STUDIES ON TWO-DIMENSIONAL TRANSONIC FLOWS OF COMPRESSIBLE FLUID.—PART III*

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13. The introduction of a second new hypothetical gas. In the foregoing sections, we have investigated some transonic fields of flow by employing a hypothetical gas which closely approximates the real gas during an isentropic flow in the transonic range. Although such a hypothetical gas has a merit that the fundamental equation governing its flow assumes a rather simple form and can be solved exactly in several cases, it has the drawback that it can approximate the real gas obeying the isentropic law only for a limited (transonic) range of velocities. Thus, the method of analysis as developed in the foregoing sections has a rather narrow application; it can be applied only to nearly uniform transonic flows.

In Part III, an attempt is made to develop a theory which is applicable even to a transonic flow containing limited supersonic regions as well as stagnation points. For this purpose, we have introduced a second hypothetical gas which is capable of representing the real gas subject to the isentropic law with a better degree of approximation than that used in Parts I and II.

For the sake of convenience, we start from the linearized equations of motion in the hodograph plane which are valid for any compressible perfect fluid. They can be written in the following forms:

$$\begin{aligned} \varphi_w &= -X\psi_\theta , \\ \varphi_\theta &= \psi_w , \end{aligned} \tag{13.1}$$

where, as before, the coefficient X and independent variable w are respectively given by

$$\begin{aligned} X &= -\frac{q^2}{\rho}\left(\frac{1}{\rho q}\right)' = \frac{q^2}{\rho^2}\left(\frac{1}{q^2} - \frac{1}{c^2}\right), \\ w &= \int_1^q \frac{\rho}{q}\,dq. \end{aligned} \tag{13.2}$$

Eliminating φ from Eqs. (13.1), we obtain the fundamental equation for determining ψ in the form:

$$\psi_{ww} + X\psi_{\theta\theta} = 0. \tag{13.3}$$

The coefficient X in the second term can evidently be expressed as a function of w alone by the use of the known equation of state of any gas together with Bernoulli's theorem. The dotted-line curve in Fig. 20 shows the behaviour of $X(w)$ for the case of the real gas obeying the isentropic law, the value of γ having been taken equal to 1.4 for air.

It is in general difficult to obtain exact solutions of the fundamental equation (13.3)

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with the coefficient $X(w)$ for the case of the real gas. It is therefore suggested that the function $X(w)$ for the case of the real gas may be replaced by a simple approximate function in order to reduce Eq. (13.3) to a tractable form. The most simple replacement of $X(w)$ by a suitable constant (as done by Chaplygin, Kármán and Tsien), however, is not appropriate for the investigation of transonic flow. In Parts I and II, we have replaced the function $X(w)$ for the case of the real gas by the tangent, as shown by the

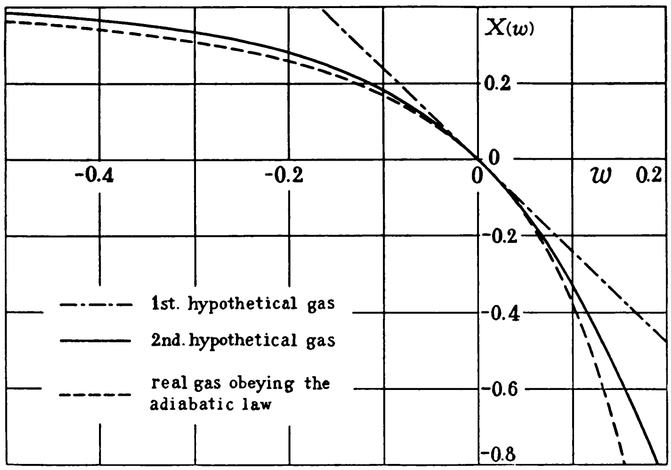

Fig. 20.

chain-line curve in Fig. 20, at the point $w = 0$ (i.e. at the critical state $q = c = 1$) for the purpose of dealing with nearly uniform transonic flow. In order to treat transonic fields of flow containing stagnation points, however, it is necessary to make use of another approximate function which can approximate more precisely the function $X(w)$ for the real gas.

In the following lines, we shall use a function of the form:

$$X(w) = a(1 - be^{2\kappa w}), \tag{13.4}$$

where a, b and κ are constants to be determined adequately. As will be seen presently, the use of this function makes our fundamental equation (13.3) tractable.

We have conveniently determined the values of the constants a, b and κ in (13.4) in such a way that the curve of $X(w)$ as given by (13.4) coincides with the corresponding curve for the real gas to the order of their tangents at $w = -\infty$ (which corresponds to the stagnation point) as well as at $w = 0$ (which corresponds to the critical state). The expression for $X(w)$ thus determined is given by

$$X(w) = a(1 - e^{2\kappa w}),$$

with

$$a = \left(\frac{2}{\gamma + 1}\right)^{2/(\gamma-1)}, \qquad \kappa = \left(\frac{\gamma + 1}{2}\right)^{(\gamma+1)/(\gamma-1)}. \tag{13.5}$$

The full-line curve in Fig. 20 shows the curve of $X(w)$ given by (13.5), by taking, as before, $\gamma = 1.4$ for air. It will readily be observed that this curve can satisfactorily approximate the curve of $X(w)$ for the real gas which is shown by the dotted curve in the figure.

Now, as mentioned already, the form of the function $X(q)$ defined by (13.2) depends upon the equation of state $\rho(q)$ of the gas concerned. Conversely, any given expression of this function $X(q)$ determines the equation of state of a corresponding gas. Thus, by introducing, in the following analysis, a second new hypothetical gas, we take the function defined by (13.5) as an exact relation valid for such a hypothetical gas, instead of considering it as an approximation to the corresponding function for the real gas obeying the isentropic law, and we shall deal, in an exact manner, with the field of flow

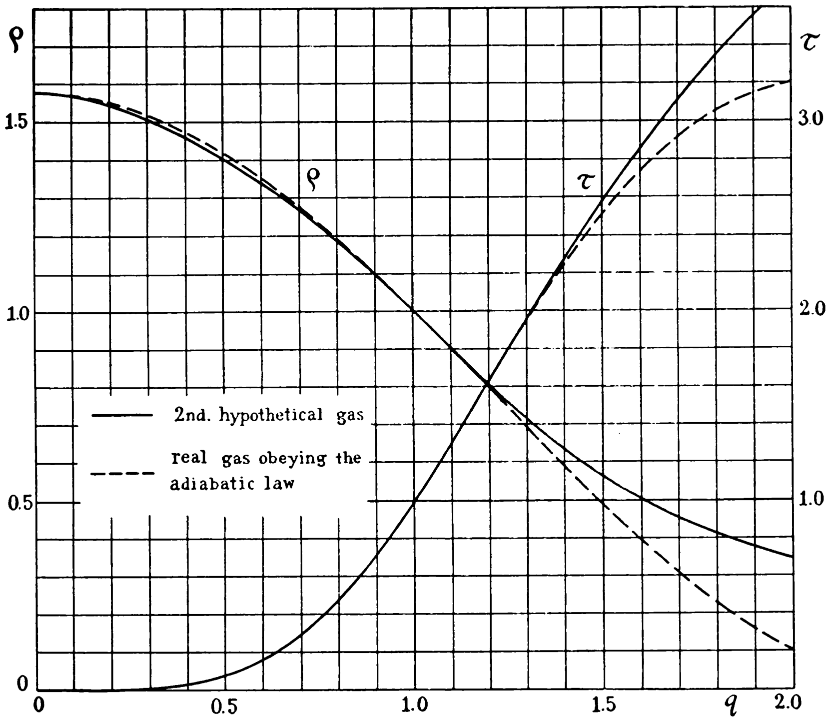

FIG. 21.

of such a hypothetical gas, as done with the previous hypothetical gas in Parts I and II.

