$$\operatorname{sn}([1+k_1']z_1;\bar{k}_1) = (1+k_1')\frac{\operatorname{sn}(z_1;k_1)\operatorname{cn}(z_1;k_1)}{\operatorname{dn}(z_1;k_1)},$$
$$\operatorname{cn}([1+k_1']z_1;\bar{k}_1) = \frac{1-(1+k_1')\operatorname{sn}^2(z_1;k_1)}{\operatorname{dn}(z_1;k_1)}, \tag{22}$$
$$\operatorname{dn}([1+k_1']z_1;\bar{k}_1) = \frac{1-(1-k_1')\operatorname{sn}^2(z_1;k_1)}{\operatorname{dn}(z_1;k_1)},$$

then equation (20) may be written

$$(\zeta_4')^2 = \frac{\operatorname{cn}(z_1;k_1)+i(k_1')^{1/2}\operatorname{sn}(z_1;k_1)}{\operatorname{cn}(z_1;k_1)-i(k_1')^{1/2}\operatorname{sn}(z_1;k_1)}, \tag{23}$$

where z_1 is given as in (1).

We observe that equation (23) is indeed the mapping function for a cross of equal lengths, for it is a special case of equation (17), namely $\epsilon = 1$.

The cross of equal lengths can easily be mapped into the cross of unequal lengths in Fig. 2. We apply a linear fractional transformation which preserves the unit circle in the two-sheeted plane. If the transformation

$$\zeta_4^2 = \frac{(\zeta_4')^2 - (\epsilon-1)/(\epsilon+1)}{1-[(\epsilon-1)/(\epsilon+1)](\zeta_4')^2} \tag{24}$$

is combined with equation (23), then equation (17) is the result.

References

[1] E. T. Whittaker and G. N. Watson, *Modern Analysis*, Cambridge Univ. Press, London, 1927, ch. 22.
[2] R. Fricke, *Die elliptische Funktionen*, vol. 2, Leipzig, 1922, p. 293.
[3] G. Holzmüller, *Einführung in die Theorie der isogonalen Verwandtschaften und der conformen Abbildung*, Teubner, Leipzig, 1882, pp. 256-258.
[4] H. J. Stewart, *The lift of a delta wing at supersonic speeds*, Quart. Appl. Math. 4, 246-254 (1946).
[5] J. Kronsbein, *Analytical expressions for some extremal schlicht functions*, J. Lond. Math. Soc., **17**, 152-157 (1942).
[6] Whittaker and Watson, loc. cit., pp. 507-508.

EFFECT OF A RIGID ELLIPTIC DISK ON THE STRESS DISTRIBUTION IN AN ORTHOTROPIC PLATE*

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A thin orthotropic plate of uniform thickness will possess two perpendicular axes of symmetry in the plane of the plate. An infinite rectangular plate of this type containing a rigid elliptic disk with major and minor axes coinciding with the axes of symmetry is discussed. A uniform tension is assumed to act along two opposite edges of the plate and a mathematical analysis of the stress distribution is given. It is assumed the

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strains are small and remain within the limits of perfect elasticity. The solution obtained is applied to a plain-sawn Sitka spruce plate.

Choose as the origin of the coordinate system the center of the ellipse. The boundary conditions may be stated mathematically using the notation of A. E. H. Love[1] as

$$\left.\begin{array}{ll} X_x \mid_{x\to\pm\infty} = S, & Y_y \mid_{y\to\pm\infty} = 0, \\ X_y \mid_{x\to\pm\infty} = 0, & X_y \mid_{y\to\pm\infty} = 0, \end{array}\right\} \tag{1}$$

and $u(x, y) = v(x, y) = 0$ on the circumference of the disk given by the parametric equations $x = a \cos \theta$, $y = b \sin \theta$. The displacements in the x and y directions are $u(x, y)$ and $v(x, y)$, respectively. S is the uniform tension applied at two edges of the plate.

