A NOTE ON BATEMAN'S VARIATIONAL PRINCIPLE FOR COMPRESSIBLE FLUID FLOW*

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1. Introduction. The variational principle for a compressible fluid was first studied by Hargreaves [1], who showed that the integrand of the variational integral is a linear function of the pressure. A variational principle for an inviscid compressible fluid was formulated by Bateman [2]. A study of Bateman's work, however, shows that his variational principle is applicable only when the domain of the flow is finite. A large class of aerodynamic problems require the study of a flow field which extends to infinity. In such cases, Bateman's principle must be modified. This fact has already been noted in references [3] and [4], in which the Rayleigh-Ritz method was used in the approximate solution of compressible flows past arbitrary bodies. The formulation of a suitable variational principle in these references was however carried out in connection with the particular problems considered, so that the derivation appears to be in a rather restricted form. In this note, a more general formulation is presented and the resulting variational integral is written in a more general form. The author is indebted to Professor K. O. Friedrichs for his kind suggestions and discussions.

2. Bateman's variational principle. For steady, inviscid, irrotational compressible flow, the governing differential equation is

$$\left[a^2 - \left(\frac{\partial \phi}{\partial x_i}\right)^2\right] \frac{\partial^2 \phi}{\partial x_i \, \partial x_i} - \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \frac{\partial^2 \phi}{\partial x_i \, \partial x_j} = 0, \tag{1}$$

where

$$a^2 = \frac{\gamma - 1}{2}\left(q_m^2 - \frac{\partial \phi}{\partial x_k} \frac{\partial \phi}{\partial x_k}\right). \qquad (i, j, k = 1, 2, 3). \tag{2}$$

In Eqs. (1) and (2), a is the velocity of sound, ϕ is the velocity potential, x_i are the Cartesian coordinates, γ is the ratio of specific heats, q_m is the maximum attainable velocity in the flow. A repetition of the subscripts in the above expression indicates summation.

Bateman's problem is to show that the variational integral

$$I_1 = \int_V p(\phi) \, dV \tag{3}$$

has Eq. (1) as its Euler's equation, where p is the pressure, $d\,V$ is the elementary volume, and the integration is extended to the whole volume of the fluid.

For barometric fluid, p may be written as

$$p = A + B\rho^\gamma,$$

where ρ is the density, and A, B are constants. In terms of ϕ, one obtains

$$p = A + C\left(q_m^2 - \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_i}\right)^k \tag{4}$$

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where

$$C = B^{-1/(\gamma-1)}(\gamma - 1/2\gamma)^k \qquad \text{and} \qquad k = \gamma/(\gamma - 1).$$

With the expression of p as given by (4), the first variation of I_1 may be carried out with respect to ϕ and the condition $\delta I_1 = 0$ leads to

$$\int_V \delta\phi \frac{\partial}{\partial x_i}\left(\rho \frac{\partial \phi}{\partial x_i}\right) dV + \int_{S_1+S_2} \delta\phi\, \rho \frac{\partial \phi}{\partial n} dS = 0, \tag{5}$$

where δ indicates first variation, $\partial/\partial n$ is the derivative in the inward normal direction, $S_1 + S_2$ is the surface that encloses the volume V, S_1 denotes the stream surface and S_2 denotes the boundary surface at infinity. Since $\delta\phi$ is arbitrary in V, the condition $\delta I_1 = 0$ gives the continuity equation

$$\frac{\partial}{\partial x_i}\left(\rho \frac{\partial \phi}{\partial x_i}\right) = 0$$

as the Euler's equation of the variational integral (3).

