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A CONVENIENT GENERAL SOLUTION OF THE CONFLUENT HYPERGEOMETRIC EQUATION, ANALYTIC AND NUMERICAL DEVELOPMENT*

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1. Introduction. The standard forms for the general solution of the Confluent Hypergeometric Equation prove too unwieldy for application to many physical problems, particularly in the field of Quantum Mechanics. The two standard power series solutions,[1] $M_{n,\pm m}(y)$, reduce to a single regular polynomial solution whenever $2m$ is an integer (the standard case for quantum mechanics), and in this case the two integral solutions[2], $W_{\pm n,m}$ $(\pm y)$, must be computed with an asymptotic expansion which is cumbersome for most physically interesting values of y.

The utility of all of these solutions is further limited because their form necessitates undertaking a complete recomputation for every physically significant value of n. This makes the labor of computation almost prohibitive in the physically important case where both n and y must be treated as continuous variables.

The possibility of achieving a more manageable form of the solutions was first indicated by the work of Wannier[3] and Jastrow.[4] Wannier showed that in theory the function $M_{n,m}(y)$ could be developed as a series in descending powers of n with coefficients given in terms of Bessel functions. Jastrow actually exhibited the first two terms of an asymptotically similar series for the solution $W_{n,m}(y)$. This paper completes the above treatments by producing analytically a *general* solution of the differential equation as a power series in $1/n^2$ with coefficients readily calculable in terms of known functions. This treatment differs from those noted above not only in the generality of its results, but also in the ease with which successive terms of the series may be explicitly generated. The method employed here makes it possible to exhibit the two particular solutions of the equation which go to zero as $y \to 0$ and as $y \to +\infty$ and to relate these analytically to the earlier solutions $W_{n,m}(y)$ and $M_{n,m}(y)$. Finally this paper will exhibit analytic and numerical values for the coefficients of several of the series of greatest physical interest.

2. The general series solution. For physical applications Whittaker's standard form of the Confluent Hypergeometric Equation,

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[1]E. T. Whittaker and G. N. Watson, *Modern analysis*, (Cambridge University Press, 1940), fourth edition, §16.1.

[2]*Ibid.* §16.12.

[3]G. H. Wannier, Phys. Rev. **64**, p. 358 (1943).

[4]R. Jastrow, Phys. Rev. **73**, p. 60 (1948).

$$\frac{d^2U}{dy^2} + \left[-\frac{1}{4} + \frac{n}{y} + \frac{1/4 - m^2}{y^2}\right]U = 0, \tag{1}$$

is conveniently transformed by the substitutions

$$r = \frac{1}{2}ny; \qquad m = l + \frac{1}{2} \tag{2}$$

to the form

$$\frac{d^2U^{(l,n)}}{dr^2} + \left[-\frac{1}{n^2} + \frac{2}{r} - \frac{l(l+1)}{r^2}\right]U^{(l,n)} = 0, \tag{3}$$

which, with $\epsilon = -1/n^2$, is just the hydrogenic radial wave equation in Rydberg units. The form (3) will be taken as standard in this paper.

The further substitutions

$$z = (8r)^{1/2}; \qquad U^{(l,n)} = \frac{1}{2}zV^{(l,n)} \tag{4}$$

reduce (3) to the form

$$\nabla_l V^{(l,n)} - n^{-2}\left(\frac{1}{2}z\right)^4 V^{(l,n)} = 0, \tag{5}$$

where ∇_l is the Bessel operator of index $2l + 1$, i.e.

$$\nabla_l \equiv z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + z^2 - (2l+1)^2. \tag{6}$$

It will be assumed here and proven in the Appendix that the general solution of (5) may be written in the form of a power series in $1/n^2$, i.e.

$$V^{(l,n)}(z) = \sum_{k=0}^{\infty} n^{-2k}V_k^{(l)}(z), \tag{7}$$

where the functions $V_k^{(l)}(z)$ are analytic functions of z and the series (7) converges absolutely and uniformly for all real l and for all real n in the region $|n| \geq n_0$, n_0 being an arbitrary positive number. Corresponding solutions of (3) may then be written in the form

$$U^{(l,n)}(z) = \sum_{k=0}^{\infty} n^{-2k}U_k^{(l)}(z), \tag{8}$$

in which the $U_k^{(l)}(z)$ are given in terms of the $V_k^{(l)}(z)$ by the equation

$$U_k^{(l)}(z) = \frac{1}{2}zV_k^{(l)}(z). \tag{9}$$

Since (7) converges uniformly and absolutely it may be inserted in the differential equation (5), and differentiated term by term. The terms may then be rearranged and the coefficients of the various powers of $1/n^2$ equated to zero, which is necessary and sufficient to make (7) a solution of (5). This procedure yields an infinite set of simultaneous differential equations for the coefficients $V_k^{(l)}(z)$:

$$\nabla_l V_0^{(l)} = 0 \tag{10a}$$

$$\nabla_l V_k^{(l)} = \left(\frac{1}{2}z\right)^4 V_{k-1}^{(l)}. \tag{10b}$$

Now (10a) is just Bessel's equation of index $2l + 1$ whose general solution is the cylindrical functions $\mathcal{C}_{2l+1}(z)$, an arbitrary linear combination of the Bessel function $J_{2l+1}(z)$, and the Weber function, $Y_{2l+1}(z)$. These cylindrical functions obey the usual recursion formulas for Bessel functions:

$$\begin{aligned} \mathcal{C}_{n-1}(z) + \mathcal{C}_{n+1}(z) &= \frac{2n}{z}\,\mathcal{C}_n(z) \\ z\mathcal{C}_n'(z) + n\mathcal{C}_n(z) &= z\mathcal{C}_{n-1}(z). \end{aligned} \tag{11}$$

By utilizing these relations and the definition (8) of the Bessel operator ∇_l, it can be shown straightforwardly that

$$\nabla_l\left\{\frac{2l+2+q}{4(2+q)}\left(\frac{1}{2}z\right)^{q+2}\mathcal{C}_{2l+3+q} - \frac{1}{4(3+q)}\left(\frac{1}{2}z\right)^{q+3}\mathcal{C}_{2l+4+q}\right\} = \left(\frac{1}{2}z\right)^{q+4}\mathcal{C}_{2l+1+q}\,. \tag{12}$$

This equation permits solutions of (10b) to be generated directly. For set $q = 0$ in (12). Then this equation becomes identical with (10b) in the case $k = 1$, provided that $V_1^{(l)}(z)$ is defined by the bracket on the left side of (12), i.e. by

$$V_1^{(l)} = \frac{l+1}{4}\left(\frac{1}{2}z\right)^2\mathcal{C}_{2l+3} - \frac{1}{12}\left(\frac{1}{2}z\right)^3\mathcal{C}_{2l+4}\,. \tag{13}$$

Thus a solution has been found for the first of equations (10b).

For $k = 2$, the right hand side of (10b) is just $(\frac{1}{2}z)^4V_1$ which, by (13), must contain two terms of the form $(\frac{1}{2}z)^{q+4}\mathcal{C}_{2l+1+q}$ with $q = 2$ and $q = 3$. These are both of the form found on the right of equation (12), so that $V_2^{(l)}(z)$ can be generated by two applications of the procedure outlined above. Higher terms are generated successively by the same process. For example, the first four coefficients of the series for $U^{(0,n)}(z)$ are

$$\begin{aligned} U_0^{(0)} &= \left(\frac{1}{2}z\right)\mathcal{C}_1 \\ U_1^{(0)} &= \frac{1}{4}\left(\frac{1}{2}z\right)^3\mathcal{C}_3 - \frac{1}{12}\left(\frac{1}{2}z\right)^4\mathcal{C}_4 \\ U_2^{(0)} &= \frac{1}{16}\left(\frac{1}{2}z\right)^5\mathcal{C}_5 - \frac{1}{30}\left(\frac{1}{2}z\right)^6\mathcal{C}_6 + \frac{1}{288}\left(\frac{1}{2}z\right)^7\mathcal{C}_7 \\ U_3^{(0)} &= \frac{1}{64}\left(\frac{1}{2}z\right)^7\mathcal{C}_7 - \frac{71}{6{,}720}\left(\frac{1}{2}z\right)^8\mathcal{C}_8 + \frac{11}{5{,}760}\left(\frac{1}{2}z\right)^9\mathcal{C}_9 - \frac{1}{10{,}368}\left(\frac{1}{2}z\right)^{10}\mathcal{C}_{10}\,. \end{aligned} \tag{14}$$

It has been assumed, and will be proven in the Appendix, that the series generated above and illustrated in (14) is itself analytic and uniformly convergent in z so that it may be differentiated term by term with respect to z to yield a series for the derivatives of the solutions of equation (3). Such a series is discussed further in section 5.