If we insert the expression for $X(w)$ given by (13.5) into Eq. (13.3), we obtain the fundamental equation for determining the flow of our new hypothetical gas in the form.

$$\psi_{ww} + a(1 - e^{2\kappa w})\psi_{\theta\theta} = 0,$$

with (13.6)

$$a = \left(\frac{2}{\gamma + 1}\right)^{2/(\gamma-1)}, \qquad \kappa = \left(\frac{\gamma + 1}{2}\right)^{(\gamma+1)/(\gamma-1)}.$$

Before proceeding further, we shall now discuss briefly the properties of our hypothetical gas. If we combine (13.2) with (13.5), we obtain the differential equation for determining the equation of state $\rho(q)$ of our new hypothetical gas. We thus have

$$2\kappa \frac{\rho}{q}\left\{a + \frac{q^2}{\rho}\left(\frac{1}{\rho q}\right)'\right\} = \left\{\frac{q^2}{\rho}\left(\frac{1}{\rho q}\right)'\right\}'. \tag{13.7}$$

Solving this equation, by the method of numerical integration or otherwise, under the conditions that $\rho = 1$ and $\rho' = -1$ at $q = 1$, we have obtained the curve of $\rho(q)$ for our hypothetical gas as shown by a full-line curve in Fig. 21. It will be seen that this curve of $\rho(q)$ coincides with the corresponding curve, shown by a dotted-line curve, for the real gas obeying the isentropic law, to the order of tangent at $q = 0$ and to the order of curvature at $q = 1$. In this figure, the curves of $\tau(q)$ for both our hypothetical gas and the real gas subject to the isentropic law are also given by a full-line curve and a dotted-line curve respectively, where $\tau = e^{\kappa w}$ is a new variable to be introduced in the following section.

14. A method of solving the fundamental equation (13.6). We now introduce new independent variables τ, β defined as:

$$\tau = e^{\kappa w} = \exp\left(\kappa \int_1^q \frac{\rho}{q}\, dq\right),$$
$$\beta = \frac{\kappa}{\sqrt{a}}\, \theta. \tag{14.1}$$

Then, the fundamental equation (13.6) for determining the flow of our hypothetical gas takes the form:

$$\tau^2 \psi_{\tau\tau} + \tau\psi_\tau + (1 - \tau^2)\psi_{\beta\beta} = 0. \tag{14.2}$$

It is evident that just as the fundamental equation for the isentropic flow of the real gas does, this equation (14.2) changes from the elliptic to the hyperbolic type according as $\tau < 1$ (i.e. $q < 1$) or $\tau > 1$ (i.e. $q > 1$), i.e. according to whether the flow is subsonic or supersonic.

The characteristic curves of Eq. (14.2) are given by the equations:

$$\pm(\beta - \beta_0) = \sqrt{\tau^2 - 1} - \cos^{-1}\frac{1}{\tau}, \tag{14.3}$$

where β_0 is an arbitrary parameter.

To solve Eq. (14.2), we first assume that

$$\psi = T(\tau)e^{-in\beta}, \tag{14.4}$$

with an arbitrary constant n. Then, we obtain an ordinary differential equation for determining the function $T(\tau)$ in the form:

$$\tau^2 \frac{d^2T}{d\tau^2} + \tau \frac{dT}{d\tau} - n^2(1 - \tau^2)T = 0. \tag{14.5}$$

The general solution of this equation can be expressed in terms of Bessel functions. Thus,

$$T(\tau) = AJ_n(n\tau) + BY_n(n\tau),$$

where A and B are arbitrary constants. Hence, the solution of (14.2) which is finite at $\tau = 0$ can be expressed in the form:

$$\psi = \sum_{n=0}^{\infty} A_n J_n(n\tau)e^{-in\beta}, \tag{14.6}$$

where n and A_n's are arbitrary constants.

If now we make use of the integral representation of $J_n(n\tau)$ of the Bessel type, namely[1]:

$$J_n(n\tau) = \frac{1}{2\pi i}\oint_{(0+)} \{t^{-1} \exp [\tau(t - 1/t)/2]\}^n \frac{dt}{t},$$

the expression for ψ becomes

$$\psi = \frac{1}{2\pi i}\oint_{(0+)} \sum_{n=0}^{\infty} A_n\{t^{-1} \exp [\tau(t - 1/t)/2 - i\beta]\}^n \frac{dt}{t} \cdot \tag{14.7}$$

Since, however, the series in the integrand:

$$\sum_{n=0}^{\infty} A_n\{t^{-1} \exp [\tau(t - 1/t)/2 - i\beta]\}^n$$

is evidently the expansion of a certain function in powers of $t^{-1} \exp [\tau(t - 1/t)/2 - i\beta]$, the above solution can be generalized in the following form:

$$\psi = \frac{1}{2\pi i}\int_C F(Z)\frac{dt}{t},$$

with (14.8)

$$Z = t^{-1} \exp [\tau(t - 1/t)/2 - i\beta],$$

where $F(Z)$ denotes an arbitrary function of the variable Z, and the path of integration C should be so chosen that this expression for ψ may really become the solution of Eq. (14.2).

In fact, inserting the above expression (14.8) for ψ into the left-hand side of Eq. (14.2), which is conveniently denoted by $D(\psi)$, we have

$$D(\psi) = \frac{1}{2\pi i}\int_C \frac{d}{dt}\left[\left\{\frac{\tau}{2}\left(t + \frac{1}{t}\right) + 1\right\}Z\frac{dF}{dZ}\right] dt.$$

Thus, it will be seen that in order that the function ψ given by (14.8) becomes in effect the solution of Eq. (14.2), i.e. $D(\psi) = 0$, the function:

$$\Delta \equiv \left\{\frac{\tau}{2}\left(t + \frac{1}{t}\right) + 1\right\}Z\frac{dF}{dZ}, \qquad (Z = t^{-1} \exp [\tau(t - 1/t)/2 - i\beta])$$

must take the same value at the two end-points of the path of integration C and that in the case where the path C is a closed curve, it is sufficient that dF/dZ should be one-valued along C.

15. A few fundamental solutions of equation (14.2). In this section, a few fundamental solutions will be constructed by applying the method explained above.