If the domain is finite and the boundary surfaces are stream surfaces, no condition has to be imposed on ϕ and $\partial\phi/\partial n = 0$ follows as the natural boundary condition. If the domain is infinite, $\partial\phi/\partial n = 0$ on S_1 can still be concluded from the condition $\int_{S_1} \delta\phi\, \rho(\partial\phi/\partial n)\, dS = 0$. At infinity ϕ must be prescribed, and since S_2 is an infinite surface, the vanishing of δI_1 requires that ϕ must be prescribed to an order of magnitude so that $\int_{S_2} \delta\phi\, \rho(\partial\phi/\partial n)\, ds = 0$. This however is not the case in fluid dynamics problems.

To clarify this point, let us consider the two-dimensional case. At infinity, the admitted velocity potential is required to behave as follows

$$\phi = Ur \cos\theta - \frac{K}{2\pi}\theta + (U + A_1)r^{-1}\cos\theta + A_2 r^{-1}\sin\theta + O(r^{-2}), \tag{6}$$

where r, θ are the polar coordinates, K is the strength of circulation A_1 and A_2 are to be determined and $O(r^{-2})$ represents terms of the order $1/r^2$ or higher. Thus

$$\delta\phi = \delta A_1 r^{-1}\cos\theta + \delta A_2 r^{-1}\sin\theta + O(r^{-2}),$$

where δA_1 and δA_2 are arbitrary. Writing ρ and $\partial\phi/\partial n$ in terms of ϕ as given in (6) and integrating, one obtains

$$\int_{s_2} \delta\phi\, \rho \frac{\partial\phi}{\partial n} ds = \left.\int_{\theta=0}^{\theta=2\pi} \delta\phi\, \rho \frac{\partial\phi}{\partial n} r\, d\theta\right]_{r=\infty} = -\rho_0 U\pi\delta A_1\ , \tag{7}$$

where ρ_0 is the density at infinity, s_1 is the boundary curve and ds is the elementary length. The vanishing of δI_1 then requires that $\rho_0 U$ must be zero because δA_1 is arbitrary. This however is not possible. It is therefore clear that Bateman's variational principle is not applicable to flows in which the domain extends to infinity. The appropriate principle in this case should be

$$\delta\left[\int_S p(\phi)\, dS + \rho_0 U\pi A_1\right] = 0, \tag{8}$$

where S is the surface of the domain and dS is the elementary surface.

3. A variational principle for steady, irrotational compressible flow with infinite domain. In applying the Rayleigh-Ritz method to the approximate solution of compressible flow problems, it is found that for most problems the velocity potential ϕ may be written in the form

$$\phi = \phi_1 + \phi_2 , \tag{9}$$

where ϕ_1 denotes the velocity potential of the corresponding incompressible flow and ϕ_2 denotes the remaining part due to the compressibility effect. Substituting ϕ as given by (9) into Eq. (4) and carrying out the expansion, the expression for p may be written in the following form

$$p = A + C\left(q_m^2 - \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_1}{\partial x_i}\right)^k - \rho_0 \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_2}{\partial x_i} + \cdots . \tag{10}$$

Since ϕ_1 is a definite function, $\delta\phi_1$ is zero, and hence

$$\delta \int_V \left[A + C\left(q_m^2 - \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_1}{\partial x_i}\right)^k\right] dV = 0, \tag{11}$$

$$\begin{aligned} \delta \int_V \rho_0 \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_2}{\partial x_i}\, dV &= -\int_{S_1+S_2} \delta\phi_2\, \rho_0 \frac{\partial \phi_1}{\partial n}\, dS - \int_V \delta\phi_2\, \rho_0 \frac{\partial^2 \phi_1}{\partial x_i\, \partial x_i}\, dV \\ &= -\int_{S_2} \delta\phi_2\, \rho_0 \frac{\partial \phi_1}{\partial n}\, dS. \end{aligned} \tag{12}$$

The last step in (12) is obtained because ϕ_1 satisfied the Laplace equation $\partial^2\phi_1/\partial x_i \partial x_i = 0$ and on the stream surface S_1 , $\partial\phi_1/\partial n = 0$ and thus $\int_{S_1} \delta\phi_2(\partial\phi_1/\partial n)\, dS = 0$.