3. Some important particular solutions. The series generated above yields a general solution of equation (3) because the linear combination $\alpha J_m(z) + \beta Y_m(z)$ to be inserted for the cylindrical function $\mathcal{C}_m$ remains entirely arbitrary, so that the coefficients $U_k^{(l)}(z)$ are in fact ambiguously defined. For physical application of the series it is necessary to examine the effect of removing this ambiguity by particular choices of the constants

α and β. This examination is facilitated by defining two sets of coefficients, ${}^0U_k^{(l)}(z)$ and ${}^1U_k^{(l)}(z)$, which are gained from the ambiguous coefficients $U_k^{(l)}(z)$ by substituting $J_m(z)$ and $Y_m(z)$ respectively for the cylindrical functions $\mathcal{C}_m(z)$. Two particular independent solutions of (3), ${}^0U^{(l,n)}(z)$ and ${}^1U^{(l,n)}(z)$, may then be defined as the result of applying the summation (8) to the coefficients ${}^0U_k^{(l)}(z)$ and ${}^1U_k^{(l)}(z)$, respectively.

The most general solution of (3) may now be written in the form

$$f(l, n)\,{}^0U^{(l,n)}(z) + g(l, n)\,{}^1U^{(l,n)}(z) \equiv \sum_{k=0}^{\infty} n^{-2k}[f(l, n)\,{}^0U_k^{(l)}(z) + g(l, n)\,{}^1U_k^{(l)}(z)], \tag{15}$$

where $f(l, n)$ and $g(l, n)$ are entirely arbitrary. Important particular solutions of (3) are gained by specifying these arbitrary functions in (15).

Since the Bessel functions all have zeros and the Weber functions all have poles at the origin, it can be shown that ${}^0U^{(l,n)}(z)$ is the only particular solution of (3) with a zero at the origin. It must therefore be identical, except in amplitude, with the particular solution $M_{n,m}(y)$ of (1), and a comparison of the leading terms (in z) of the expansions of the two series yields

$$M_{n,l+1/2}(z^2/4n) = n^{-l-1}\Gamma(2l + 2)\,{}^0U^{(l,n)}(z). \tag{16}$$

A second solution of physical interest, the only particular solution which goes to zero as z goes to infinity, may be discovered by a comparison of the series developed above with a series solution given by Wannier. In the paper previously noted, Wannier defines two solutions of (5), $J_{2l+1}^n(z)$ and $N_{2l+1}^n(z)$, by the formulas

$$\begin{aligned} J_{2l+1}^n(z) &\equiv \frac{n^{l+1}}{(1/2\,z)\Gamma(2l + 2)} M_{n,l+1/2}(z^2/4n) \\ N_{2l+1}^n(z) &\equiv \frac{1}{\sin(2l + 1)\pi}\left[\frac{\Gamma(n + l + 1)}{\Gamma(n - l)n^{2l+1}} J_{2l+1}^n(z)\cos(2l + 1)\pi - J_{-2l-1}^n(z)\right]. \end{aligned} \tag{17}$$

These solutions are shown to be independent and well defined for all values of l and n. Wannier further proves that the particular solution which goes to zero at infinity may be written[5]

$$\begin{aligned} W_{n,l+1/2}(z^2/4n) = (z^2/4n)^{1/2}[&\Gamma(n + l + 1)n^{-l-1/2}J_{2l+1}^n(z)\cos(n - l - 1)\pi \\ &+ \Gamma(n - l)n^{l+1/2}N_{2l+1}^n(z)\sin(n - l - 1)\pi]. \end{aligned} \tag{18}$$

Since the series expansion of $M_{n,l+1/2}$ is well known, equations (17) and (18) completely determine the expansion of $W_{n,l+1/2}$ for any value of l. It follows that if, for a given fixed value of l, the first $m + 1$ coefficients ${}^0U_k^{(l)}$ and ${}^1U_k^{(l)}$ have been developed by the generating procedure, the functions $f(l, n)$ and $g(l, n)$ may be determined (to terms in n^{-2m}) by explicit comparison of the series expansions (in z) of (15) and (18). If $2l + 1$ is not an integer, it is convenient to compare the coefficients of the terms in $(\frac{1}{2}z)^{-2l}$ and in $(\frac{1}{2}z)^{2l+2}$ in the two series; if $2l + 1$ is an integer the coefficient of the terms $(\frac{1}{2}z)^{2l+2}$ and $(\frac{1}{2}z)^{2l+2}\log(\frac{1}{2}z)$ are most conveniently compared. In the latter case, $2l + 1$ an integer, these two terms of (18) are given by

[5]Wannier's paper has $-\Gamma(n - l)n^{l+1/2}\cdots$, a discrepancy which I assume to be due to a misprint in the original.

$$W_{n,l+1/2}(z^2/4n) = \cdots + \frac{\Gamma(n+l+1)}{n^{l+1}} \frac{(1/2\,z)^{2l+2}}{(2l+1)!} \Big\{ \cos(n-l-1)\pi$$

$$+ \frac{1}{\pi} \sin(n-l-1)\pi \Big[2 \log\left(\frac{1}{2}z\right) + 2\gamma - \sum_{m=1}^{2l+1} \frac{1}{m} + \Psi(n-l) - \log(n) \tag{19}$$

$$+ \Gamma(n-l)(2l+1)! \sum_{r=0}^{2l} (-1)^r \frac{(2l-r)!}{2^{2l+1-r}(2l+1-r)!\Gamma(n+l+1-r)} \Big] \Big\} + \cdots ,$$

in which $\Psi(x) = d/dx \log \Gamma(x)$ and $\gamma =$ Euler's constant.

This procedure has been carried out for the two most important cases, $l = 0$ and $l = 1$. In both cases the manipulation yields (to terms in n^{-10})

$$\begin{aligned} W_{n,l+1/2}(z^2/4n) = n^{-l-1}\Gamma(n+l+1)[\cos(n-l-1)\pi\, {}^0U^{(l,n)}(z) \\ + \sin(n-l-1)\pi\, {}^1U^{(l,n)}(z)]. \end{aligned} \tag{20}$$

There are additional theoretical reasons for supposing the equation (20) is in fact valid for all values of l, integral and non-integral, but a general proof of this result has not yet been given. Until such a proof is produced the method outlined above may be used to produce an equivalent result for any value of l for which the coefficients ${}^0U_k^{(l)}$ and ${}^1U_k^{(l)}$ have been developed.

4. An alternate generating procedure. Since the Weber functions have not been tabulated for large indices, it is convenient to develop the formulas for $U_k^{(l)}(z)$ so that they involve only $\mathcal{C}_0(z)$ and $\mathcal{C}_1(z)$. This may be accomplished by repeated application of the first of the recursion formulas (11) to the functions $V_k^{(l)}(z)$ generated by the method of section 2, but this reduction is arduous and may conveniently be replaced by the procedure sketched below.

The function $V_0^{(l)}(z)$ is just $\mathcal{C}_{2l+1}(z)$, and this may, by application of the recursion formulas, be rewritten in the form

$$V_0^{(l)}(z) = \sum_{i=m}^{M} a_i \left(\frac{1}{2}z\right)^{2i} \mathcal{C}_0 + \sum_{i=n}^{N} b_i \left(\frac{1}{2}z\right)^{2i+1} \mathcal{C}_1 , \tag{21}$$

where the constants a_i and b_i are known rational numbers and the constants m, n, M, and N are known integers.

The function $V_1^{(l)}(z)$ must be expressible in the form

$$V_1^{(l)}(z) = \sum_{i=m}^{M+2} \alpha_i \left(\frac{1}{2}z\right)^{2i} \mathcal{C}_0 + \sum_{i=n}^{N+2} \beta_i \left(\frac{1}{2}z\right)^{2i+1} \mathcal{C}_1 , \tag{22}$$

where the constants α_i and β_i are unknown rational numbers which can be determined by applying the differential equation (10b) to (21) and (22). This application of the differential equation is facilitated by the use of the equations

$$\begin{aligned} \nabla_l\left[\left(\frac{1}{2}z\right)^q \mathcal{C}_0\right] &= -4q\left(\frac{1}{2}z\right)^{q+1} \mathcal{C}_1 + [q^2 - (2l+1)^2]\left(\frac{1}{2}z\right)^q \mathcal{C}_0 \\ \nabla_l\left[\left(\frac{1}{2}z\right)^q \mathcal{C}_1\right] &= 4q\left(\frac{1}{2}z\right)^{q+1} \mathcal{C}_0 + [(q-1)^2 - (2l+1)^2]\left(\frac{1}{2}z\right)^q \mathcal{C}_1 , \end{aligned} \tag{23}$$

which are immediate consequences of (6) and (11). This procedure reduces $V_1^{(l)}(z)$ to the form (21) after which the method may be reapplied to the generation of $V_2^{(l)}(z)$, etc.