The components of stress and strain are connected by the following relations if the x and y axes are taken as the axes of elastic symmetry of the orthotropic plate:[2]

$$e_{xx} = \frac{\partial u}{\partial x} = \frac{1}{E_x} X_x - \frac{\sigma_{yx}}{E_y} Y_y , \tag{2}$$

$$e_{yy} = \frac{\partial v}{\partial y} = -\frac{\sigma_{xy}}{E_x} X_x + \frac{1}{E_y} Y_y , \tag{3}$$

$$e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{\mu_{xy}} X_y . \tag{4}$$

It is desirable to find a stress function $F(x, y)$[3] such that

$$X_x = \frac{\partial^2 F}{\partial y^2}, \qquad Y_y = \frac{\partial^2 F}{\partial x^2}, \qquad \text{and} \qquad X_y = -\frac{\partial^2 F}{\partial x\, \partial y}. \tag{5}$$

For the problem of a thin orthotropic plate in a state of plane stress F must satisfy the differential equation[4]

$$\frac{\partial^4 F}{\partial x^4} + 2K \frac{\partial^4 F}{\partial x^2\, \partial \eta^2} + \frac{\partial^4 F}{\partial \eta^4} = 0 \tag{6}$$

where

$$K = \tfrac{1}{2}(E_x E_y)^{1/2} \left(\frac{1}{\mu_{xy}} - \frac{2\sigma_{xy}}{E_x}\right), \tag{7}$$

$$\eta = \epsilon y, \qquad \text{and} \qquad \epsilon = (E_x/E_y)^{1/4}. \tag{8}$$

A suitable stress function is

$$F = R\left\{\frac{A}{2\gamma_1^2}\left[\frac{1}{2}(Z_1 - W_1)^2 + \gamma_1^2 \ln (Z_1 + W_1)\right] + \frac{B}{2\gamma_2^2}\left[\frac{1}{2}(Z_2 - W_2)^2 + \gamma_2^2 \ln (Z_2 + W_2)\right]\right\} + \frac{S\eta^2}{2\epsilon^2} \tag{9}$$

[1]A. E. H. Love, *A treatise on the mathematical theory of elasticity,* Dover Publications, New York, 1944.

[2]H. W. March, *Stress-strain relations in wood and plywood considered as orthotropic materials,* Forest Products Laboratory Rep. No. R1503, p. 2 (1944).

where the symbol $R\{\quad\}$ denotes the real part,

$$A = A_1 + iA_2 , \qquad B = B_1 + iB_2 , \tag{10}$$

$$Z_1 = x + i\alpha\eta, \qquad Z_2 = x + i\beta\eta, \qquad \eta = \epsilon y, \tag{11}$$

$$W_1 = (Z_1^2 - \gamma_1^2)^{1/2}, \qquad W_2 = (Z_2^2 - \gamma_2^2)^{1/2}, \tag{12}$$

$$\gamma_1^2 = a^2 - \alpha^2\epsilon^2b^2, \qquad \gamma_2^2 = a^2 - \beta^2\epsilon^2b^2, \tag{13}$$

$$\alpha = \{K + (K^2 - 1)^{1/2}\}^{1/2},$$

and

$$\beta = \{K - (K^2 - 1)^{1/2}\}^{1/2}.$$

Using the transformation

$$K = \cosh \phi,$$

$$\alpha = e^{\phi/2} \quad \text{and} \quad \beta = e^{-\phi/2}. \tag{14}$$

In order that the stresses may be single valued W_1 and W_2 are to be assigned values so that the inequalities

$$| Z_1 + W_1 | \geqq \gamma_1 \quad \text{and} \quad | Z_2 + W_2 | \geqq \gamma_2 \tag{15}$$

are satisfied.

It follows that

$$Y_y = \frac{\partial^2 F}{\partial x^2} = R\left\{\frac{-A}{W_1(Z_1 + W_1)} + \frac{-B}{W_2(Z_2 + W_2)}\right\}, \tag{16}$$

$$X_x = \epsilon^2 \frac{\partial^2 F}{\partial \eta^2} = R\left\{\frac{\alpha^2\epsilon^2 A}{W_1(Z_1 + W_1)} + \frac{\beta^2\epsilon^2 B}{W_2(Z_2 + W_2)}\right\} + S, \tag{17}$$

and

$$X_y = -\epsilon \frac{\partial^2 F}{\partial x\, \partial \eta} = R\left\{\frac{i\alpha\epsilon A}{W_1(Z_1 + W_1)} + \frac{i\beta\epsilon B}{W_2(Z_2 + W_2)}\right\}. \tag{18}$$

It can be shown that the exterior boundary conditions are satisfied so long as A and B are finite.

