

As for the speed with which the limit in (b) is approached, we get, again using Taylor's expansion, the following:

II. In I(b), the error is  $O(t^3)$ . If  $f$  has continuous fourth derivatives near  $P$ , the error can be made  $O(t^4)$  by a further restriction on  $S$ . Whatever set  $S$  is chosen, the error cannot be made  $o(t^4)$  even if the class of functions is restricted to polynomials.

## REFERENCES

1. L. Hopf, *Differential equations of physics*, Dover Publications, 1948, p. 62.
2. Garrett Birkhoff and David Young, *Numerical quadrature of analytic and harmonic functions*, J. of Math. and Physics 29, 217-221 (1950).

## A NOTE ON ASYMPTOTIC STABILITY\*

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1. In this note we shall develop a stability criterion for a vector differential equation of the form

$$\frac{dx}{dt} = A(t)x, \quad (1)$$

where the elements of the matrix  $A(t) = (a_{ij}(t))$ ,  $i, j = 1, 2, \dots, n$ , are real continuous and uniformly bounded functions for all positive  $t \geq t_0$ .

A. Wintner\*\* recently established the following criterion: Let  $\lambda_1(t)$  be the greatest, and  $\lambda_2(t)$  the least characteristic value of the matrix  $\frac{1}{2}[A(t) + A'(t)]$ , and let  $\|x(t)\|$  denote the Euclidean length of the vector  $x(t)$ . If  $\int_{t_0}^{\infty} \lambda_1(t) dt < \infty$ ,  $\int_{t_0}^{\infty} \lambda_2(t) dt < \infty$ , then  $\|x(t)\| \rightarrow \kappa \neq 0$  as  $t \rightarrow \infty$  for every non-trivial solution  $x(t)$  of (1).

It is to be noted that the condition of integrability of  $\lambda_1(t)$ ,  $\lambda_2(t)$  over  $(t_0, \infty)$  implies  $\int_{t_0}^{\infty} [\text{trace } A(t)] dt < \infty$ . Furthermore, this condition automatically excludes the important case  $A(t) = \text{const.}$  unless  $A(t) = \text{const.}$  is skew-symmetric.

In the following we shall establish a stability criterion which is free of the above objection, i.e. which will also apply to the general case  $A(t) = \text{const.}$  We shall consider a condition to be satisfied by the matrix  $A(t)$  which will suffice to insure that  $\|x(t)\|$  of every non-trivial solution  $x(t)$  of (1) tends to zero as  $t \rightarrow \infty$ . According to Liapounoff†, the trivial solution  $x(t) \equiv 0$  is then said to be asymptotically stable.

2. Consider a function  $V(x, t)$  which is defined and continuous for all  $x$  and  $t$  in  $R$ :  $|x_i| \leq c$ ,  $t \geq T$  ( $i = 1, 2, \dots, n$ ). If for equation (1) there exists in  $R$  a function  $V(x, t)$  which is of fixed sign and admits of an infinitely small upper bound, and for which  $dV/dt$  by virtue of (1) is opposite in sign to  $V(x, t)$  in  $R$ , then the trivial solution  $x(t) \equiv 0$  of (1) is asymptotically stable. Liapounoff proved that the existence of such a function

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\*\*A. Wintner, *On free vibrations with amplitudinal limits*, Quart. Applied Math. **8**, 102-104 (1950).

†A. Liapounoff, *Problème général de la stabilité du mouvement*, Ann. Math. Studies, No. 17, 1949.

$V(x, t)$  is sufficient for asymptotic stability; it is, however, not necessary as was shown by J. Malkin.\*

We shall make use of Malkin's results to establish the following theorem:

Let  $\lambda_1(t)$  be the greatest, and  $\lambda_2(t)$  the least characteristic value of the matrix  $\frac{1}{2}[A(t) + A'(t)]$ . If  $\int^t \lambda_1(\tau) d\tau \rightarrow -\infty$ ,  $\int^t \lambda_2(\tau) d\tau \rightarrow -\infty$  as  $t \rightarrow \infty$ , then  $\|x(t)\| \rightarrow 0$  as  $t \rightarrow \infty$  for every non-trivial solution  $x(t)$  of (1), i.e. the trivial solution  $x(t) \equiv 0$  is asymptotically stable.

Note that now  $\int^\infty [\text{trace } A(t)]dt$  diverges.

3. First, we transform (1) into diagonal form. Let  $x_1, x_2, \dots, x_n$  be a base of solutions of (1), and use this base to construct an orthogonal matrix  $C(t)$ . If  $y = C^{-1}(t)x$ , then (1) reduces to

$$\frac{dy}{dt} = B(t)y, \quad B(t) = C^{-1}AC + \frac{dC^{-1}}{dt}C \quad (2)$$

where the matrix  $B(t) = (b_{ij}(t))$ ,  $i, j = 1, 2, \dots, n$ , is diagonal, i.e.  $b_{ij}(t) \equiv 0$  for all  $i > j$ . If  $y_1(t), y_2(t), \dots, y_n(t)$  is that base of solutions of (2) for which  $y_i(t_0) = I^i$ , the  $i$ -th column vector of the identity matrix  $I$ , then  $\|x_i(t)\| = \|y_i(t)\|$  as is easily verified. Evidently,  $C(t)$  and  $C^{-1}(t)$  have bounded elements and  $|C(t)| = |C^{-1}(t)| = 1$ ; hence stability properties are preserved in both directions.