It must, however, be noted that this procedure, in contrast to that described in section 2, does not always uniquely determine the functions $V_k^{(l)}(z)$, for the differential equation (10b) is normally satisfied by functions of the form (21) and (22) for all values of certain of the coefficients α_i and β_i . This will be understood when it is observed that if, with $V_{k-1}^{(l)}(z)$ given, a function $V_k^{(l)}(z)$ is found to satisfy (10b), then the new function $V_k^{(l)}(z) + \alpha V_0^{(l)}(z)$ will also satisfy (10b) for any α, a fact which follows directly from (10a). This ambiguity does not affect the legitimacy of the generating procedure, for although the quantity $\alpha V_0^{(l)}(z)$ may be added to any $V_k^{(l)}(z)$, the quantity thus added will, by (10b), affect the formulas for $V_{k+1}^{(l)}(z)$, $V_{k+2}^{(l)}(z)$, etc. It is in fact readily seen from (10b) that the net effect of adding $\alpha V_0^{(l)}(z)$ to the coefficient $V_k^{(l)}(z)$ is just to increase the amplitude of the sum of the series, i.e. of $V^{(l,n)}(z)$, by the factor $(1 + \alpha/n^{2k})$.

This new generating procedure was used in the preparation of the tables which follow. For simplicity of computation all those coefficients, α_i and β_i , which were not explicitly determined by the procedure were set equal to zero, thus reducing the complexity of the formulas for the coefficients $V_k^{(l)}(z)$. It is readily seen that, for $l = 0$, the tabulated coefficients are just those which would have been gained using the original generating procedure. For $l = 1$ the tabulated coefficients differ from those provided by the original procedure, but in this case the sums of the series gained with the tabulated coefficients may be made equal to the sums of the series gained with the standard coefficients described in section 2 by multiplying the former with the amplitude factor $n^2/(n^2 - 1)$. This modification of the amplitudes of the solutions is of no significance except when it is necessary to use (16) and (20) for explicit computation of Whittaker's functions, $M_{n,m}$ and $W_{n,m}$.

5. The tables. Table I below lists the formulas for $U_k^{(0)}(z)$ with $k = 0$ through 7. Table II gives a similar list for the functions $U_k^{(1)}(z)$. Tables III and IV list corresponding formulas for the functions $D_k^{(0)}(z)$ and $D_k^{(1)}(z)$ which are defined by

$$D_k^{(l)}(z) \equiv \left(\frac{1}{2} z\right) \frac{d}{dz} U_k^{(l)}(z), \tag{24}$$

from which the functions

$$D^{(l,n)}(z) \equiv \sum_{k=0}^{\infty} n^{-2k} D_k^{(l)}(z) = \frac{d}{dz} U^{(l,n)}(z) \tag{25}$$

may be computed.

Tables V through VIII list values of the functions ${}^0U_k^{(l)}(z)$, ${}^1U_k^{(l)}(z)$, ${}^0D_k^{(l)}(z)$, and ${}^1D_k^{(l)}(z)$ for $l = 0$ and 1, $k = 0$ through 7, and $z = 3.5(0.5)7.5$. As before a superscript zero preceding the function indicates substitution of a Bessel function for the corresponding cylindrical function, and a superscript one connotes use of the Weber function. Persons interested in utilizing these numerical results may also find useful the *Tables of Coulomb Wave Functions* recently prepared by the National Applied Mathematics Laboratory of the National Bureau of Standards. These supply a single irregular solution of the wave equation for $l = 0$ and for values of z smaller than those listed in our tables.

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Since completing this paper, my attention has been called to the fact that the generating procedure developed in section 2 above has been previously discussed by Yost, Wheeler, and Breit[6] in treating the formally identical problem of the wave equation with a repulsive Coulomb field (proton-proton scattering). The additional analytic and numerical material displayed here seems more than sufficient to justify the presentation of this independent investigation.

APPENDIX

The existence of an expansion in $1/n^2$. We may seek a solution of equation (5) in the form of a power series in z,

$$V^{(l,n)}(z) = z^{\alpha} \sum_{k=0}^{\infty} c_k z^k \tag{26}$$

Throughout its circle of convergence this series may be inserted in the differential equation, and manipulations identical with those used in producing solutions of Bessel's equation show that the two series

$$V^{(l,n)}(z) = \left(\frac{1}{2}z\right)^{\pm(2l+1)} \sum_{k=0}^{\infty} (-1)^k a_k \left(\frac{1}{2}z\right)^{2k} \tag{27}$$

are solutions of (5) for an arbitrary value of a_0, providing that the coefficients a_k are generated by the formulas

$$a_1 = a_0[1 \pm (2l + 1)]^{-1} \tag{28a}$$

$$a_k = \frac{a_{k-1} + (a_{k-2}/4n^2)}{k[k \pm (2l + 1)]} \tag{28b}$$

In equations (28a) and (28b) the positive or the negative signs are to be taken together, so that, if $2l + 1$ is not an integer, these equations define two independent solutions of (5). That choice of sign which makes $\pm(2l + 1) > 0$ produces a solution which is regular at $z = 0$; the other choice of sign yields a solution which is irregular there. These two solutions will hereafter be distinguished as the "regular" and the "irregular" series, respectively.

When $2l + 1$ is an integer, equations (27) and (28) define only the regular solution of the equation, but in this event a second irregular solution may be defined by any of the usual[7] devices developed for Bessel functions. In this paper we make the expedient assumption that $2l + 1$ is not an integer and then produce two independent solutions of (5), ${}^0V^{(l,n)}(z)$ and ${}^1V^{(l,n)}(z)$, which remain finite and independent as $2l + 1$ is varied continuously through any integer or zero. Because of their continuity as functions of l these solutions may be assumed valid for $2l + 1$ an integer, and this assumption may be rigorously justified by standard methods.

We now investigate the convergence of the two series subject to the simplifying restriction that n and l be real variables, z remaining complex. It then appears from (28b) that all the coefficients a_k of the regular series have the same sign as a_0, so it follows directly that

[6] F. L. Yost, J. A. Wheeler, and G. Breit, Phys. Rev. 49, p. 174 (1936).

[7] G. N. Watson, *Theory of Bessel functions*, (Cambridge University Press, 1945), second edition, §§3.5 ff.

$$\frac{a_{k+1}}{a_k} = \frac{1}{(k+1)[k+1 \pm (2l+1)]}\left\{1 + \frac{k[k \pm (2l+1)]}{4n^2 + (a_{k-2}/a_{k-1})}\right\}$$
$$< \frac{1}{(k+1)[k+1 \pm (2l+1)]} + \frac{1}{4n^2}. \tag{29}$$

The inequality in (29) provides a very weak condition on a_{k+1}/a_k, because it is gained by dropping the entire positive term $a_{k-2}/4n^2a_{k-1}$ from the positive denominator $1 + a_{k-2}/4n^2a_{k-1}$ in the equality. Stronger conditions may be gained by successive application of (29) to itself, for (29) may be used to set a lower bound on a_{k-2}/a_{k-1}, and this bound may be used to write a second and stronger form of (29). The process may be repeated indefinitely, and the p-th form of (29) gained in this manner is

$$\frac{a_{k+1}}{a_k} < \frac{(p+2)}{2(k+1-2p)[k+1-2p \pm (2l+1)]} + \frac{1}{(p+1)4n^2} \tag{30}$$

a formula which reduces to (29) when $p = 0$ and which is valid for all $k > 2p - 1 \mp (2l + 1)$.