In the first place, we shall consider a solution which is obtained by taking the arbitrary constant A_n in the general solution (14.6) to be equal to $1/\lambda^n$, where λ is a certain positive constant. By comparison with the results obtained in §§8, 9 and 10 of Part II, it is expected that such a solution will have a branch-point of order $-1/2$ at a certain point in the hodograph plane, the position of the singular point depending however upon the value of the parameter λ.

[1]G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, 1922, p. 20.

Since $A_n = 1/\lambda^n$, the function F in (14.8) becomes

$$F = \frac{1}{1 - e^z}$$

with (15.1)

$$z = \frac{\tau}{2}\left(t - \frac{1}{t}\right) - \log t - i\beta - \log \lambda.$$

Thus, if we denote the solution under consideration by $\psi_{(-1/2)}$, we have

$$\psi_{(-1/2)} = \frac{1}{2\pi i}\int_C \frac{dt}{t(1 - e^z)}, \tag{15.2}$$

where C is a path of integration to be so chosen that this $\psi_{(-1/2)}$ actually becomes the solution of the fundamental equation (14.2).

Let $t = t_0$ be a root of the equation $z(t) = 0$.[2] Then, in the neighbourhood of the point $t = t_0$, the integrand in (15.2) can be expanded in the following form:

$$\frac{1}{t(1 - e^z)} = -\frac{1}{t_0(dz/dt)_{t=t_0}}\frac{1}{t - t_0} + O(1),$$

and therefore it will be seen that the integrand has a pole of the first order at the point $t = t_0$, provided that $(dz/dt)_{t=t_0} \neq 0$.

It is readily found that if we take, as the path of integration C in (15.2), a small closed curve enclosing only the point $t = t_0$ but not any other singular points, the function $\psi_{(-1/2)}$ given by (15.2) actually becomes the solution of the fundamental equation (14.2). The value of the integral can then be obtained by evaluating the residue of the integrand at the point $t = t_0$ and thus, taking the integral along the path of integration in the positive sense, we have

$$\psi_{(-1/2)} = -\frac{1}{t_0(dz/dt)_{t=t_0}} = \frac{1}{1 - (\tau/2)(t_0 + 1/t_0)}.$$

If we here put $t_0 = e^{i\omega}$ for the sake of convenience, the expression for $\psi_{(-1/2)}$ becomes ultimately

$$\psi_{(-1/2)} = \frac{1}{1 - \tau\cos\omega}. \tag{15.3[3]}$$

On the other hand, since $t = t_0$ has been assumed to be a zero-point of the function $z(t)$ as defined by (15.1), we have $z(t_0) = 0$, and when use is made of the above substitution $t_0 = e^{i\omega}$, this equation becomes

$$\beta - i\log\lambda + \tau\sin\omega - \omega = 0. \tag{15.4}$$

By eliminating ω from the above two equations (15.3) and (15.4), we can express $\psi_{(-1/2)}$ in terms of τ and β.

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[2]It may be remarked here that this equation has in general two different roots.

[3]It can be ascertained without difficulty that this $\psi_{(-1/2)}$ really becomes a solution of the fundamental equation (14.2).

For a special set of values τ_∞, β_∞ of the independent variables τ, β, the two equations:

$$\begin{aligned} \beta - i \log \lambda + \tau \sin \omega - \omega &= 0, \\ 1 - \tau \cos \omega &= 0 \end{aligned} \tag{15.5}$$

are satisfied simultaneously,[4] and the function $\psi_{(-1/2)}$ as given by (15.3) becomes infinite. Thus, it is found that as has been expected from the outset, the solution $\psi_{(-1/2)}$ has a branch-point of order $-1/2$ at the point $(\tau_\infty, \beta_\infty)$ in the τ, β plane.

Elimination of ω from the two equations in (15.5) gives the equation for determining τ_∞ and β_∞ in the form:

$$\beta_\infty + i\left\{-\log \lambda \pm \sqrt{1 - \tau_\infty^2} \mp \cosh^{-1} \frac{1}{\tau_\infty}\right\} = 0. \tag{15.6}$$

In case when $\tau_\infty \leqq 1$, both $\sqrt{1 - \tau_\infty^2}$ and $\cosh^{-1}(1/\tau_\infty)$ are real and therefore, separating the real and imaginary parts, we have

$$\begin{aligned} \beta_\infty &= 0, \\ -\log \lambda \pm \sqrt{1 - \tau_\infty^2} \mp \cosh^{-1} \frac{1}{\tau_\infty} &= 0. \end{aligned} \tag{15.7}$$

From these equations it will be seen that the singular point of the solution $\psi_{(-1/2)}$ in the τ, β plane is, in this case, situated at an isolated point $\tau = \tau_\infty$ on the axis $\beta = 0$. The second equations in (15.7) give the relationship between the coordinate τ_∞ and the parameter λ. The curve of τ_∞ plotted against λ is shown in Fig. 22, from which it is

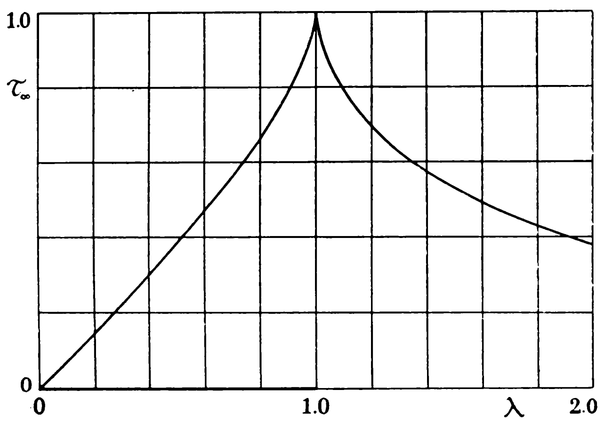

FIG. 22.

readily found that in order to make τ_∞ to have any value in the range from 0 to 1, it is quite sufficient to take for λ only a value in the restricted range from 0 to 1.

Further, for the hypothetical gas under consideration, we have, from (13.2), (13.5) and (14.1),

$$\frac{q^2}{\rho^2}\left(\frac{1}{c^2} - \frac{1}{q^2}\right) = a(\tau^2 - 1).$$

[4]It is readily found that this corresponds to the case when $(z)_{t=t_0} = 0$ and $(dz/dt)_{t=t_0} = 0$, i.e. when the two roots of the equation $z(t) = 0$ coincide with each other.

Therefore, the Mach number M associated with the state at the singular point $\tau = \tau_\infty$, $\beta = 0$ is calculated by the formula:

$$M = \frac{q_\infty}{c_\infty} = \sqrt{1 + a\rho_\infty^2(\tau_\infty^2 - 1)}. \tag{15.8}$$

Next, we shall consider the case when $\tau_\infty \geqq 1$. In this case, both $\sqrt{1 - \tau_\infty^2}$ and $\cosh^{-1}(1/\tau_\infty)$ are purely imaginary and therefore, by separating the real and imaginary parts on the right-hand side of Eq. (15.6), we have the following two equations:

$$\begin{aligned} \log \lambda &= 0, \\ \beta_\infty \mp \sqrt{\tau_\infty^2 - 1} \pm \cos^{-1}\frac{1}{\tau_\infty} &= 0. \end{aligned} \tag{15.9}$$

Hence, it will be seen that in the supersonic region where $\tau \geqq 1$, the singularity occurs only at a particular value $\lambda_{max} = 1$ of the parameter λ, and this time the solution $\psi_{(-1/2)}$ becomes infinite, not at an isolated point but along two curves in the τ, β plane which satisfy the second equations in (15.9).