As long as the Euler equation of a variational integral is not affected, it is permissible to change the original integral by adding or subtracting other integrals. Since (11) is zero and (12) gives only a boundary integral, the following variational integral will have the same Euler's equation as Bateman's integral (4).

$$I_2 = \int_V \left\{p(\phi) - \left[A + C\left(q_m^2 - \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_1}{\partial x_i}\right)^k\right]\right\} dV + \int_V \rho_0 \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_2}{\partial x_i}\, dV. \tag{13}$$

Noting that $[A + C(q_m^2 - (\partial\phi_1/\partial x_i)(\partial\phi_1/\partial x_i)^k] = p(\phi_1)$ and integrating the second integral in (13) by Green's formula, (13) becomes

$$I_2 = \int_V [p(\phi) - p(\phi_1)]\, dV - \int_{S_2} \phi_2 \rho_0 \frac{\partial \phi_1}{\partial n}\, dS. \tag{14}$$

In the above expression, $p(\phi_1)$ subtracted thusly insures the boundedness of I_2 . The last integral in (14) must be subtracted so that the boundary integral vanishes when the first variation of (14) is taken. In the two-dimensional case, the first variation of the last integral in (14) indeed reduces to (7).

In references [2]-[9], variational methods have been carried out to solve compressible flow problems following the Rayleigh-Ritz, the Galerkin, and Biezieno-Koch procedures. In all the problems solved, excellent results were obtained. In the case of the Galerkin method and the Biezieno-Koch method, the formulation of a variational principle is not necessary. However, in performing the numerical computation for potential flows past arbitrary bodies, it was found that the Rayleigh-Ritz method requires the least amount of labor.

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ON A GEOMETRICAL METHOD OF DERIVING THREE-DIMENSIONAL HARMONIC FLOWS FROM TWO-DIMENSIONAL ONES*

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The Flow Operators Ω. Let D be a domain in a (u,v)-plane, E a domain in an (x,y,z)-space, and let $\varphi = \varphi(u, v)$, $\psi = \psi(x, y, z)$ denote (real-valued and regular) solutions of $\Delta_2\varphi = 0$, $\Delta_3\psi = 0$ on D, E, respectively, where Δ_2 and Δ_3 denote the two- and three-dimensional euclidean Laplace operators,

$$\Delta_2 = \partial^2/\partial u^2 + \partial^2/\partial v^2 \qquad \text{and} \qquad \Delta_3 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2. \tag{1}$$

All harmonic functions φ are accessible in principle, since all of them are given by $\varphi(u, v) = Re\chi(w)$, where χ is any function which is regular-analytic in $w = u + iv$ on D. In contrast, there does not exist anything like this rule for the harmonic functions $\psi(x, y, z)$ on a three-dimensional E. Hence it is natural to ask for flow operators, say $\Omega = \Omega(D)$, which, from every regular solution $\varphi = \varphi(u, v)$ of $\Delta_2\varphi = 0$ on a two-dimensional (u,v)-domain D, will manufacture a regular solution,

$$\psi(x, y, z) = \Omega\varphi(u, v), \tag{2}$$

of $\Delta_3\psi = 0$ on a three-dimensional (x,y,z)-domain $E = E(D)$. The latter should not depend on the particular choice of the function $\varphi(u, v)$, but merely on the operator $\Omega = \Omega(D)$ and on the domain D on which $\varphi(u, v)$ is supposed to be harmonic.

A trivial instance of such "harmonic flow operators" Ω is supplied by the cylindrical flow which, from a given $\varphi(u, v)$, manufactures the corresponding $\psi(x, y, z)$ as follows:

$$\psi(x, y, z) = \varphi(x, y). \tag{3}$$

In fact, (3) is of the type (2), since $\Delta_3\psi(x, y, z) = \Delta_2\psi(x, y)$ if $\partial^2\psi/\partial z^2 = 0$; cf. (1). The

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