It follows that for all $|n| \geq n_0$ and for all $|z| < R$ (where n_0 and R are arbitrary positive numbers) the regular series in (27) converges absolutely and uniformly, for if p in (30) is chosen to be the largest integer less than $R^2/16n_0^2$, an integer k_0 may always be found such that $a_{k+1}|\frac{1}{2}z|^2/a_k < 1$ for all $k > k_0$. It follows that the regular series defined by (27) converges to an analytic function of z in any bounded domain in the z plane and that this function is uniformly continuous in n for all real n such that $|n| \geq n_0 > 0$.

It may next be noted that equation (28) generates coefficients a_k which are given by finite polynomials of the form

$$a_k = \sum_{m=0}^{M_k} b_m^k n^{-2m} \tag{31}$$

where $M_k = \frac{1}{2}k$ or $\frac{1}{2}(k-1)$, whichever is an integer. Further, if a_k is a coefficient of the regular series, the polynomial coefficients b_m^k must be positive quantities.

By considering (27) and (31), we may now define the "complete series" for $V^{(l,n)}(z)$ as the series in which each $b_m^k\, n^{-2m}\, (\frac{1}{2}z)^{2k+(2l+1)}$ is considered a separate term and in which for each value of k, the summation is carried out over values of m from 0 to M_k before k is increased by unity. It then follows from the absolute convergence of (27) and from the uniformly positive values of the b_m^k that the "complete series" for the regular solution also converges absolutely and uniformly, so that the terms of the "complete" series may be rearranged to provide an analytic solution of (5) in the form

$$V^{(l,n)}(z) = \sum_{k=0}^{\infty} V_k^{(l)}(z) n^{-2k} \tag{7}$$

The functions $V_k^{(l)}(z)$ are defined by uniformly and absolutely convergent power series in z, so that they are analytic functions of z, and the entire series (7) converges uniformly whenever $|n| \geq n_0$.

The existence of the series (7) has so far been proven only in the case of the regular solution of (5), but the proof is readily extended to the irregular solution and hence to

the general solution, which is an arbitrary linear combination of the two. For consider a series (27) in which the coefficients a_k are replaced by coefficients a'_k determined by

$$a'_1 = a'_0 \mid 1 \pm (2l + 1) \mid^{-1} \tag{32a}$$

$$a'_k = \frac{a'_{k-1} + (a'_0/4n^2)}{k \mid k \pm (2l + 1) \mid} \tag{32b}$$

This series contains only positive terms, so that the entire proof applied to the regular series holds for it. Further, if the choice of sign which produced the irregular series in (27) is utilized in (32), then every a'_k and every b'^k_m is greater than or equal to the corresponding a_k or b^k_m of the irregular series. It follows from the comparison test that the "complete series" for the irregular solution must converge absolutely and uniformly, so that the rearrangement of terms is again permissible. This is sufficient to justify entirely the assumption made in section 2, above.

TABLE I. Formulas for the coefficients $U_k^{(0)}(z)$ with $k = 0$ through 7.

$$U_0^{(0)} = \left(\frac{z}{2}\right)\mathfrak{c}_1$$

$$U_1^{(0)} = -\frac{1}{12}\left(\frac{z}{2}\right)^4\mathfrak{c}_0 + \frac{1}{12}\left(\frac{z}{2}\right)^3\mathfrak{c}_1$$

$$U_2^{(0)} = \left[\frac{1}{120} - \frac{1}{120}\left(\frac{z}{2}\right)^2\right]\left(\frac{z}{2}\right)^4\mathfrak{c}_0 - \left[\frac{1}{120} - \frac{1}{80}\left(\frac{z}{2}\right)^2 + \frac{1}{288}\left(\frac{z}{2}\right)^4\right]\left(\frac{z}{2}\right)^3\mathfrak{c}_1$$

$$U_3^{(0)} = -\left[\frac{1}{252} - \frac{1}{252}\left(\frac{z}{2}\right)^2 + \frac{79}{60{,}480}\left(\frac{z}{2}\right)^4 - \frac{1}{10{,}368}\left(\frac{z}{2}\right)^6\right]\left(\frac{z}{2}\right)^4\mathfrak{c}_0 + \left[\frac{1}{252} - \frac{1}{168}\left(\frac{z}{2}\right)^2 + \frac{179}{60{,}480}\left(\frac{z}{2}\right)^4 - \frac{13}{25{,}920}\left(\frac{z}{2}\right)^6\right]\left(\frac{z}{2}\right)^3\mathfrak{c}_1$$

$$U_4^{(0)} = \left[\frac{1}{240} - \frac{1}{240}\left(\frac{z}{2}\right)^2 + \frac{19}{12{,}096}\left(\frac{z}{2}\right)^4 - \frac{97}{362{,}880}\left(\frac{z}{2}\right)^6 + \frac{7}{414{,}720}\left(\frac{z}{2}\right)^8\right]\left(\frac{z}{2}\right)^4\mathfrak{c}_0 - \left[\frac{1}{240} - \frac{1}{160}\left(\frac{z}{2}\right)^2 + \frac{5}{1{,}512}\left(\frac{z}{2}\right)^4 - \frac{115}{145{,}152}\left(\frac{z}{2}\right)^6 + \frac{403}{4{,}838{,}400}\left(\frac{z}{2}\right)^8 - \frac{1}{497{,}664}\left(\frac{z}{2}\right)^{10}\right]\left(\frac{z}{2}\right)^3\mathfrak{c}_1$$

$$U_5^{(0)} = -\left[\frac{1}{132} - \frac{1}{132}\left(\frac{z}{2}\right)^2 + \frac{21}{7{,}040}\left(\frac{z}{2}\right)^4 - \frac{5}{8{,}448}\left(\frac{z}{2}\right)^6 + \frac{1{,}977}{31{,}933{,}440}\left(\frac{z}{2}\right)^8 - \frac{131}{43{,}545{,}600}\left(\frac{z}{2}\right)^{10} + \frac{1}{29{,}859{,}840}\left(\frac{z}{2}\right)^{12}\right]\left(\frac{z}{2}\right)^4\mathfrak{c}_0 + \left[\frac{1}{132} - \frac{1}{88}\left(\frac{z}{2}\right)^2 + \frac{389}{63{,}360}\left(\frac{z}{2}\right)^4 - \frac{17}{10{,}560}\left(\frac{z}{2}\right)^6 + \frac{479}{2{,}128{,}896}\left(\frac{z}{2}\right)^8 - \frac{701}{43{,}545{,}600}\left(\frac{z}{2}\right)^{10} + \frac{13}{29{,}859{,}840}\left(\frac{z}{2}\right)^{12}\right]\left(\frac{z}{2}\right)^3\mathfrak{c}_1$$

$$U_6^{(0)} = \left[\frac{691}{32{,}760} - \frac{691}{32{,}760}\left(\frac{z}{2}\right)^2 + \frac{229}{27{,}027}\left(\frac{z}{2}\right)^4 - \frac{3{,}313}{1{,}853{,}280}\left(\frac{z}{2}\right)^6 + \frac{43{,}037}{197{,}683{,}200}\left(\frac{z}{2}\right)^8 - \frac{160{,}361}{10{,}378{,}368{,}000}\left(\frac{z}{2}\right)^{10} + \frac{509}{870{,}912{,}000}\left(\frac{z}{2}\right)^{12} - \frac{1}{119{,}439{,}360}\left(\frac{z}{2}\right)^{14}\right]\left(\frac{z}{2}\right)^4 \mathfrak{c}_0 - \left[\frac{691}{32{,}760} - \frac{691}{21{,}840}\left(\frac{z}{2}\right)^2 + \frac{14{,}929}{864{,}864}\left(\frac{z}{2}\right)^4 - \frac{6{,}977}{1{,}482{,}624}\left(\frac{z}{2}\right)^6 + \frac{47{,}951}{65{,}894{,}400}\left(\frac{z}{2}\right)^8 - \frac{393{,}847}{5{,}930{,}496{,}000}\left(\frac{z}{2}\right)^{10} + \frac{461{,}819}{134{,}120{,}448{,}000}\left(\frac{z}{2}\right)^{12} - \frac{59}{696{,}729{,}600}\left(\frac{z}{2}\right)^{14} + \frac{1}{2{,}149{,}908{,}480}\left(\frac{z}{2}\right)^{16}\right]\left(\frac{z}{2}\right)^3 \mathfrak{c}_1$$