Further, by comparing (15.9) with (14.3), it is found that these curves of singularity are nothing less than the two characteristic curves passing through the point $\tau = 1$, $\beta = 0$.

Thus, summarizing the above results we see that when the parameter λ assumes a value in the range $0 < \lambda < 1$, the singularity of the solution $\psi_{(-1/2)}$ in the hodograph plane (i.e., the τ, β plane) occurs at an isolated point on the axis $\beta = 0$, but in the limit $\lambda \to 1$, the singularity is prolonged along the two characteristics passing through the point $\tau = 1$, $\beta = 0$. This remarkable characteristic change of the singularity of the solution occurring in the hodograph plane at the stage of transition from the subsonic to the supersonic region is quite similar to what has been already found in Part II in the case of the more simple fundamental equation (7.6) of the mixed type.

Lastly, we shall derive from the preceding solution $\psi_{(-1/2)}$ another solution which will have a branch-point of order 1/2 at a certain point in the hodograph plane.

Now, in general, the form of our fundamental equation (14.2) suggests that any new solution can be obtained by differentiating or integrating one of the known solutions with respect to β, and as inferred from what has been pointed out in Part II, the order of singularity of the solution thus derived would differ by unity from the order of the original solution. Hence, the required solution, denoted by $\psi_{(1/2)}$, can be derived from the preceding solution $\psi_{(-1/2)}$ given by (15.3) by integrating it with respect to β, and we have

$$\psi_{(1/2)} = \int \frac{d\beta}{1 - \tau\cos\omega} = \int \frac{1}{1 - \tau\cos\omega}\frac{d\omega}{d\omega/d\beta} = \int d\omega = \omega,$$

where use has been made of the relation $d\omega/d\beta = (1 - \tau\cos\omega)^{-1}$ which can easily be obtained from (15.4). It is thus found that the variable ω itself is a solution having the singularity of order 1/2.

On the other hand, the variable β itself also becomes evidently a solution of the fundamental equation (14.2). Therefore, it will be seen from equation (15.4) that the function $\tau \sin \omega$ becomes also a solution having the singularity of order 1/2. Thus, denoting this solution by $\psi_{(1/2)}$ as before, we have

$$\psi_{(1/2)} = \tau \sin \omega. \tag{15.10}$$

It can easily be proved that the singularity of these solutions $\psi_{(1/2)}$ show also the characteristic change as mentioned above, just as in the case of the preceding solution $\psi_{(-1/2)}$.

As will be shown in later lines, we can discuss a uniform flow past an obstacle by making use of an appropriate linear combination of the above solutions $\psi_{(-1/2)}$ and $\psi_{(1/2)}$.

16. An alternative method of solving the fundamental equation (14.2). In the analysis developed in §14, we have used the integral representation of Bessel's type for the Bessel function $J_n(n\tau)$ in (14.6). However, if, instead of employing the integral representation of Bessel's type, we make use of the integral representation of Poisson's type for $J_n(n\tau)$, we can develop an alternative analysis similar to the preceding one, which enables us to calculate conveniently the limiting case of the incompressible fluid flow.

Now, according to Poisson, the Bessel function $J_n(n\tau)$ can be expressed in the form:[5]

$$J_n(n\tau) = \frac{(n\tau/2)^n}{\Gamma(n + 1/2)\Gamma(1/2)} \int_0^\pi \exp\{in\tau \cos\theta\} \sin^{2n}\theta \, d\theta.$$

If we insert this expression in the right-hand side of (14.6), we have

$$\psi = \int_0^\pi \sum_{n=0}^{\infty} B_n\left(\frac{\tau}{\mu} \exp\{-i\beta\} \exp\{i\tau \cos\theta\} \sin^2\theta\right)^n d\theta,$$

where B_n's and μ are arbitrary constants.[6]

Thus, summing up the integrand into the form of an arbitrary function $G(e^{\zeta})$, as done in the previous case, we get the general solution of our fundamental equation (14.2) in the form:

$$\psi = \int_0^\pi G(e^{\zeta})\, d\theta,$$

with (16.1)

$$\zeta(\theta) = \log\frac{\tau}{\mu} - i\beta + i\tau\cos\theta + 2\log\sin\theta,$$

where the path of integration from $\theta = 0$ to $\theta = \pi$ should be taken suitably in conformity with remarks which will be given presently in the next section.

We now consider a limiting case in which

$$\left.\begin{array}{l} \tau \to 0, \\ \mu \to 0, \end{array}\right\} \frac{\tau}{\mu} \to \xi(\text{finite}). \tag{16.2}$$

Such a limiting case corresponds evidently to the case of the incompressible fluid flow, and therefore, if we denote the corresponding limiting value of ψ by ψ_{inc}, we have

$$\psi_{\text{inc.}} = 2\int_0^{\pi/2} G(W\sin^2\theta)\, d\theta,$$

with (16.3)

$$W = \xi e^{-i\beta}.$$

[5]G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, 1922, p. 48.

[6]Like the previous constant λ, this constant μ is a parameter determining the Mach number of the flow.

We assume that the stream function $\psi_{\text{inc.}}$ for a known incompressible fluid flow is given by

$$\psi_{\text{inc.}} = g(W). \tag{16.4}$$

Then, substituting this in the left-hand side of (16.3), we have an integral equation for determining the arbitrary function G in the form:

$$g(W) = 2\int_0^{\pi/2} G(W\sin^2\theta)\,d\theta. \tag{16.5}$$

If we put

$$W\sin^2\theta = \chi,$$

this equation reduces to the form:

$$g(W) = \int_0^W \frac{1}{\sqrt{W-\chi}}\frac{G(\chi)}{\sqrt{\chi}}\,d\chi. \tag{16.6}$$

This is nothing but an integral equation of Abel's type and the solution is obtained by making use of the well-known formula as:

$$\begin{aligned} G(\chi) &= \frac{1}{\pi}\sqrt{\chi}\frac{\partial}{\partial\chi}\int_0^\chi \frac{g(W)}{\sqrt{\chi-W}}\,dW \\ &= \frac{1}{\pi}\left\{g(0) + \sqrt{\chi}\int_0^\chi \frac{g'(W)}{\sqrt{\chi-W}}\,dW\right\}. \end{aligned} \tag{16.7}$$

Thus, if we put this expression for G into the integrand in the formula (16.1), we shall obtain the stream function for the corresponding flow of a compressible fluid. In the following lines, we shall give a few examples.

