—NOTES—

A METHOD OF VARIATION FOR FLOW PROBLEMS—II*

By A. R. Manwell (*University College, Swansea, England*)

Summary. The method of variation of reference [1] is developed afresh in a slightly different manner which enables the main principle used in [1] to be derived directly and also makes the actual calculations much simpler. It is shown how a variety of problems concerning aerofoils possessing minimal properties may be reduced to the solution of integro-differential equations which determine the mapping of the aerofoil onto a circular region. It is briefly indicated how the method may be extended to three dimensional flows.

1. Introduction. In [1] the author has given an elementary method of variation suitable for treating a type of extremal problems suggested by two-dimensional aerofoil theory. Briefly, the method is to make small elliptic bulges in the boundary and after calculating the changes of the flow functions to equate to zero the variation of the functional which is to be minimized, for the case of infinitely flat ellipses. This process was justified by appealing to the fact that for such flat bulges the velocity changes as well as the geometrical changes were in effect infinitesimal. Such a physical argument being not quite convincing the author also tested the principle by showing that *all* small bulges in the *hodograph* plane gave equivalent results. It will however be seen that it is quite easy to prove that for infinitesimal velocity changes, although not for general perturbations of the boundary, the first variations may be equated to zero.

The results of the method appear in a conveniently compact form viz. as integro-differential equations (sometimes just differential equations) from which the mapping of the aerofoil on the standard unit circle may be determined. In some problems it is necessary to make auxiliary restrictions on the aerofoil, such as limitations on the chord or the velocity in the field.

In three dimensional fields the device of conformal mapping is of course not available but, analogous to the above equations for aerofoil problems, the method yields certain relations between geometrical and field properties on the boundary of the field. A discussion of the determination of the field from such conditions is left for a future paper.

2. The method of variation. In [1] the following principle was treated as physically obvious:—

If a functional of the geometry of a closed curve and the velocities of an associated hydrodynamic field is maximized by a certain curve, this functional is stationary for all variations in which both physical coordinates and the velocities are changed infinitesimally.

On the other hand small bulges giving rise to finite changes of velocity will change the functional by a small quantity of the *first* order. This situation may be illustrated by the following simple case. It is easily shown that the ratio of the area of an ellipse to the strength of the doublet, giving an equivalent disturbance in the stream at large distances, depends on the shape of the ellipse. Let it be admitted that there is some aerofoil problem in which the ratio of area to disturbance at infinity is stationary (c.f. §5).

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Then by taking two small ellipses of the same area it would be possible to find an equation corresponding to zero variation of area but a non-zero disturbance at infinity. Equally one could determine a variation corresponding to no disturbance but non-zero change of area. The ratio cannot be stationary for such variations. Further illustration may be found in the detailed calculations of [1] for small elliptic bulges. It is also interesting to note that, since velocity changes are *finite*, small elliptic bulges give something more than a second variation and so the sign of the changes is rather strong evidence, although not proof, of a true maximum or minimum.

The principle at the beginning of this paragraph will now be embodied in a simple lemma capable of direct proof.

Main lemma. If ψ satisfies a second order partial differential equation of elliptic type together with the boundary conditions on a simple closed curve C (and suitable auxiliary conditions) whilst I is a functional of C, in the geometrical sense, and of the velocities at points in the field and on C, then, for variations giving infinitesimal velocity changes everywhere, $\delta I = 0$ if I is a maximum or minimum.

Proof. If a small perturbing stream function is added to ψ it is readily seen that $\delta\psi$ satifies a homogeneous linear partial differential equation and so, if $\delta\psi$ is a solution so is $-\delta\psi$. Now the new boundary C': $\psi + \delta\psi = 0$ is to be derived from C by drawing a normal to C of length

$$\delta n = -\left[\delta\psi \Big/ \frac{\partial\psi}{\partial n}\right]_c = -\left(\frac{\delta\psi}{q_s}\right)_c. \tag{2.1}$$

Except at the isolated stagnation points the velocity q_s is not zero so the perturbation is infinitesimal provided $\delta\psi$ is made zero at such points.

Geometrical changes may be expressed linearly in terms of δn. For example, if A is the area enclosed by C whilst L is the length of C and ds the line element

$$\delta A = \int_c \delta n \, ds, \tag{2.2}$$

$$\delta L = \int_c K \, \delta n \, ds, \tag{2.3}$$

where K is the curvature of C.