$$U_7^{(0)} = -\left[\frac{273}{3{,}276} - \frac{273}{3{,}276}\left(\frac{z}{2}\right)^2 + \frac{26{,}609}{786{,}240}\left(\frac{z}{2}\right)^4 - \frac{6{,}953}{943{,}488}\left(\frac{z}{2}\right)^6 + \frac{3{,}569}{3{,}706{,}560}\left(\frac{z}{2}\right)^8 - \frac{11{,}717}{148{,}262{,}400}\left(\frac{z}{2}\right)^{10} + \frac{404{,}561}{99{,}632{,}332{,}800}\left(\frac{z}{2}\right)^{12} - \frac{65{,}539}{536{,}481{,}792{,}000}\left(\frac{z}{2}\right)^{14} + \frac{11}{6{,}270{,}566{,}400}\left(\frac{z}{2}\right)^{16} - \frac{1}{180{,}592{,}312{,}320}\left(\frac{z}{2}\right)^{18}\right]\left(\frac{z}{2}\right)^4 \mathfrak{c}_0 + \left[\frac{273}{3{,}276} - \frac{273}{2{,}184}\left(\frac{z}{2}\right)^2 + \frac{53{,}909}{786{,}240}\left(\frac{z}{2}\right)^4 - \frac{45{,}011}{2{,}358{,}720}\left(\frac{z}{2}\right)^6 + \frac{5{,}363}{1{,}729{,}728}\left(\frac{z}{2}\right)^8 - \frac{279{,}187}{889{,}574{,}400}\left(\frac{z}{2}\right)^{10} + \frac{286{,}511}{14{,}233{,}190{,}400}\left(\frac{z}{2}\right)^{12} - \frac{919{,}637}{1{,}162{,}377{,}216{,}000}\left(\frac{z}{2}\right)^{14} + \frac{353}{20{,}901{,}888{,}000}\left(\frac{z}{2}\right)^{16} - \frac{61}{451{,}480{,}780{,}800}\left(\frac{z}{2}\right)^{18}\right]\left(\frac{z}{2}\right)^3 \mathfrak{c}_1$$

TABLE II. Formulas for the coefficients $U_k^{(1)}(z)$ with $k = 0$ through 7.

$$U_0^{(1)} = -2\mathfrak{c}_0 + \left[2 - \left(\frac{z}{2}\right)^2\right]\left(\frac{z}{2}\right)^{-1} \mathfrak{c}_1$$

$$U_1^{(1)} = \frac{1}{12}\left(\frac{z}{2}\right)^4 \mathfrak{c}_0 - \frac{1}{4}\left(\frac{z}{2}\right)^3 \mathfrak{c}_1$$

$$U_2^{(1)} = \frac{11}{720}\left(\frac{z}{2}\right)^6 \mathfrak{c}_0 - \left[\frac{11}{720} - \frac{1}{288}\left(\frac{z}{2}\right)^2\right]\left(\frac{z}{2}\right)^5 \mathfrak{c}_1$$

$$U_3^{(1)} = -\left[\frac{31}{15{,}120} - \frac{31}{20{,}160}\left(\frac{z}{2}\right)^2 + \frac{1}{10{,}368}\left(\frac{z}{2}\right)^4\right]\left(\frac{z}{2}\right)^6 \mathfrak{c}_0 + \left[\frac{31}{15{,}120} - \frac{31}{12{,}096}\left(\frac{z}{2}\right)^2 + \frac{1}{1{,}440}\left(\frac{z}{2}\right)^4\right]\left(\frac{z}{2}\right)^5 \mathfrak{c}_1$$

$$U_4^{(1)} = \left[\frac{41}{30{,}240} - \frac{41}{40{,}320}\left(\frac{z}{2}\right)^2 + \frac{481}{1{,}814{,}400}\left(\frac{z}{2}\right)^4 - \frac{13}{622{,}080}\left(\frac{z}{2}\right)^6\right]\left(\frac{z}{2}\right)^6 \mathfrak{c}_0 - \left[\frac{41}{30{,}240} - \frac{41}{24{,}192}\left(\frac{z}{2}\right)^2 + \frac{799}{1{,}209{,}600}\left(\frac{z}{2}\right)^4 - \frac{2{,}111}{21{,}772{,}800}\left(\frac{z}{2}\right)^6 + \frac{1}{497{,}664}\left(\frac{z}{2}\right)^8\right]\left(\frac{z}{2}\right)^5 \mathfrak{c}_1$$

$$U_5^{(1)} = -\left[\frac{31}{15{,}840} - \frac{31}{21{,}120}\left(\frac{z}{2}\right)^2 + \frac{25}{59{,}136}\left(\frac{z}{2}\right)^4 - \frac{5{,}557}{95{,}800{,}320}\left(\frac{z}{2}\right)^6 + \frac{11}{3{,}225{,}600}\left(\frac{z}{2}\right)^8 - \frac{1}{29{,}859{,}840}\left(\frac{z}{2}\right)^{10}\right]\left(\frac{z}{2}\right)^6 \mathfrak{c}_0 + \left[\frac{31}{15{,}840} - \frac{31}{12{,}672}\left(\frac{z}{2}\right)^2 + \frac{661}{665{,}280}\left(\frac{z}{2}\right)^4 - \frac{3{,}599}{19{,}160{,}064}\left(\frac{z}{2}\right)^6 + \frac{1{,}471}{87{,}091{,}200}\left(\frac{z}{2}\right)^8 - \frac{1}{1{,}990{,}656}\left(\frac{z}{2}\right)^{10}\right]\left(\frac{z}{2}\right)^5 \mathfrak{c}_1$$

$$U_6^{(1)} = \left[\frac{10{,}331}{2{,}162{,}160} - \frac{10{,}331}{2{,}882{,}880}\left(\frac{z}{2}\right)^2 + \frac{79{,}021}{74{,}131{,}200}\left(\frac{z}{2}\right)^4 - \frac{68{,}543}{415{,}134{,}720}\left(\frac{z}{2}\right)^6 + \frac{24{,}931}{1{,}761{,}177{,}600}\left(\frac{z}{2}\right)^8 - \frac{3{,}239}{5{,}225{,}472{,}000}\left(\frac{z}{2}\right)^{10} + \frac{1}{107{,}495{,}424}\left(\frac{z}{2}\right)^{12}\right]\left(\frac{z}{2}\right)^6 \mathfrak{c}_0 - \left[\frac{10{,}331}{2{,}162{,}160} - \frac{10{,}331}{1{,}729{,}728}\left(\frac{z}{2}\right)^2 + \frac{2{,}251}{915{,}200}\left(\frac{z}{2}\right)^4 - \frac{147{,}967}{296{,}524{,}800}\left(\frac{z}{2}\right)^6 + \frac{650{,}857}{11{,}623{,}772{,}160}\left(\frac{z}{2}\right)^8 - \frac{229{,}793}{67{,}060{,}224{,}000}\left(\frac{z}{2}\right)^{10} + \frac{221}{2{,}351{,}462{,}400}\left(\frac{z}{2}\right)^{12} - \frac{1}{2{,}149{,}908{,}480}\left(\frac{z}{2}\right)^{14}\right]\left(\frac{z}{2}\right)^5 \mathfrak{c}_1$$

$$U_7^{(1)} = -\left[\frac{3{,}421}{196{,}560} - \frac{3{,}421}{262{,}080}\left(\frac{z}{2}\right)^2 + \frac{40{,}907}{10{,}378{,}368}\left(\frac{z}{2}\right)^4 - \frac{57{,}131}{88{,}957{,}440}\left(\frac{z}{2}\right)^6 + \frac{2{,}743}{43{,}929{,}600}\left(\frac{z}{2}\right)^8 - \frac{1{,}837{,}343}{498{,}161{,}664{,}000}\left(\frac{z}{2}\right)^{10} + \frac{200{,}159}{1{,}609{,}445{,}376{,}000}\left(\frac{z}{2}\right)^{12} - \frac{1}{522{,}547{,}200}\left(\frac{z}{2}\right)^{14} + \frac{1}{180{,}592{,}312{,}320}\left(\frac{z}{2}\right)^{16}\right]\left(\frac{z}{2}\right)^6 \mathfrak{c}_0 + \left[\frac{3{,}421}{196{,}560} - \frac{3{,}421}{157{,}248}\left(\frac{z^2}{2}\right) + \frac{9{,}749}{1{,}081{,}080}\left(\frac{z}{2}\right)^4 - \frac{235{,}111}{124{,}540{,}416}\left(\frac{z}{2}\right)^6 + \frac{819{,}631}{3{,}558{,}297{,}600}\left(\frac{z}{2}\right)^8 - \frac{407{,}413}{23{,}721{,}984{,}000}\left(\frac{z}{2}\right)^{10} + \frac{800{,}593}{1{,}046{,}139{,}494{,}400}\left(\frac{z}{2}\right)^{12} - \frac{751}{41{,}803{,}776{,}000}\left(\frac{z}{2}\right)^{14} + \frac{11}{75{,}246{,}796{,}800}\left(\frac{z}{2}\right)^{16}\right]\left(\frac{z}{2}\right)^5 \mathfrak{c}_1$$