(a) As a first example, we consider the case in which

$$\psi_{\text{inc.}} = g(W) = (1-W)^{-1/2}.$$

Putting this in the above formula (16.7), we readily have

$$G(\chi) = \frac{1}{\pi}\frac{1}{1-\chi}.$$

Thus, inserting this expression for G in (16.1), we have ultimately

$$\psi = \frac{1}{\pi}\int_0^\pi \frac{d\theta}{1-e^\zeta},$$

with (16.8)

$$\zeta(\theta) = \log\frac{\tau}{\mu} - i\beta + i\tau\cos\theta + 2\log\sin\theta.$$

(b) As a second example, we next consider the case in which

$$\psi_{\text{inc.}} = g(W) = \log(1-W).$$

Substituting this expression for $g(W)$ into (16.7), we get

$$G(\chi) = -\frac{2}{\pi}\left(\frac{1}{\chi}-1\right)^{-1/2}\tan^{-1}\left(\frac{1}{\chi}-1\right)^{-1/2}.$$

Therefore, by (16.1), we obtain a solution of the form:

$$\psi = -\frac{2}{\pi}\int_0^{\pi} (e^{-\zeta} - 1)^{-1/2} \tan^{-1} (e^{-\zeta} - 1)^{-1/2}\, d\theta,$$

with (16.9)

$$\zeta(\theta) = \log\frac{\tau}{\mu} - i\beta + i\tau\cos\theta + 2\log\sin\theta.$$

17. Remarks on the path of integration for the integral (16.1). From the above examples it will be seen that when any solution for the incompressible fluid flow reveals singularity at the point $W = 1$, the integrand of the corresponding solution of the integral form (16.1) for the compressible fluid flow has in general a singular point at $\theta = \theta_0$, where θ_0 is a root of the equation:

$$\zeta(\theta) = \log\frac{\tau}{\mu} - i\beta + i\tau\cos\theta + 2\log\sin\theta = 0, \tag{17.1}$$

and the position of such a singular point $\theta = \theta_0$ in the θ-plane varies with the values of τ and β.

Now, it is well known that in case a singular point of the integrand of any function defined by a definite integral moves across the path of integration with the variation of a variable, the said function loses in general its analytic continuity with regard to the variable. Therefore, in order that the solution (16.1) be capable of maintaining its analytic continuity, it is necessary that the path of integration should be such a curve connecting the two points $\theta = 0$ and $\theta = \pi$ which can be deformed, with the variation of τ and β, without being cut across by any singular point of the integrand (Fig. 23).

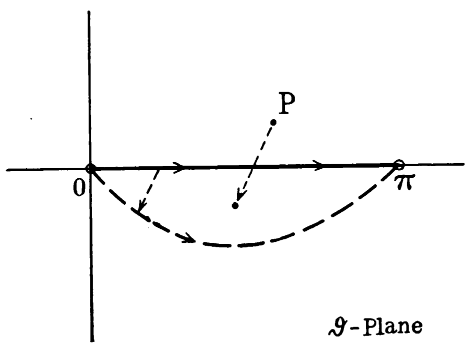

Fig. 23.

18. The singular point of the solution (16.1). Next, we shall discuss the singular point of the solution ψ given by (16.1), which corresponds to the singular point $W = 1$ of the stream function $\psi_{\text{inc.}}$ for the incompressible fluid flow.

Now, it is found that Eq. (17.1) has in general two roots, and the corresponding two singular points of the integrand of ψ are usually situated in such a way that the one, denoted by P, lies on one side of the path of integration, while the other, denoted by Q, on the opposite side (Fig. 24). However, for a particular set of values of τ, β, which will be denoted here by τ_∞, β_∞ as before, the confluence of these two singular points may occur. In this particular case, then, the path of integration must pass through the confluent singular points, and the solution ψ has a singularity.

Thus, it is found that the singular point $(\tau_\infty, \beta_\infty)$ of the solution (16.1) is determined by the two equations $\zeta = 0$ and $d\zeta/d\theta = 0$, namely:

$$\begin{aligned} \log \frac{\tau}{\mu} - i\beta + i\tau \cos \theta + 2 \log \sin \theta = 0, \\ - i\tau \sin \theta + 2 \cot \theta = 0. \end{aligned} \tag{18.1}$$

Eliminating θ from these equations, we obtain the equation for determining τ_∞, β_∞ in the form:

$$\beta_\infty - i \mp \sqrt{\tau_\infty^2 - 1} + i \log \left\{ \frac{2}{\mu\tau_\infty} (1 \mp i\sqrt{\tau_\infty^2 - 1}) \right\} = 0, \tag{18.2}$$

and this equation corresponds to Eq. (15.6) in the preceding analysis.

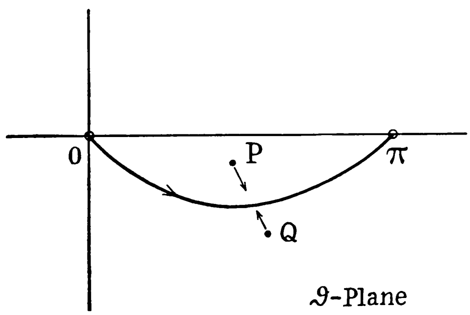

FIG. 24.

After some calculations it can be shown however that this equation becomes in accord with the preceding equation (15.6), if we take the value of the parameter μ equal to $2e^{-1}\lambda$. Therefore, further development of the analysis can be made along the same lines as in the case of Eq. (15.6), and we thus arrive at the conclusion that the singularity of the present solution (16.1) has also the same characteristic features as those of the preceding solution $\psi_{(-1/2)}$ which have been described in detail in §15.

19. The solution giving the uniform flow past an obstacle. In the following lines, we shall discuss the flow past an obstacle by making use of the two fundamental solutions $\psi_{(-1/2)}$ and $\psi_{(1/2)}$ obtained in §15, namely:

$$\psi_{(-1/2)} = \frac{1}{1 - \tau \cos \omega}, \tag{19.1}$$

$$\psi_{(1/2)} = \tau \sin \omega, \tag{19.2}$$

with

$$\beta - i \log \lambda + \tau \sin \omega - \omega = 0. \tag{19.3}$$

We put

$$\omega = r + is.$$

Then, inserting this in the left-hand side of (19.3) and separating the real and imaginary parts, we obtain the relations between τ, β and r, s in the forms:

$$\tau = \frac{s + \log \lambda}{\cos r \sinh s}, \qquad \beta = r - (s + \log \lambda) \tan r \coth s. \tag{19.4}$$

In the first place, we shall examine a solution as expressed by the imaginary part of $\psi_{(-1/2)}$ given by (19.1), namely:

$$\psi = I\{\psi_{(-1/2)}\} = \frac{-(s + \log \lambda) \tan r}{\{1 - (s + \log \lambda) \coth s\}^2 + (s + \log \lambda)^2 \tan^2 r}. \tag{19.5}$$

Putting the denominator equal to zero, we have the equations for determining the coordinates τ_∞, β_∞ of the singular point of ψ in the forms:

$$1 - (s_\infty + \log \lambda) \coth s_\infty = 0,$$

$$(s_\infty + \log \lambda) \tan r_\infty = 0,$$

or

$$\tanh s_\infty = s_\infty + \log \lambda,$$

$$r_\infty = 0.$$

Thus, taking (19.4) into account, we obtain the result that

$$\tau_\infty = \operatorname{sech} s_\infty, \qquad \beta_\infty = 0,$$

and therefore it is seen that the solution ψ under consideration has a singularity at the point $\tau = \tau_\infty = \operatorname{sech} s_\infty$, $\beta = \beta_\infty = 0$.

