where f is an arbitrary harmonic function and $\mathbf{A}$ a conjugate harmonic vector such that

$$\text{grad } f + \text{curl } \mathbf{A} = 0. \tag{2}$$

Proof. We have

$$\int_V (\mathbf{A} \cdot \text{curl } \mathbf{v} - \mathbf{v} \cdot \text{curl } \mathbf{A}) \, dx \, dy \, dz = -\iint_S \mathbf{A} \cdot (\mathbf{v} \times \mathbf{n}) \, dS$$

(here $\mathbf{n}$ is the normal unit vector) and

$$\int_V (f \text{ div } \mathbf{v} + \mathbf{v} \cdot \text{grad } f) \, dx \, dy \, dz = \iint_R f\mathbf{v} \cdot \mathbf{n} \, dS,$$

and therefore it follows that

$$\iint_B (\mathbf{A} \cdot (\mathbf{v} \times \mathbf{n}) + f\mathbf{v} \cdot \mathbf{n}) \, dS \tag{3}$$

vanishes if and only if Eq. (1) is satisfied. However, (3) will vanish for arbitrary harmonic f only if a continuous $\mathbf{v}$ vanishes everywhere on the surface B, and thus (1) is necessary and sufficient for the satisfaction of the viscous boundary condition.

NOTE ON ITERATIONS WITH CONVERGENCE OF HIGHER DEGREE*

By HASKELL B. CURRY (*Pennsylvania State College*)

In a recent paper[1] E. Bodewig has derived a general expression for a function $F(x)$ such that the sequence

$$x_{n+1} = F(x_n) \tag{A}$$

converges in a degree at least m in the neighborhood of every root of a polynomial $f(x)$, provided the latter has only simple roots. A part of this argument can be simplified, while another part can be made somewhat more natural.

In regard to the first point, the operator P used by Bodewig is the same as d/dy where $y = f(x)$. Further, since his r is $1/f'$, we have $r = dx/dy = Px$. Hence the identity between Bodewig's formula (14a) and the Euler inverse Taylor expansion (15) follows immediately; it is not necessary to use the recurrence formula for the higher derivatives of an inverse function.

In regard to the second point, if we have a sequence of functions $F_1(x)$, $F_2(x)$, $\cdots$ such that the sequence (A) converges in the degree at least m for $F = F_m$, then any $F(x)$ giving rise to a sequence converging at least in the degree m must be of the form

$$F = F_m + g_m f^m.$$

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[1]E. Bodewig, *On types of convergence and on the behavior of approximations in the neighborhood of a multiple root of an equation*, Q. Appl. Math. **7**, 325-333 (1949).

Now F_{m+1} is itself such an F. Hence, by induction, we have, for suitable g_k ,

$$F_m = x + g_1 f + g_2 f^2 + \cdots + g_{m-1} f^{m-1}.$$

This suggests a change of variable to $y = f(x)$ (which is possible since $f'(X) \neq 0$). If $x = u(y)$ is the inverse function, and

$$\Phi_m(y) = F_m[u(y)], \qquad \psi_k(y) = g_k[u(y)],$$

then

$$\Phi_m = u + \psi_1 y + \psi_2 y^2 + \cdots + \psi_{m-1} y^{m-1}. \tag{B}$$

The conditions which must be satisfied by Φ_m are

$$\Phi(0) = X,$$

$$\Phi'(0) = \Phi''(0) = \cdots = \Phi^{m-1}(0) = 0.$$

The first of these is automatically satisfied.

Now a function $\Phi_m(y)$ satisfying these conditions is given immediately by the inverse Taylor expansion of $X = u(y - y)$. In fact, if we set $\Phi_m(y)$ equal to the sum of the first m terms of this expansion, viz.:

$$\Phi_m(y) = \sum_{k=0}^{m-1} \frac{(-y)^k}{k!} u^{(k)}(y),$$

then

$$X = \Phi_m(y) + \frac{(-y)^m}{m!} u^{(m)}(\eta),$$

and hence

$$\Phi_m(y) = X - \frac{(-y)^m}{m!} u^{(m)}(\eta)$$

satisfies the above conditions.

This method avoids the necessity of slapping down Bodewig's formula (14) or of motivating it by tedious experimenting with small values of m.

BOUNDARIES FOR THE LIMIT CYCLE OF VAN DER POL'S EQUATION*

By R. GOMORY AND D. E. RICHMOND (*Williams College*)

1. Introduction. In non-linear mechanics much interest centers on the Van der Pol (VDP) equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0 \tag{1}$$

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