TABLE III. Formulas for the coefficients $D_k^{(0)}(z)$ with $k = 0$ through 7.

$$D_0^{(0)} = \left(\frac{z}{2}\right)^2 \mathfrak{c}_0$$

$$D_1^{(0)} = -\frac{1}{12}\left(\frac{z}{2}\right)^4 \mathfrak{c}_0 + \left[\frac{1}{12} + \frac{1}{12}\left(\frac{z}{2}\right)^2\right]\left(\frac{z}{2}\right)^3 \mathfrak{c}_1$$

$$D_2^{(0)} = \left[\frac{1}{120} - \frac{1}{80}\left(\frac{z}{2}\right)^2 - \frac{1}{288}\left(\frac{z}{2}\right)^4\right]\left(\frac{z}{2}\right)^4 \mathfrak{c}_0 - \left[\frac{1}{120} - \frac{1}{60}\left(\frac{z}{2}\right)^2 + \frac{1}{480}\left(\frac{z}{2}\right)^4\right]\left(\frac{z}{2}\right)^3 \mathfrak{c}_1$$

$$D_3^{(0)} = -\left[\frac{1}{252} - \frac{1}{168}\left(\frac{z}{2}\right)^2 + \frac{137}{60{,}480}\left(\frac{z}{2}\right)^4 + \frac{1}{51{,}840}\left(\frac{z}{2}\right)^6\right]\left(\frac{z}{2}\right)^4 \mathfrak{C}_0$$
$$+ \left[\frac{1}{252} - \frac{1}{126}\left(\frac{z}{2}\right)^2 + \frac{11}{2{,}240}\left(\frac{z}{2}\right)^4 - \frac{127}{181{,}440}\left(\frac{z}{2}\right)^6 - \frac{1}{10{,}368}\left(\frac{z}{2}\right)^8\right]\left(\frac{z}{2}\right)^3 \mathfrak{C}_1$$

$$D_4^{(0)} = \left[\frac{1}{240} - \frac{1}{160}\left(\frac{z}{2}\right)^2 + \frac{1}{336}\left(\frac{z}{2}\right)^4 - \frac{79}{145{,}152}\left(\frac{z}{2}\right)^6 + \frac{87}{4{,}838{,}400}\left(\frac{z}{2}\right)^8\right.$$
$$\left. + \frac{1}{497{,}664}\left(\frac{z}{2}\right)^{10}\right]\left(\frac{z}{2}\right)^4 \mathfrak{C}_0 - \left[\frac{1}{240} - \frac{1}{120}\left(\frac{z}{2}\right)^2 + \frac{29}{5{,}040}\left(\frac{z}{2}\right)^4 - \frac{29}{18{,}144}\left(\frac{z}{2}\right)^6\right.$$
$$\left. + \frac{433}{2{,}903{,}040}\left(\frac{z}{2}\right)^8 + \frac{1}{207{,}360}\left(\frac{z}{2}\right)^{10}\right]\left(\frac{z}{2}\right)^3 \mathfrak{C}_1$$

$$D_5^{(0)} = -\left[\frac{1}{132} - \frac{1}{88}\left(\frac{z}{2}\right)^2 + \frac{367}{63{,}360}\left(\frac{z}{2}\right)^4 - \frac{19}{14{,}080}\left(\frac{z}{2}\right)^6 + \frac{1{,}559}{10{,}644{,}480}\left(\frac{z}{2}\right)^8\right.$$
$$\left. - \frac{1}{201{,}600}\left(\frac{z}{2}\right)^{10} - \frac{1}{5{,}971{,}968}\left(\frac{z}{2}\right)^{12}\right]\left(\frac{z}{2}\right)^4 \mathfrak{C}_0 + \left[\frac{1}{132} - \frac{1}{66}\left(\frac{z}{2}\right)^2\right.$$
$$\left. + \frac{229}{21{,}120}\left(\frac{z}{2}\right)^4 - \frac{73}{21{,}120}\left(\frac{z}{2}\right)^6 + \frac{1{,}135}{2{,}128{,}896}\left(\frac{z}{2}\right)^8 - \frac{791}{22{,}809{,}600}\left(\frac{z}{2}\right)^{10}\right.$$
$$\left. + \frac{41}{1{,}045{,}094{,}400}\left(\frac{z}{2}\right)^{12} + \frac{1}{29{,}859{,}840}\left(\frac{z}{2}\right)^{14}\right]\left(\frac{z}{2}\right)^3 \mathfrak{C}_1$$

$$D_6^{(0)} = \left[\frac{691}{32{,}760} - \frac{691}{21{,}840}\left(\frac{z}{2}\right)^2 + \frac{14{,}383}{864{,}864}\left(\frac{z}{2}\right)^4 - \frac{6{,}275}{1{,}482{,}624}\left(\frac{z}{2}\right)^6 + \frac{38{,}123}{65{,}894{,}400}\left(\frac{z}{2}\right)^8\right.$$
$$\left. - \frac{247{,}597}{5{,}930{,}496{,}000}\left(\frac{z}{2}\right)^{10} + \frac{165{,}269}{134{,}120{,}448{,}000}\left(\frac{z}{2}\right)^{12} + \frac{13}{1{,}393{,}459{,}200}\left(\frac{z}{2}\right)^{14}\right.$$
$$\left. - \frac{1}{2{,}149{,}908{,}480}\left(\frac{z}{2}\right)^{16}\right]\left(\frac{z}{2}\right)^4 \mathfrak{C}_0 - \left[\frac{691}{32{,}760} - \frac{691}{16{,}380}\left(\frac{z}{2}\right)^2 + \frac{14{,}747}{480{,}480}\left(\frac{z}{2}\right)^4\right.$$
$$\left. - \frac{26{,}855}{2{,}594{,}592}\left(\frac{z}{2}\right)^6 + \frac{19{,}957}{10{,}782{,}720}\left(\frac{z}{2}\right)^8 - \frac{29{,}777}{164{,}736{,}000}\left(\frac{z}{2}\right)^{10}\right.$$
$$\left. + \frac{2{,}154{,}983}{249{,}080{,}832{,}000}\left(\frac{z}{2}\right)^{12} - \frac{1}{10{,}752{,}000}\left(\frac{z}{2}\right)^{14} - \frac{1}{238{,}878{,}200}\left(\frac{z}{2}\right)^{16}\right]\left(\frac{z}{2}\right)^3 \mathfrak{C}_1$$

$$D_7^{(0)} = -\left[\frac{273}{3{,}276} - \frac{273}{2{,}184}\left(\frac{z}{2}\right)^2 + \frac{17{,}509}{262{,}080}\left(\frac{z}{2}\right)^4 - \frac{83{,}803}{4{,}717{,}440}\left(\frac{z}{2}\right)^6 + \frac{7{,}717}{2{,}882{,}880}\left(\frac{z}{2}\right)^8\right.$$
$$\left. - \frac{1{,}489}{6{,}220{,}800}\left(\frac{z}{2}\right)^{10} + \frac{111{,}901}{9{,}057{,}484{,}800}\left(\frac{z}{2}\right)^{12} - \frac{716{,}747}{2{,}324{,}754{,}432{,}000}\left(\frac{z}{2}\right)^{14}\right.$$
$$\left. + \frac{41}{62{,}705{,}664{,}000}\left(\frac{z}{2}\right)^{16} + \frac{67}{902{,}961{,}561{,}600}\left(\frac{z}{2}\right)^{18}\right]\left(\frac{z}{2}\right)^4 \mathfrak{C}_0$$
$$+ \left[\frac{273}{3{,}276} - \frac{273}{1{,}638}\left(\frac{z}{2}\right)^2 + \frac{32{,}069}{262{,}080}\left(\frac{z}{2}\right)^4 - \frac{100{,}217}{2{,}358{,}720}\left(\frac{z}{2}\right)^6\right.$$
$$\left. + \frac{84{,}407}{10{,}378{,}368}\left(\frac{z}{2}\right)^8 - \frac{136{,}427}{148{,}262{,}400}\left(\frac{z}{2}\right)^{10} + \frac{176{,}149}{2{,}846{,}638{,}080}\left(\frac{z}{2}\right)^{12}\right.$$

$$- \frac{7,911,653}{3,487,131,648,000}\left(\frac{z}{2}\right)^{14} + \frac{4,001}{134,120,448,000}\left(\frac{z}{2}\right)^{16}$$

$$+ \frac{13}{32,248,627,200}\left(\frac{z}{2}\right)^{18} - \frac{1}{180,592,312,320}\left(\frac{z}{2}\right)^{20}\Bigg]\left(\frac{z}{2}\right)^{3}\mathfrak{c}_1$$