Also, it will be shown without difficulty that ψ is always equal to zero along the axis $\beta = 0$ from $\tau = 0$ to $\tau = \tau_\infty$.

Along any streamline $\psi = \text{const.}$, we can in general express r as a function of s, with the aid of (19.5). Therefore, if we substitute the function $r(s)$ so obtained into Eqs. (19.4), we can obtain the relation between τ and β with s as the parameter. In other words, we can thus obtain the flow pattern in the τ, β plane, which is shown in Fig. 25.

In the next place, we shall consider a solution which is obtained by taking the real part of $\psi_{(1/2)}$ as given by (19.2), namely:

$$\psi = R\{\psi_{(1/2)}\} = \tau \sin r \cosh s. \tag{19.6}$$

It can easily be shown that this solution possesses a branch-point of order 1/2 at the same point ($\tau = \tau_\infty = \operatorname{sech} s_\infty$, $\beta = \beta_\infty = 0$) as the preceding solution (19.5), and that ψ becomes always equal to zero along the axis $\beta = 0$ from $\tau = 0$ to $\tau = \tau_\infty$. In this case, the flow pattern in the τ, β plane becomes as shown in Fig. 26.

From these figures it is naturally expected that if we superpose the above two solutions (19.5) and (19.6) appropriately, we can obtain a solution which would represent the flow pattern in the hodograph plane (i.e., the τ, β plane) as shown in Fig. 27(a), and such a solution would give a required field of flow past an obstacle in the physical plane as shown in Fig. 27(b), as would be expected from the physical meaning of the hodograph plane.

The superposed solution is expressed in the following form:

$$\psi = I\left\{iK\tau \sin\omega + \frac{1}{1 - \tau\cos\omega}\right\},$$

with (19.7)

$$\beta - i\log\lambda + \tau\sin\omega - \omega = 0,$$

where K is a certain constant to be determined appropriately.

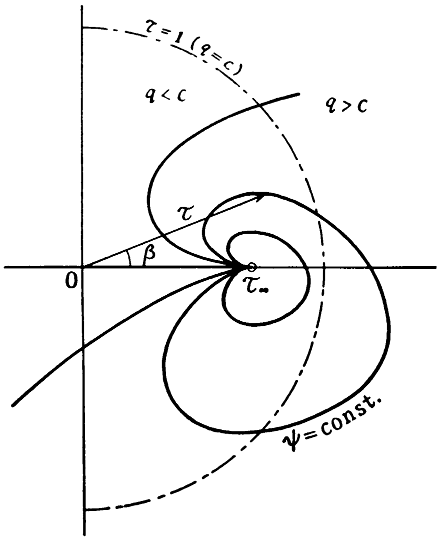

FIG. 25. $\psi_{(-1/2)}$

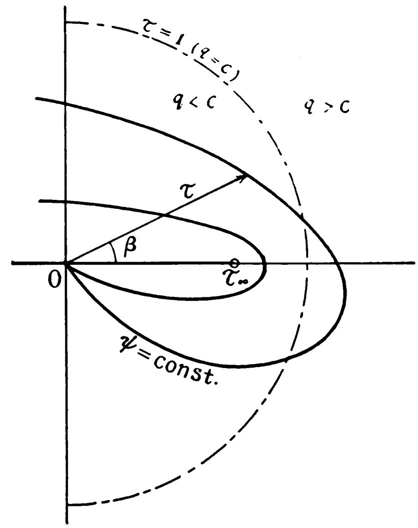

FIG. 26. $\psi_{(1/2)}$

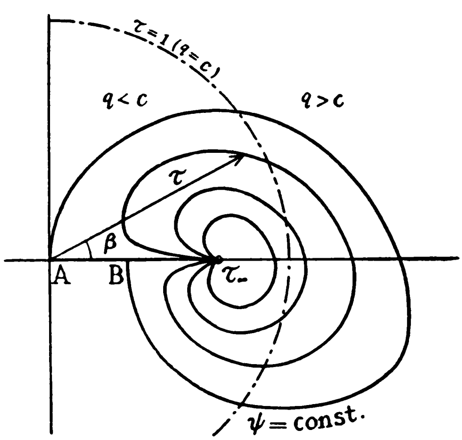

FIG. 27. (a) Hodograph plane.

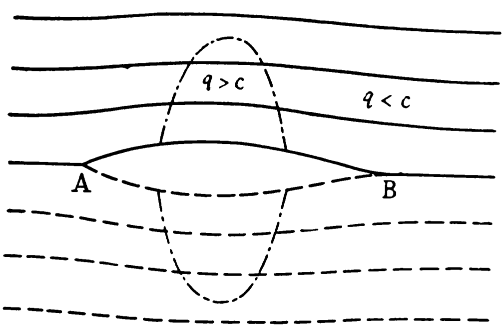

FIG. 27. (b) Physical plane.

If we determine the value of K in such a way that the leading edge of the obstacle coincides with the point $\tau = 0$ in the hodograph plane, we have

$$K = \frac{1 - \lambda^2}{1 + \lambda^2}\cdot \qquad (19.8)$$

In this case, it is easily found that the leading edge of the body becomes a stagnation point and that the upper and lower surfaces of the body meet at a finite angle $(2\sqrt{a}/\kappa)\pi$ there.

If, in the fundamental equations (13.1), we change the independent variables from w, θ to τ, β by the aid of (13.5) and (14.1), we have

$$\begin{aligned} \varphi_\tau &= \sqrt{a}\,(\tau - 1/\tau)\psi_\beta\,, \\ \varphi_\beta &= \sqrt{a}\;\tau\psi_\tau\,. \end{aligned} \tag{19.9}$$

Thus, putting the expression (19.7) for ψ into the right-hand sides of these equations and carrying out simple integrations, we obtain the corresponding velocity potential φ in the form:

$$\varphi = \sqrt{a}\tau I\left\{-iK\left(\frac{\tau}{2} + \cos\omega\right) + \frac{\sin\omega}{1 - \tau\cos\omega}\right\}. \tag{19.10}$$

We shall next consider the transformation equations from the τ, β plane to the physical x, y plane. In general, we have

$$\begin{aligned} dx &= x_\varphi\, d\varphi + x_\psi\, d\psi, \\ dy &= y_\varphi\, d\varphi + y_\psi\, d\psi. \end{aligned}$$