Substitution of the stresses given by (16) and (17) into (2) and integration gives

$$u = R\left\{A\left[\frac{\alpha^2\epsilon^2}{E_x} + \frac{\sigma_{yx}}{E_y}\right]\left[\frac{-1}{Z_1 + W_1}\right] + B\left[\frac{\beta^2\epsilon^2}{E_x} + \frac{\sigma_{xy}}{E_y}\right]\left[\frac{-1}{Z_2 + W_2}\right]\right\} + \frac{Sx}{E_x} + u_0(y)$$

where $u_0(y)$ is an arbitrary function which can be shown to be identically zero. A similar expression for v may be found by substituting the stresses in (3) and integrating. From the conditions $u = v = 0$ on the boundary of the disk four equations are obtained which uniquely determine A_1 , A_2 , B_1 and B_2 .

[3]G. B. Airy, Rep. Brit. Assoc. Adv. Sci., 1862.
[4]C. Bassel Smith, Q. App. Math. 6, 452-456 (1949).

It follows that

$$A_2 = B_2 = 0;$$

$$A_1 = \frac{(a + \alpha\epsilon b)[Sa\alpha(1 + \epsilon^2\beta^2\sigma_{yx}) + Sb\epsilon\sigma_{yx}(\epsilon^2\beta^2 + \sigma_{xy})]}{\alpha(\epsilon^2\alpha^2 + \sigma_{xy})(1 + \epsilon^2\beta^2\sigma_{yx}) - \beta(1 + \epsilon^2\alpha^2\sigma_{yx})(\epsilon^2\beta^2 + \sigma_{xy})}; \tag{19}$$

and

$$B_1 = -\frac{(a + \beta\epsilon b)[Sa\beta(1 + \epsilon^2\alpha^2\sigma_{yx}) + Sb\epsilon\sigma_{yx}(\epsilon^2\alpha^2 + \sigma_{xy})]}{\alpha(\epsilon^2\alpha^2 + \sigma_{xy})(1 + \epsilon^2\beta^2\sigma_{yx}) - \beta(1 + \epsilon^2\alpha^2\sigma_{yx})(\epsilon^2\beta^2 + \sigma_{xy})}. \tag{20}$$

The stress function is now determined and it can be shown that it contains as a special case the stress function for an isotropic plate. Let $a = b$ and set $\epsilon = 1$. Then from (6) it is seen that $E_x = E_y$. This is a necessary condition but not a sufficient condition that the plate in question be an isotropic plate. For the isotropic case, it is now sufficient that $\phi \to 0$; since by (14) $K \to 1$ and (6) reduces to the biharmonic equation for the isotropic case. Parts of the stress function become indeterminate but they may be evaluated by successive applications of L'Hospitals rule.

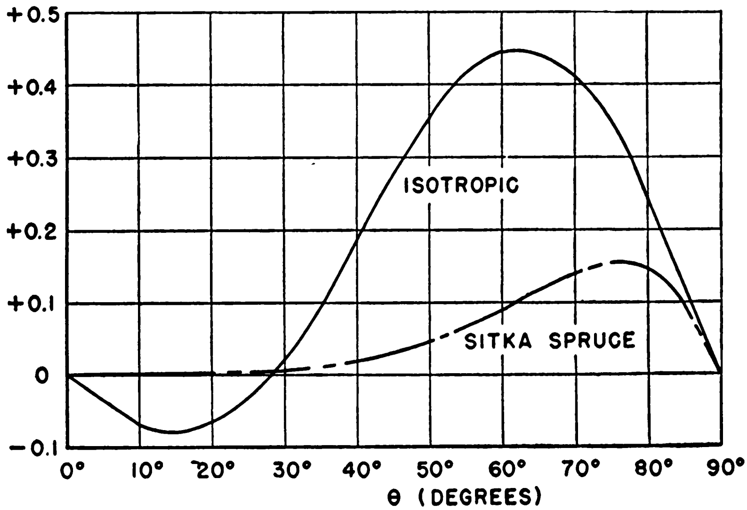

FIG. 1. Variation of the shear stress component X_y at points along the boundary of a rigid circular disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.—Ordinates: Ratio of shear stress component X_y at points on the boundary of a rigid circular disk to S the normal stress at infinity.

