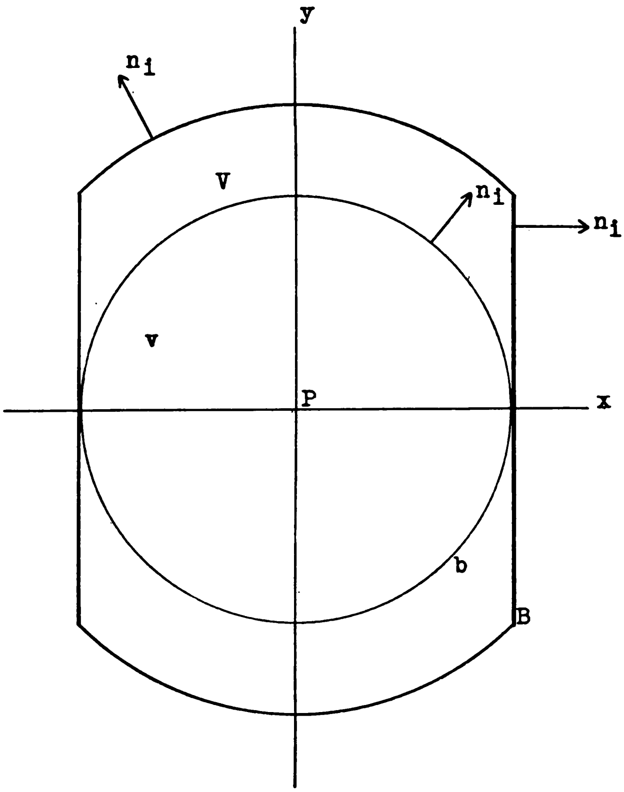

FIG. 4. $\Delta\psi = 0$ in V, $(\psi)_B = \frac{1}{2}(x^2 + y^2)$. The domain V is located in the Euclidean plane and is bounded by B: $(x^2 + y^2 - 2)(x^2 - 1) = 0$. The point P is taken at the origin and v consists of the interior of the unit circle.

For the purpose of improving the bounds, we use a set of vectors of the class (6.3) and introduce another class of vector fields p_i' in V satisfying the conditions

$$p_i' = \psi_{,i}', \qquad (\psi')_B = 0. \tag{6.4}$$

Thus our problem is that of selection of the three types of functions from which we can derive the vector fields p_i^* , p_i'', p_i' satisfying the appropriate conditions. The functions selected for use in our particular example are given in Table I. This table shows the vector in F, the vector field in V to which it corresponds, and the particular function from which the vector field is derived. The functions given in Table I are used to obtain only two approximations for the solution function ψ. It is, of course, possible to improve the approximation last obtained by the selection of additional functions.

TABLE I.

Vector in F	Vector field in V	Function from which vector field is derived
$\mathbf{S}^*$	$p_1^* = x$ $p_2^* = y$	$\psi^* = (x^2 + y^2)/2$
$\mathbf{S}_1'$	$p'_{(1)1} = 2x(2x^2 + y^2 - 3)$ $p'_{(1)2} = 2y(x^2 - 1)$	$\psi'_{(1)} = (x^2 - 1)(x^2 + y^2 - 2)$
$\mathbf{S}_2'$	$p'_{(2)1} = 2x(3x^4 + 2x^2y^2 - 6x^2 - y^2 + 2)$ $p'_{(2)2} = 2x^2y(x^2 - 1)$	$\psi'_{(2)} = x^2(x^2 - 1)(x^2 + y^2 - 2)$
$\mathbf{S}_3'$	$p'_{(3)1} = 2xy^2(2x^2 + y^2 - 3)$ $p'_{(3)2} = 2y(x^2 - 1)(x^2 + 2y^2 - 2)$	$\psi'_{(3)} = y^2(x^2 - 1)(x^2 + y^2 - 2)$
$\mathbf{S}_1''$	$p''_{(1)1} = x$ $p''_{(1)2} = -y$	$\chi''_{(1)} = xy$
$\mathbf{S}_2''$	$p''_{(2)1} = x^3 - 3xy^2$ $p''_{(2)2} = y^3 - 3x^2y$	$\chi''_{(2)} = x^3y - xy^3$

To determine what computations are required, we examine the quantities contained in the inequality (3.8) of Theorem II, Section 3; namely, for $N = 2$,

$$\left| \int fG_in_i\, dB - \mathbf{C}\cdot\mathbf{G} - 2\pi\psi(P) \right| \leq MR. \tag{6.5}$$

It is convenient to have all the scalar products available before computing the various quantities present in this inequality. These scalar products have been tabulated in a triangular array in Table II. The number in Table II which represents the scalar product of any two vectors $\mathbf{S}$, $\mathbf{S}'$ is located at the intersection of row $\mathbf{S}$ and column $\mathbf{S}'$ (or vice versa).

TABLE II. Scalar products.