Inserting in the right-hand sides of the well-known expressions for x_φ, y_φ, x_ψ, y_ψ as given in Part I, namely:

$$\begin{aligned} x_\varphi &= \frac{1}{q}\cos\theta, \qquad & x_\psi &= -\frac{1}{\rho q}\sin\theta, \\ y_\varphi &= \frac{1}{q}\sin\theta, \qquad & y_\psi &= \frac{1}{\rho q}\cos\theta, \end{aligned}$$

and carrying out integrations, we obtain the coordinates x, y on any streamline $\psi =$ const. $\equiv \psi_1$ in the physical plane in the following forms:

$$\begin{aligned} x &= \int_{\varphi_0}^{\varphi}\left(\frac{1}{q}\cos\theta\right)_{\psi=\psi_1} d\varphi - \int_0^{\psi_1}\left(\frac{1}{\rho q}\sin\theta\right)_{\varphi=\varphi_0} d\psi, \\ y &= \int_{\varphi_0}^{\varphi}\left(\frac{1}{q}\sin\theta\right)_{\psi=\psi_1} d\varphi + \int_0^{\psi_1}\left(\frac{1}{\rho q}\cos\theta\right)_{\varphi=\varphi_0} d\psi, \end{aligned} \tag{19.11}$$

where the point $\varphi = \varphi_0$, $\psi = 0$ has been adjusted so as to correspond to the origin of the x, y plane.

In particular, the coordinates on a particular streamline $\psi = 0$, a part of which coincides with the surface of the body, are calculated by the following formulas:

$$\begin{aligned} x &= \int_{\varphi_0}^{\varphi}\left(\frac{1}{q}\cos\theta\right)_{\psi=0} d\varphi, \\ y &= \int_{\varphi_0}^{\varphi}\left(\frac{1}{q}\sin\theta\right)_{\psi=0} d\varphi. \end{aligned} \tag{19.12}$$

Since Eqs. (19.7) and (19.10) give the relations between φ, ψ and τ, β (and consequently, between φ, ψ and q, θ), all the integrands in the formulas (19.11) and (19.12) can be expressed as functions of φ, ψ. Therefore, carrying out integrations, we can obtain the coordinates (x, y) on each of various streamlines and the flow pattern in the physical plane can thus be found.

20. Numerical computations. By assuming $\gamma = 1.4$ for air, detailed numerical computations have been carried out for three cases in which the Mach number M of the

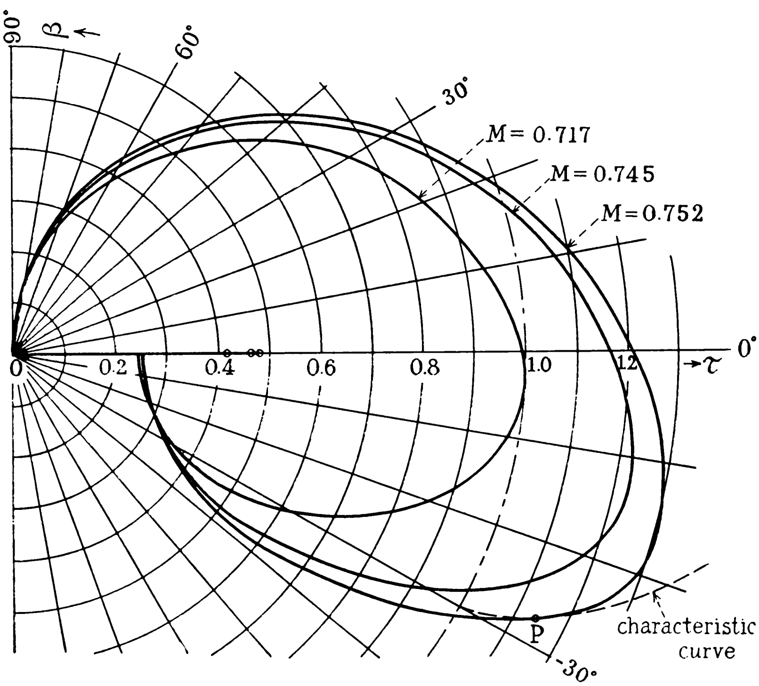

Fig. 28.

undisturbed uniform flow is equal to 0.717, 0.745 and 0.752 respectively, paying special attention to the state of affairs on the surface of the body, the corresponding values of the parameter λ being 0.542, 0.600 and 0.616 respectively. Here, $M = 0.717$ is the so-called critical Mach number at which the maximum local Mach number in the field

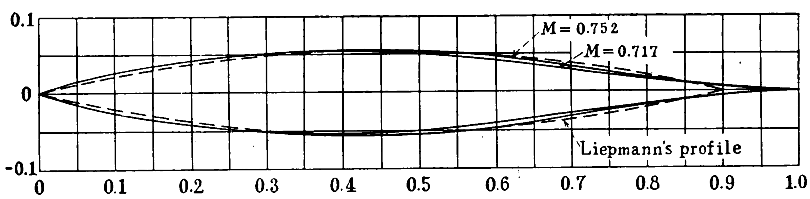

Fig. 29.

of flow becomes equal to unity. Fig. 28 shows the streamlines $\psi = 0$ in the τ, β plane for these three cases.

The coordinates (x, y) of the point on the surface of the obstacle in the physical plane have been calculated by the formulas (19.12). Denoting the chord length of the body by l, the values of x/l and y/l are shown in Table II. The profiles of the obstacles for the two cases in which $M = 0.717$ and $M = 0.752$ respectively are shown in Fig. 29. The profile for the case $M = 0.745$ has been omitted, however.

It will be seen that there is a very satisfactory coincidence, especially in the forward part as well as in the vicinity of the trailing edge, between the profiles for the three cases, and therefore it may be assumed that the shape of the obstacle is fixed in spite of the variation of the Mach number M of the undisturbed flow.

Values of the fluid velocity q on the surface of the obstacle have been calculated

TABLE II

$M = 0.717$ ($\lambda = 0.542$)			$M = 0.745$ ($\lambda = 0.600$)			$M = 0.752$ ($\lambda = 0.616$)		
x/l	y/l	q	x/l	y/l	q	x/l	y/l	q
0	0	0	0	0	0	0	0	0
0.0299	0.0104	0.686	0.0323	0.0106	0.706	0.0338	0.0111	0.709
0.0874	0.0236	0.793	0.0943	0.0246	0.821	0.0669	0.0191	0.780
0.1399	0.0330	0.854	0.1506	0.0346	0.881	0.1274	0.0310	0.862
0.1897	0.0402	0.898	0.2040	0.0420	0.930	0.1837	0.0396	0.917
0.2376	0.0455	0.927	0.2553	0.0475	0.967	0.2375	0.0460	0.961
0.2843	0.0493	0.951	0.3051	0.0514	0.997	0.2894	0.0506	0.994
0.3299	0.0518	0.972	0.3535	0.0539	1.023	0.3397	0.0537	1.022
0.3747	0.0531	0.989	0.4008	0.0552	1.043	0.3888	0.0556	1.048
0.4000	0.0533	0.997	0.4264	0.0553	1.053	0.4367	0.0561	1.069
0.4190	0.0532	0.999	0.4472	0.0552	1.062	0.4753	0.0555	1.083
0.4632	0.0520	0.988	0.4930	0.0541	1.077	0.5217	0.0538	1.097
0.5086	0.0493	0.947	0.5385	0.0517	1.075	0.5681	0.0506	1.067
0.5563	0.0450	0.896	0.5621	0.0496	0.986	0.5730	0.0501	1.000
0.6065	0.0395	0.853	0.6142	0.0432	0.893	0.5940	0.0475	0.938
0.6590	0.0333	0.817	0.6704	0.0357	0.841	0.6502	0.0397	0.864
0.7136	0.0267	0.785	0.7298	0.0275	0.800	0.7103	0.0311	0.815
0.7704	0.0199	0.757	0.7919	0.0192	0.765	0.7736	0.0223	0.777
0.8293	0.0132	0.729	0.8569	0.0113	0.733	0.8401	0.0137	0.742
0.8904	0.0070	0.704	0.9246	0.0044	0.703	0.9098	0.0060	0.709
0.9536	0.0020	0.682	0.9954	0.0001	0.678	0.9827	0.0005	0.681
0.9861	0.0004	0.672	1.0000	0	0.675	1.0000	0	0.676
1.0000	0	0.667						