TABLE IV. Formulas for the coefficients $D_k^{(1)}(z)$ with $k = 0$ through 7.

$$D_0^{(1)} = \left[2 - \left(\frac{z}{2}\right)^2\right]\mathfrak{c}_0 - \left[2 - 2\left(\frac{z}{2}\right)^2\right]\left(\frac{z}{2}\right)^{-1}\mathfrak{c}_1$$

$$D_1^{(1)} = -\frac{1}{12}\left(\frac{z}{2}\right)^4\mathfrak{c}_0 - \left[\frac{1}{4} + \frac{1}{12}\left(\frac{z}{2}\right)^2\right]\left(\frac{z}{2}\right)^3\mathfrak{c}_1$$

$$D_2^{(1)} = \left[\frac{11}{360} + \frac{1}{288}\left(\frac{z}{2}\right)^2\right]\left(\frac{z}{2}\right)^6\mathfrak{c}_0 - \left[\frac{11}{360} + \frac{7}{1,440}\left(\frac{z}{2}\right)^2\right]\left(\frac{z}{2}\right)^5\mathfrak{c}_1$$

$$D_3^{(1)} = -\left[\frac{31}{7,560} - \frac{31}{8,640}\left(\frac{z}{2}\right)^2 - \frac{11}{51,840}\left(\frac{z}{2}\right)^4\right]\left(\frac{z}{2}\right)^6\mathfrak{c}_0$$

$$+ \left[\frac{31}{7,560} - \frac{341}{60,480}\left(\frac{z}{2}\right)^2 + \frac{5}{4,032}\left(\frac{z}{2}\right)^4 + \frac{1}{10,368}\left(\frac{z}{2}\right)^6\right]\left(\frac{z}{2}\right)^5\mathfrak{c}_1$$

$$D_4^{(1)} = \left[\frac{41}{15,120} - \frac{41}{17,280}\left(\frac{z}{2}\right)^2 + \frac{2,413}{3,628,800}\left(\frac{z}{2}\right)^4 - \frac{619}{21,772,800}\left(\frac{z}{2}\right)^6\right.$$

$$\left. - \frac{1}{497,664}\left(\frac{z}{2}\right)^8\right]\left(\frac{z}{2}\right)^6\mathfrak{c}_0 - \left[\frac{41}{15,120} - \frac{451}{120,960}\left(\frac{z}{2}\right)^2 + \frac{983}{604,800}\left(\frac{z}{2}\right)^4\right.$$

$$\left. - \frac{4,783}{21,772,800}\left(\frac{z}{2}\right)^6 - \frac{11}{1,244,160}\left(\frac{z}{2}\right)^8\right]\left(\frac{z}{2}\right)^5\mathfrak{c}_1$$

$$D_5^{(1)} = -\left[\frac{31}{7,920} - \frac{217}{63,360}\left(\frac{z}{2}\right)^2 + \frac{271}{241,920}\left(\frac{z}{2}\right)^4 - \frac{15,347}{95,800,320}\left(\frac{z}{2}\right)^6 + \frac{608}{87,091,200}\left(\frac{z}{2}\right)^8\right.$$

$$\left. + \frac{7}{29,859,840}\left(\frac{z}{2}\right)^{10}\right]\left(\frac{z}{2}\right)^6\mathfrak{c}_0 + \left[\frac{31}{7,920} - \frac{31}{5,760}\left(\frac{z}{2}\right)^2 + \frac{667}{266,112}\left(\frac{z}{2}\right)^4\right.$$

$$- \frac{9,859}{19,160,064}\left(\frac{z}{2}\right)^6 + \frac{10,379}{239,500,800}\left(\frac{z}{2}\right)^8 - \frac{37}{348,364,800}\left(\frac{z}{2}\right)^{10}$$

$$\left. - \frac{1}{29,859,840}\left(\frac{z}{2}\right)^{12}\right]\left(\frac{z}{2}\right)^5\mathfrak{c}_1$$

$$D_6^{(1)} = \left[\frac{10,331}{1,081,080} - \frac{10,331}{1,235,520}\left(\frac{z}{2}\right)^2 + \frac{106,387}{37,065,600}\left(\frac{z}{2}\right)^4 - \frac{1,020,521}{2,075,673,600}\left(\frac{z}{2}\right)^6\right.$$

$$+ \frac{313,097}{7,264,857,600}\left(\frac{z}{2}\right)^8 - \frac{308,233}{201,180,672,000}\left(\frac{z}{2}\right)^{10} - \frac{193}{18,811,699,200}\left(\frac{z}{2}\right)^{12}$$

$$\left. + \frac{1}{2,149,908,480}\left(\frac{z}{2}\right)^{14}\right]\left(\frac{z}{2}\right)^6\mathfrak{c}_0 - \left[\frac{10,331}{1,081,080} - \frac{10,331}{786,240}\left(\frac{z}{2}\right)^2\right.$$

$$+ \frac{45,079}{7,207,200}\left(\frac{z}{2}\right)^4 - \frac{423,751}{296,524,800}\left(\frac{z}{2}\right)^6 + \frac{992,969}{5,811,886,080}\left(\frac{z}{2}\right)^8$$

$$- \frac{4,285,159}{435,891,456,000}\left(\frac{z}{2}\right)^{10} + \frac{887}{6,718,464,000}\left(\frac{z}{2}\right)^{12}$$

$$+ \frac{11}{2,149,908,480}\left(\frac{z}{2}\right)^{14}\Bigg]\left(\frac{z}{2}\right)^{5} \mathfrak{e}_1$$

$$D_7^{(1)} = -\Bigg[\frac{3,421}{98,280} - \frac{3,421}{112,320}\left(\frac{z}{2}\right)^{2} + \frac{42,671}{3,991,680}\left(\frac{z}{2}\right)^{4} - \frac{1,223,947}{622,702,080}\left(\frac{z}{2}\right)^{6}$$

$$+ \frac{14,713}{71,165,952}\left(\frac{z}{2}\right)^{8} - \frac{558,461}{45,287,424,000}\left(\frac{z}{2}\right)^{10} + \frac{7,406,743}{20,922,789,888,000}\left(\frac{z}{2}\right)^{12}$$

$$- \frac{7}{5,971,968,000}\left(\frac{z}{2}\right)^{14} - \frac{11}{128,994,508,800}\left(\frac{z}{2}\right)^{16}\Bigg]\left(\frac{z}{2}\right)^{6} \mathfrak{e}_0$$

$$+ \Bigg[\frac{3,421}{98,280} - \frac{37,631}{786,240}\left(\frac{z}{2}\right)^{2} + \frac{39,815}{1,729,728}\left(\frac{z}{2}\right)^{4} - \frac{52,667}{9,580,032}\left(\frac{z}{2}\right)^{6}$$

$$+ \frac{1,316,273}{1,779,148,800}\left(\frac{z}{2}\right)^{8} - \frac{1,370,671}{23,721,984,000}\left(\frac{z}{2}\right)^{10} + \frac{25,463,237}{10,461,394,944,000}\left(\frac{z}{2}\right)^{12}$$

$$- \frac{961}{25,751,126,016}\left(\frac{z}{2}\right)^{14} - \frac{17}{37,623,398,400}\left(\frac{z}{2}\right)^{16}$$

$$+ \frac{1}{180,592,312,320}\left(\frac{z}{2}\right)^{18}\Bigg]\left(\frac{z}{2}\right)^{5} \mathfrak{e}_1$$