	S^*	S_1'	S_2'	S_3'	S_1''	S_2''	G
S^*	4.474926						
S_1'	−8.633233	21.474728					
S_2'	−1.412487	1.458289	3.253382				
S_3'	−3.170796	6.953487	.395429	11.049406			
S_1''	−1.333333	0	0	0	4.474926		
S_2''	− .533333	0	0	0	−2.133333	7.654613	
G	2.000000	−3.141593	−.785398	−2.356194	− .858407	−.050148	1.493808

We next orthonormalize the vectors S_i', S_i''. It is inherent in the method that $S_i' \cdot S_j'' = 0$ for all i, j (this may easily be checked by integration), so it is only necessary to require that

$$\begin{aligned} S_i' \cdot S_j' &= \delta_{ij} , \\ S_i'' \cdot S_j'' &= \delta_{ij} . \end{aligned} \tag{6.6}$$

A systematic method of orthonormalization has been given by Peach [7]. We apply his method to the two preceding classes of vectors S_i', S_i'' to obtain the two orthonormal

TABLE III. Scalar products.

	I_1'	I_2'	I_3'	I_1''	I_2''
S^*	−1.863019	.465193	−.133665	−.630298	.453794
G	− .677944	.322103	−.456269	−.405789	.178329

$$\begin{aligned}
(S^* \cdot I_1')^2 &= 3.470839 & \quad (G \cdot I_1')^2 &= .459608 \\
(S^* \cdot I_2')^2 &= .216405 & \quad (G \cdot I_2')^2 &= .103750 \\
(S^* \cdot I_3')^2 &= .017866 & \quad (G \cdot I_3')^2 &= .208181 \\
(S^* \cdot I_1'')^2 &= .397275 & \quad (G \cdot I_1'')^2 &= .164665 \\
(S^* \cdot I_2'')^2 &= .205929 & \quad (G \cdot I_2'')^2 &= .031801
\end{aligned}$$

classes $\mathbf{I}'_i$, $\mathbf{I}''_i$ (obviously, $\mathbf{I}'_i \cdot \mathbf{I}''_j = 0$, for all i, j). When these computations have been carried out, we find:

$$
\begin{aligned}
\mathbf{I}'_1 &= .215792\ \mathbf{S}'_1, \\
\mathbf{I}'_2 &= -.038235\ \mathbf{S}'_1 + .563047\ \mathbf{S}'_2, \\
\mathbf{I}'_3 &= -.109735\ \mathbf{S}'_1 + .008205\ \mathbf{S}'_2 + .337178\ \mathbf{S}'_3, \\
\mathbf{I}''_1 &= .472723\ \mathbf{S}''_1, \\
\mathbf{I}''_2 &= .185041\ \mathbf{S}''_1 + .388146\ \mathbf{S}''_2 .
\end{aligned}
\tag{6.7}
$$

We also need the scalar products of $\mathbf{S}^*$ with each $\mathbf{I}'$, $\mathbf{I}''$, and the square of all such products. These are easily obtained from (6.7) and Table II: the results are given in Table III.

The approximations for R^2 are given by direct substitution from Table III into the first of (2.4). The first two approximations for R^2 are as follows:

$$
\begin{aligned}
R_1^2 &= .151703, &&\text{(approximation based on } \mathbf{S}^*, \mathbf{I}'_1, \mathbf{I}''_1\text{)},\\
R_2^2 &= .041653, &&\text{(approximation based on all vectors given above)}.
\end{aligned}
\tag{6.8}
$$

It should be pointed out that the above quantities are independent of the position of the point at which we are seeking to bound ψ. In fact, both the radius of the hypercircle and the position of its center depend only on the domain V and on the functions from which the vector fields are obtained. Consequently, we would write down the first two approximations for $\mathbf{C}$ if we so desired. However, we are not directly interested in $\mathbf{C}$, so we proceed to the computation of $\mathbf{C} \cdot \mathbf{G}$ and M.

We use the Green's vector defined by (3.3) for $N = 2$ and $a = 1$; that is,

$$
\left.\begin{aligned}
G_1 &= xr^{-2}\\
G_2 &= yr^{-2}
\end{aligned}\right\} \text{ for } r \geq 1,
\qquad
G_1 = G_2 = 0, \qquad \text{for } r < 1.
\tag{6.9}
$$

The necessary scalar products of $\mathbf{G}$ with each of the other vectors have been computed and are included in the appropriate tables. We then form the scalar product $\mathbf{C} \cdot \mathbf{G}$ by dotting $\mathbf{G}$ into the second of (2.4). Then direct substitution from Tables II and III gives the first two numerical approximations for $\mathbf{C} \cdot \mathbf{G}$. Substitution from Table III into (2.9) gives the approximations for M^2. These results are as follows:

$$
\begin{aligned}
M_1^2 &= .869535, & M_1 &= .932488, &&\text{(first approximation)},\\
\mathbf{G} \cdot \mathbf{C}_1 &= .496371, & & &&\text{(first approximation)},\\
M_2^2 &= .525803, & M_2 &= .725122, &&\text{(second approximation)},\\
\mathbf{G} \cdot \mathbf{C}_2 &= .431422, & & &&\text{(second approximation)}.
\end{aligned}
\tag{6.10}
$$

The line integral over the boundary B of the domain V is found to be

$$\int fG_{,i}n_i \, dB = 2 + \pi. \tag{6.11}$$

We may now substitute from (6.11), (6.10), (6.8) into (6.5) to obtain bounds for the function ψ at the point $P(0, 0)$. This gives

$$\begin{aligned} .682 &\leq \psi(P) \leq .797, \\ .726 &\leq \psi(P) \leq .773, \end{aligned} \tag{6.12}$$

as the first and second approximations respectively.