The stress function for a plate in a state of plane strain was also determined and it was shown that it also contained as a special case the stress function for an isotropic plate. It is interesting to note that using a different method Professor I. S. Sokolnikoff[5] obtained the same stress function for an isotropic plate.

Consider now a large plain-sawn rectangular plate of Sitka spruce containing a rigid elliptic disk at its center. It is possible to apply the results to a finite plate since the

[5]I. S. Sokolnikoff, *Mathematical theory of elasticity* (mimeographed lecture notes, Brown University, 1941) Chap. V.

stress concentration is localized near the rigid disk. The x- and y- axes are parallel and perpendicular to the grain, respectively.

Shear stresses are probably most important in producing failure in a wooden plate. The shear stress given by (18) evaluated on the boundary of the rigid disk is

$$X_y \Big|_{0 \leq \theta \leq \pi/2} = \frac{\alpha\epsilon A_1 \sin\theta \cos\theta}{\alpha^2\epsilon^2 b^2 \cos^2\theta + a^2 \sin^2\theta} + \frac{\beta\epsilon B_1 \sin\theta\cos\theta}{\beta^2\epsilon^2 b^2 \cos^2\theta + a^2\sin^2\theta}. \tag{21}$$

In Figure 1 the curve for Sitka spruce was plotted utilizing formulas (19), (20) and (21). The isotropic curve was computed for $\sigma = 0.3$.

Throughout the Sitka spruce plate containing a rigid circular disk $|Y_y|$ is less than $0.03S$. The maximum value of X_x is $1.23S$ at the point $(a, 0)$.

From these results, it is probable that, for S sufficiently large, failure in the plate will occur approximately along the lines $y = \pm a \sin 77\frac{1}{2}° = \pm 0.976a$ with the crack beginning at the edge of the disk.

BOOK REVIEWS

The mathematical theory of plasticity. By R. Hill. Oxford at the Clarendon Press, 1950. ix + 356 pp. $7.00.

Time, temperature, Bauschinger, hysteresis, and size effects are all explicitly ruled out and the major but not exclusive emphasis is on ideal plasticity. However, a wide range of topics is discussed in this interesting and invaluable treatise. Mathematical proofs, experimental evidence and the practical evaluation of theory are kept in excellent balance from the study of stress-strain relations, variational principles, and the questions of uniqueness, to the solving of practical metal forming problems. A few relatively simple elastic-plastic solutions are included for prismatic bars, and for thick cylindrical tubes and spherical shells. However, most of the chapters deal with plastic-rigid techniques and solutions developed by the author and E. H. Lee. An extensive discussion is given of the slip line fields for plane problems. The necessity for complete solutions is stated strongly and repeated warning is given against the error of thinking in terms of static determinacy only and not considering velocity conditions as well. A number of interesting and truly amazing solutions are given in detail for which the configuration remains geometrically similar as the plastic deformation proceeds. Among the miscellaneous subjects covered are: machining, hardness tests, notched bars, normal and oblique necking, earing and anisotropy. Tensor notation is used throughout without apology.

As is entirely proper for a book on the mathematical theory of plasticity, the physics of metals is covered only by reference to treatises on the subject. A very brief Appendix on suffix notation, the summation convention, and hyperbolic differential equations is considered sufficient for the reader who is assumed to be familiar with the elementary theory of elasticity.

The reviewer regretted to detect an apparent desire on the part of the author to have written a truly definitive text at a stage in the development of plasticity when too many basically new facts are being discovered. As the author himself is in the forefront of many of these developments he could have done a greater service by indicating more clearly the needed fundamentally new directions rather than glossing over our present shortcomings and often embarrassing lack of knowledge. He then might not have allowed himself to get so worried by Southwell and Allen's elastic-plastic solution for a notched bar that