TABLE V. Values of the coefficients ${}^0U_k^{(0)}(z)$ and ${}^1U_k^{(0)}(z)$ with $k = 0$ through 7 and with $z = 3.5(0.5)7.5$

z \ k	0	1	2	3	4	5	6	7
							${}^0U_k^{(0)}(z)$	
3.5	+ .2404	+ .3585	+ .0593	+ .0040	+ .0001			
4.0	− .1321	+ .4855	+ .1662	+ .0204	+ .0013			
4.5	− .5199	+ .4653	+ .3677	+ .0803	+ .0088	+ .0006		
5.0	− .8189	+ .1516	+ .6405	+ .2512	+ .0451	+ .0048		
5.5	− .9390	− .5591	+ .8194	+ .6326	+ .1844	+ .0304	+ .0033	
6.0	− .8301	−1.6394	+ .5094	+1.2653	+ .6126	+ .1543	+ .0248	+ .0028
6.5	− .5000	−2.8582	− .9198	+1.8590	+1.6560	+ .6394	+ .1500	+ .0241
7.0	− .0016	−3.7693	−4.1461	+1.2775	+3.5502	+2.1826	+ .7462	+ .1703
7.5	+ .5072	−3.7948	−9.4364	−3.0898	+5.4044	+6.0816	+3.0829	+ .9871
							${}^1U_k^{(0)}(z)$	
3.5	+ .7178	+ .0355	− .0362	− .0043	− .0003			
4.0	+ .7959	+ .2879	− .0374	− .0138	− .0020			
4.5	+ .6772	+ .7016	+ .0522	− .0286	− .0073	+ .0009		
5.0	+ .3697	+1.1968	+ .3751	− .0125	− .0200	− .0018		
5.5	− .0653	+1.5768	+1.1173	+ .1787	− .0276	− .0124	− .0067	
6.0	− .5250	+1.5515	+2.3930	+ .9167	+ .0838	− .0297	− .0128	+ .0091
6.5	− .8908	+ .8266	+4.0212	+2.8948	+ .7845	+ .0375	− .0299	− .0127
7.0	−1.0593	− .7569	+5.2477	+6.9946	+3.4453	+ .7529	+ .0207	− .0512
7.5	− .9717	−3.0720	+4.5695	+13.5834	+11.1065	+4.3933	+ .8567	− .0145

TABLE VI. Values of the coefficients ${}^0U_k^{(1)}(z)$ and ${}^1U_k^{(1)}(z)$ with $k = 0$ through 7 and with $z = 3.5(0.5)7.0$

z \ k	0	1	2	3	4	5	6	7
3.5	+ .6768	− .4812	− .1773	− .0175	− .0008			
4.0	+ .8603	− .3974	− .3854	− .0712	− .0060	− .0003	${}^0U_k^{(1)}(z)$	
4.5	+ .9556	− .0266	− .6660	− .2273	− .0328	− .0027		
5.0	+ .9121	+ .7015	− .8679	− .5861	− .1402	− .0184	− .0015	
5.5	+ .7043	+1.7426	− .6349	−1.2141	− .4851	− .0990	− .0126	
6.0	+ .3443	+2.8845	+ .6039	−1.9323	−1.3738	− .4327	− .0822	− .0106
6.5	− .1149	+3.7384	+3.4891	−1.8745	−3.1495	−1.5606	− .4354	− .2526
7.0	− .5864	+3.8027	+8.3606	+1.0917	−5.5007	−4.6443	−1.9128	− .5038
3.5	− .6271	− .4019	+ .0517	+ .0143	+ .0024			
4.0	− .3640	− .8184	− .0342	+ .0343	+ .0067	− .0019	${}^1U_k^{(1)}(z)$	
4.5	− .0203	−1.2730	− .3460	+ .0369	+ .0196	+ .0006		
5.0	+ .3657	−1.5819	−1.0580	− .0954	+ .0392	+ .0102	+ .0052	
5.5	+ .7270	−1.4944	−2.2842	− .6801	− .0028	+ .0331	+ .0081	
6.0	+ .9847	− .7640	−3.8890	−2.3339	− .4337	+ .0339	+ .0256	+ .0134
6.5	+1.0686	+ .7416	−5.2455	−5.9138	−2.2843	− .2992	+ .0439	+ .1615
7.0	+ .9383	+2.9197	−5.0617	−12.0115	−8.0027	−2.3942	− .2510	+ .0885

TABLE VII. Values of the coefficients ${}^0D_k^{(0)}(z)$ and ${}^1D_k^{(0)}(z)$ with $k = 0$ through 7 and with $z = 3.5(0.5)7.5$

z \ k	0	1	2	3	4	5	6	7
3.5	−1.1641	+ .5464	+ .2399	+ .0248	+ .0012			
4.0	−1.5886	+ .3094	+ .6046	+ .1219	+ .0108	+ .0005	${}^0D_k^{(0)}(z)$	
4.5	−1.6227	− .6451	+1.1299	+ .4524	+ .0696	+ .0059		
5.0	−1.1100	−2.5143	+1.3511	+1.2957	+ .3423	+ .0473	+ .0041	
5.5	− .0518	−5.0341	+ .1217	+2.8315	+1.3213	+ .2896	+ .0387	
6.0	+1.3558	−7.2422	−4.5006	+4.2666	+4.0382	+1.4073	+ .2830	+ .0381
6.5	+2.7472	−7.5067	−14.4887	+1.9408	+9.5432	+5.5044	+1.6547	+ .3222
7.0	+3.6760	−3.9742	−29.9598	−13.2338	+15.4978	+17.2609	+7.8914	+2.2065
7.5	+3.7454	+4.5633	−46.3111	−56.8456	+5.9852	41.6448	+30.6623	+12.3593
3.5	+ .5789	+ .5965	− .0599	− .0197	− .0035			
4.0	− .0678	+1.3490	+ .1059	− .0547	− .0115	+ .0030	${}^1D_k^{(0)}(z)$	
4.5	− .9857	+2.1480	+ .7959	− .0496	− .0386	− .0030		
5.0	−1.9282	+2.4001	+2.5091	+ .3255	− .0799	− .0280	− .0042	
5.5	−2.5673	+1.2654	+5.5298	+2.0747	+ .0874	− .0904	− .0119	
6.0	−2.5938	−1.9924	+9.1250	+7.2941	+1.6581	− .0485	− .0705	− .0921
6.5	−1.8299	−7.4553	+10.4872	+18.8387	+8.7051	+1.4192	− .1036	− .1935
7.0	− .3179	−14.0041	+4.1085	+37.7230	+31.3594	+10.8055	+1.4318	− .2809
7.5	+1.6497	−19.0857	−17.2733	+56.5553	+87.7617	+51.1071	+14.8618	+1.7063

TABLE VIII. Values of the coefficients ${}^0D_k^{(1)}(z)$ and ${}^1D_k^{(1)}(z)$ with k = o through 7 and with z = 3.5(0.5)7.0

z \ k	0	1	2	3	4	5	6	7
							${}^0D_k^{(1)}(z)$	
3.5	+ .7277	− .0749	− .5522	− .0941	− .0063			
4.0	+ .5962	+ .8377	−1.0240	− .3628	− .0441	− .0031		
4.5	+ .1473	+2.4529	−1.2668	−1.0756	− .2312	− .0281		
5.0	− .6210	+4.5236	− .3164	−2.4620	− .9419	− .1797	− .0172	
5.5	−1.5915	+6.2828	+3.4468	−4.0832	−3.0436	− .9013	− .1378	
6.0	−2.5302	+6.4536	+11.7834	−3.3613	−7.7458	−3.3690	− .8705	− .1361
6.5	−3.1324	+3.5506	+25.1749	+6.6506	−14.5558	−12.1739	−4.4488	−1.0093
7.0	−3.1059	−3.4974	+40.5401	+38.8142	−14.2714	−32.1868	−18.6006	−6.1044
							${}^1D_k^{(1)}(z)$	
3.5	+ .7660	−1.2584	− .0823	+ .0565	+ .0074			
4.0	+1.2277	−1.8344	− .6849	+ .0815	+ .0304	+ .0051		
4.5	+1.6832	−1.8877	−2.1735	− .1481	+ .0840	+ .0260		
5.0	+1.9322	− .7766	−4.8160	−1.4141	+ .0598	+ .0853	− .0010	
5.5	+1.7750	+2.0529	−8.0904	−5.4407	− .7879	+ .1463	+ .0442	
6.0	+1.0840	+6.6706	−9.8249	−14.8121	−5.2657	− .4507	+ .1660	+ .1452
6.5	− .1295	+12.2448	−5.5858	−31.2672	−20.9246	−5.4457	− .1685	+ .2918
7.0	−1.6797	+16.8159	+10.8370	−50.8593	−62.7531	−29.3100	−5.7721	+ .0160