7. An alternative method. The reading of the work of H. J. Greenberg [8] on the same subject suggested an alternative method of attack on the problem of obtaining bounds at an interior point for the solution of the Dirichlet problem. The method is outlined below for the special case $N = 2$, but may obviously be extended to the N-dimensional case.

We let ϕ represent the solution we seek to bound at the point P and $\mathbf{S}$ the solution vector in function space corresponding to ϕ. We use two functions: the first is the free Green's function G, and the second any function H which is regular (the function and its first derivatives continuous) throughout V and is equal to G on the boundary B of V. Thus we have:

$$\begin{aligned} G &= \log r, \\ (H)_B &= (G)_B \, . \end{aligned} \tag{7.1}$$

We consider the scalar product $\mathbf{S}\cdot\mathbf{G}$ of the solution vector $\mathbf{S}$ and the Green's vector $\mathbf{G}$. We cannot apply the divergence theorem to $\mathbf{S}\cdot\mathbf{G}$ directly so we surround P with a small sphere b with center at P and radius a and remove the interior v of b. Then applying the divergence theorem to $\mathbf{S}\cdot\mathbf{G}$ for the region $V - v$ in both possible ways and making use of the fact that both ϕ and G are harmonic, we have

$$\begin{aligned} \int_{V-v} \phi_{,i}G_{,i} \, dV &= \int_B \phi G_{,i}n_i \, dB - \int_b \phi G_{,i}n_i \, db \\ &= \int_B G\phi_{,i}n_i \, dB - \int_b G\phi_{,i}n_i \, db. \end{aligned} \tag{7.2}$$

Upon taking the limit as the radius a goes to zero, we obtain the following well known expression for the value of ϕ at point P:

$$2\pi\phi(P) = \int_B \phi G_{,i}n_i \, dB - \int_B G\phi_{,i}n_i \, dB. \tag{7.3}$$

The first integral is calculable since ϕ is known on B. The essential point of the method concerns the second integral. Since the regular function H is equal to G on B, we replace G by H and again apply the divergence theorem. Since ϕ is harmonic, this gives,

$$\int_B G\phi_{,i}n_i \, dB = \int_B H\phi_{,i}n_i \, dB = \int_V H_{,i}\phi_{,i} \, dV = \mathbf{H}\cdot\mathbf{S}. \tag{7.4}$$

Thus equation (7.3) may be written:

$$2\pi\phi(P) = \int_B \phi G_{,i} n_i \, dB - \mathbf{H}\cdot\mathbf{S}. \tag{7.5}$$

Now if the extremity of $\mathbf{S}$ is confined to a known hypercircle (cf. Section 2), we know that $\mathbf{H}\cdot\mathbf{S}$ is bounded above and below. Hence (7.5) gives bounds for $\phi(P)$.

It will be recalled that in the work of Section 2 the norm of the Green's vector $|\,\mathbf{G}\,|$ enters into the bounds for $\mathbf{G}\cdot\mathbf{S}$ via the number M (see equation (2.9)); and in order to insure the existence of $|\,\mathbf{G}\,|$, it was necessary to define our vector field to be zero in a region v containing P. This difficulty does not now arise, because it is $|\,\mathbf{H}\,|$, not $|\,\mathbf{G}\,|$, that enters, and $|\,\mathbf{H}\,|$ is finite since H is regular.

The question of practical importance now is whether we can find the function H which has the required properties. This question has already been answered by Greenberg since one of his functions has these precise properties. The function referred to is defined in the following way:

$$\begin{aligned} H &= \log r, && \text{for } r \geq a, \\ &= c_1 r^2 + c_2 r + c_3\,, && \text{for } r < a, \end{aligned} \tag{7.6}$$

where the coefficients c_1, c_2, c_3 are determined so that H has continuous second derivatives across b, that is:

$$\begin{aligned} c_1 &= -a^{-2}/2, \\ c_2 &= 2a, \\ c_3 &= \log a - 3/2. \end{aligned} \tag{7.7}$$

But this is not the only way in which H may be defined, at least for convex regions V. For any point P interior to V, the distance r from P to any point on the boundary B may be defined by a single valued function of a polar angle θ at P; explicitly,

$$r = \rho(\theta), \qquad 0 \leq \theta \leq 2\pi. \tag{7.8}$$

Then, if we define H by the relation

$$H = \frac{r^2 \log r}{(\rho(\theta))^2} + h, \tag{7.9}$$

where h is harmonic in V and vanishes on B, we have a function such that $(H)_B = (G)_B$ and has the required regularity at P.

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