Observing that  $C^{-1}(t) = C'(t)$  by construction, we find by differentiating the identity  $C(t)C^{-1}(t) \equiv I$  that  $(dC^{-1}/dt)C$  is skew-symmetric. Therefore  $B(t) + B'(t) = C^{-1}[A(t) + A'(t)]C$ , and thus the characteristic values of  $\frac{1}{2}[B(t) + B'(t)]$  are identical with those of  $\frac{1}{2}[A(t) + A'(t)]$ . Hence it is sufficient to prove our theorem for the reduced equation (2). We shall show that there exists a function  $V(y, t)$  which satisfies Liapounoff's criterion for asymptotic stability.

Consider the diagonal elements  $b_{ii}(t)$ ,  $i = 1, 2, \dots, n$ , of the matrix  $B(t)$ . Since  $(dC^{-1}/dt)C$  is skew-symmetric,  $\text{trace } (dC^{-1}/dt)C \equiv 0$ , and thus

$$b_{ii}(t) = (C^{-1})_i A C^i = (C^i)' A C^i = \frac{1}{2}(C^i)'[A(t) + A'(t)]C^i. \quad (3)$$

All diagonal elements of  $B(t)$  are quadratic forms in the components of the column vectors  $C^i$  of the matrix  $C(t)$  for which we evidently have  $\|C^i\| = 1$ . These quadratic forms attain their maximum and minimum on the unit sphere  $\|C^i\| = 1$  (compact set); if  $\lambda_1(t)$  is the greatest,  $\lambda_2(t)$  the least characteristic value of  $\frac{1}{2}[B(t) + B'(t)]$ , then  $\lambda_1(t)$  is the maximum,  $\lambda_2(t)$  the minimum. From (3) we then obtain

$$\lambda_1(t) \geq b_{ii}(t) \geq \lambda_2(t) \quad (4)$$

whence for all  $t \geq t_0$

$$\exp \left( \int_{t_0}^t \lambda_1(\tau) d\tau \right) \geq \exp \left( \int_{t_0}^t b_{ii}(\tau) d\tau \right) \geq \exp \left( \int_{t_0}^t \lambda_2(\tau) d\tau \right). \quad (5)$$

By hypothesis  $\int^t \lambda_k(\tau) d\tau \rightarrow -\infty$  as  $t \rightarrow \infty$ ,  $k = 1, 2$ , and thus

$$\varphi_i(t) = \exp \left( \int_{t_0}^t b_{ii}(\tau) d\tau \right) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty. \quad (6)$$

As Malkin has shown, (6) involves for all  $t \geq t_0$

$$\varphi_i(t) \int_{t_0}^t \frac{d\tau}{\varphi_i(\tau)} \leq c \quad (7)$$

\*J. Malkin, *Certain questions on the theory of the stability of motion in the sense of Liapounoff*, American Math. Soc., Translation No. 20, 1950.

and (6) and (7) together, in turn, imply  $\int^\infty [\varphi_i(t)]^2 dt < \infty$ . Hence the functions

$$\psi_i(t) = [\varphi_i(t)]^{-2} \int_t^\infty [\varphi_i(\tau)]^2 d\tau \quad (8)$$

exist for all  $t \geq t_0$  and are uniformly bounded; in fact,  $a^2 \leq \psi_i(t) \leq b^2$  where  $a$  and  $b$  are certain constants.

Now consider the function

$$V(y, t) = \psi_1(t)y_1^2 + \psi_2(t)y_2^2 + \cdots + \psi_n(t)y_n^2.$$

It evidently satisfies Liapounoff's criterion for asymptotic stability; it is a positive definite quadratic form, admitting of an infinitely small upper bound, and its derivative, by virtue of (2), becomes

$$\frac{dV}{dt} = -(y_1^2 + y_2^2 + \cdots + y_n^2) + W(y, t)$$

where  $W(y, t)$  is a quadratic form whose coefficients depend upon those elements  $b_{ij}(t)$  of  $B(t)$  for which  $i < j$ ,  $i, j = 1, 2, \cdots n$ . Since these elements can always be made sufficiently small by a transformation with constant coefficients (which will not affect stability properties) the derivative  $dV/dt$  will be a negative definite quadratic form. Hence the trivial solution  $y(t) \equiv 0$  of (2) is asymptotically stable, and therefore the trivial solution  $x(t) \equiv 0$  of (1) is asymptotically stable. This establishes our theorem.

O. Perron\* was the first to prove directly that the conditions

$$\varphi_i(t) \leq C_1, \quad \varphi_i(t) \int_{t_0}^t \frac{d\tau}{\varphi_i(\tau)} \leq C_2$$

are necessary and sufficient for the trivial solution  $x(t) \equiv 0$  of (1) to be asymptotically stable.

\*O. Perron, *Die Stabilitätsfrage bei Differentialgleichungen*, Math. Zeitschrift **32**, 703-728 (1930).

## CONDITIONS SATISFIED BY THE EXPANSION AND VORTICITY OF A VISCOUS FLUID IN A FIXED CONTAINER\*

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**1. Introduction.** In plane motion of a viscous fluid inside a fixed container, the expansion  $\theta$  and the vorticity  $\omega$  cannot be arbitrarily assigned. A necessary and sufficient condition<sup>1</sup> for the consistency of given  $\theta$  and  $\omega$  with vanishing velocity on the walls is

$$\int (\theta U + 2\omega V) dS = 0, \quad (1.1)$$

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<sup>1</sup>J. L. Synge, *Quarterly of Applied Mathematics*, **8**, 107-108 (1950). The condition with  $\theta = 0$  was originally due to G. Hamel, *Göttinger Nachr. Math.-Phys. Kl.* **1911**, 261-270.