and they are given in Table II, and the velocity distributions on the surface of the body are shown in Fig. 30.

It will be seen clearly from Fig. 28 that when $M = 0.752$, the streamline $\psi = 0$ becomes in contact with one of the characteristic curves of the fundamental equation (14.2) at some point P in the τ, β plane. In other words, the singularity $J = \partial(x, y)/\partial(q, \theta) = 0$ makes its first appearance at this Mach number at some point in the field of flow in the hodograph plane, and the velocity gradient becomes infinite at the corresponding singular point P on the surface of the body as shown in Fig. 30. We shall denote by M_s the Mach number at which infinite velocity gradient occurs on the surface of the body and a trace of the so-called shock line (i.e., the singular point $J = 0$) first appears, and call it provisionally "the shock Mach number" for the sake of convenience. Thus, for our obstacle we have $M_s = 0.752$.

From Fig. 30 it will readily be found that the curve of velocity distribution does not reveal any peculiarity even at the critical Mach number $M = 0.717$. But, it becomes rapidly steeper in the vicinity of the point P as the Mach number increases until, at the above-mentioned shock Mach number, it has an infinite gradient at the point P.

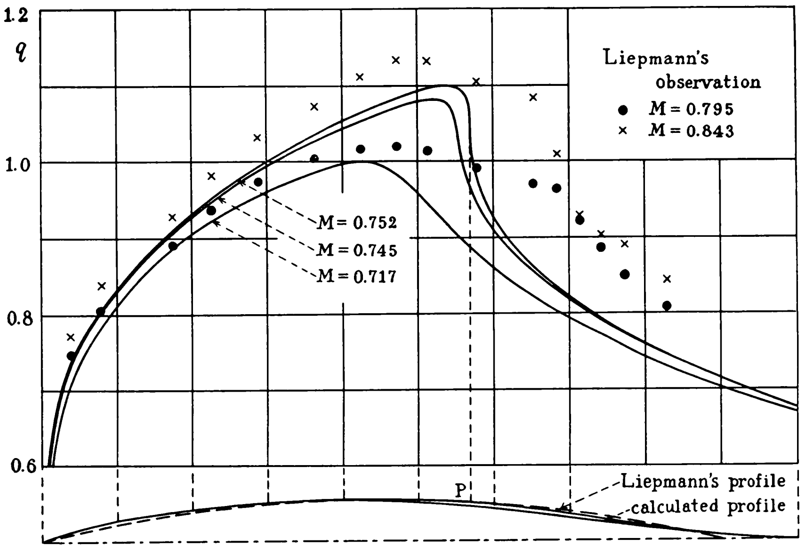

Fig. 30.

For still higher Mach number, two curves of singularity $J = 0$ grow up from the surface of the body. Then, the theoretical field of flow becomes many-valued in the neighbourhood of such curves. At this stage, shock waves must form in the actual field of flow so as to avoid the appearance of the many-valued region which is expected theoretically.

21. A comparison with observation. Quite recently, Hans W. Liepmann[7] has measured the pressure distributions on the surface of a biconvex circular arc profile placed in a high-speed wind tunnel. The dotted-line curve in Fig. 29 shows the biconvex profile of thickness ratio of 0.12 used by Liepmann in his observations, by adjusting it, for the sake of comparison, so as to have the same position of the leading edge as well as the same maximum thickness with those of our profiles derived theoretically, which have been shown by full-line curves.

From Liepmann's observed pressure distributions, we have calculated, by the use of Bernoulli's theorem together with the isentropic law, the velocity distributions on the surface of his obstacle in two cases in which $M = 0.795$ and $M = 0.843$ respectively. The observed values of the velocity q thus found are shown in Fig. 30 by small black circles and crosses respectively. It may be remarked here that at the former Mach number $M = 0.795$, the first appearance of shock waves was observed. Taking account

[7]Hans W. Liepmann, *The interaction between boundary layer and shock waves in transonic flow*, J. Aero. Sci., **13**, 623–637 (1946).

of the discrepancy between the calculated profile and the biconvex profile used in Liepmann's observations, it may be said that the agreement between the theory and observation is satisfactory.

It seems worth noticing here that (a) the observed position of the main shock wave as appeared at an earlier stage falls within the many-valued region derived theoretically in the above and that (b) the main shock wave inclined obliquely forwards as observed by Liepmann, which is enveloping Mach waves starting from the inside of the field of flow but not from the surface of the body, should be just compared with the curve of singularity $J = 0$ found theoretically, which is as well an envelope of one family of Mach waves starting from the inside of the field of flow.

Now, as will be seen from Fig. 30, the theoretical field of flow for the case in which $M = 0.745$, for example, is evidently partially supersonic in limited regions, but is still capable of being continuous and irrotational throughout the whole field of flow. Hence, we arrive at the affirmative positive answer to the so-called Taylor's problem enquiring about whether there is any theoretical possibility of the existence of a continuous irrotational flow of a compressible fluid past an obstacle such that it flows uniformly at a great distance from the body and at the same time contains limited supersonic regions in the neighbourhood of the obstacle; namely, the theoretical results of our analysis show that when a body is placed in a uniform stream of a compressible fluid moving at speeds less than that of sound, the local speed of flow can exceed that of sound in some limited regions in the vicinity of the body without violating the irrotationality as well as the analytical continuity of the flow.

In this connection, it may be emphasized that the solution used here of our fundamental equation (14.2) is expressed in a closed form but not in a form of infinite series and therefore it is quite free from the question of convergence.

Lastly, it may be added that the theoretical results obtained in the above for the flow of our hypothetical gas seem to be still valid, not only qualitatively but also quantitatively, for the flow of the real gas subject to the exact isentropic law, because, in the flow treated above, the maximum speed of flow exceeds the local speed of sound by only about 10 per cent even at the so-called shock Mach number and consequently, our second hypothetical gas as employed in the present Part III can approximate very satisfactorily the real gas obeying the isentropic law throughout the whole field of flow, as is seen clearly from Fig. 21.