It is therefore clear that both the velocities at a *given* point and geometrical changes are linear in $\delta\psi$. This is also true of velocities on the variable boundary C and for the present purpose this is more important.

Thus, the variation of velocity on $\psi = 0$ may be found as the sum of :

(i) $\partial/\partial n(\delta\psi)$ calculated at C,

(ii) the change in the undisturbed field due to displacing C a distance δn (i) is clearly linear in $\delta\psi$ and (ii) is linear in δn and so again in $\delta\psi$. If $(\partial^2\psi/\partial n^2)_C$ were zero then the contribution would be zero to the first order. In fact, for elliptic equations, this case is easily shown never to arise.

If therefore I is a maximum or minimum the existence of a $-\ \delta I$ for every $+\delta I$ shows immediately that for such variations $\delta I = 0$ as in the better known geometrical problems of the calculus of variations. Simple as the proof appears the result is not trivial since it implies such results as those of §5 [1] which are difficult to derive by direct calculation.

3. Laplace's equation in two variables. Let z be the aerofoil plane, ζ that of the unit circle and t that of the strip. For non-circulating flow, U being the stream velocity, the complex potential is

$$w = Ut = U(\zeta + \zeta^{-1}). \tag{3.1}$$

A suitable perturbing function is

$$\delta w = \mu^2 U \sum_1^\infty a_n \zeta^{-n}, \tag{3.2}$$

where μ is small and the a_n are real so that $I(\delta w) = 0$ on the real axis.

If the image in the ζ-plane of the disturbed boundary streamline C' is given by the vector $\zeta = e^{i\theta}[1 + \delta R]$ it follows that

$$\delta R = \frac{\mu^2 \eta(\theta)}{2 \sin \theta}, \tag{3.3}$$

where

$$\eta(\theta) = \sum a_n \sin n\theta.$$

Following the method of the previous section, the change of velocity is the sum $\delta_1 v + \delta_2 v$, where

$$\delta_1 v = \mu^2 \sum n a_n \sin n\theta / 2 \sin \theta, \tag{3.4}$$

$$\delta_2 v / v = -\delta R(1 + \alpha) \tag{3.5}$$

with

$$\alpha = R\left(\zeta \frac{d^2 z}{d\zeta^2} \Big/ \frac{dz}{d\zeta}\right)_c,$$

(3.5) being most conveniently found by writing

$$\left|\frac{dw}{dz}\right|_{c'} = \left|\frac{dw}{d\zeta}\right|_{c'} \Big/ \left|\frac{dz}{d\zeta}\right|_{c'}$$

and expanding numerator and denominator.

If C' is mapped on the unit circle $|\zeta'| = 1$ in a new plane the complex potential is known in two forms and by comparison of both it and the velocities on C' found from the two methods it is quite easy to establish the relation.

$$\log \left|\frac{d\zeta}{d\zeta'}\right| = \Delta\theta \cot \theta' + \frac{\delta_2 v(\theta')}{v(\theta')},$$

where

$$2 \sin \theta' \Delta\theta = \mu^2 \sum a_n \cos n\theta' \quad \text{and} \quad \zeta'_c = e^{i\theta'}.$$

These relations together with Poissons' formula for the function $\log (d\zeta/d\zeta')$ regular in $|\zeta'| \geq 1$ and with the real part given on $|\zeta'| = 1$ determine the mapping of the new region on the old. For applications however 3.3, 3.4, 3.5 are sufficient and involve only one linear transformation of $\eta(\theta)$ whereas the explicit mapping requires three such transformations.

4. Aerofoils minimizing a surface integral. Let A be the area contained in C a symmetrical profile with axis along the stream and suppose

$$I = \int_c F(v)\, ds,$$

where v is the velocity on the boundary. Then $J \equiv IA^{-1/2}$ is non-dimensional and it has been shown in [1] that for $F(v) = kv$ with k constant J is a minimum for the circle. In fact I is not changed by conformal mapping between the aerofoil and plane of the circle but depends only on the potential function. The problem therefore reduces to making A as large as possible under conditions for which the circle is well known to be the result. It is natural to assume the existence of solutions for more general F and the method of variation shows that these must satisfy a certain integro-differential relation. From (3.5) since the equipotentials are normal to C it readily follows that $\delta(ds) = -\delta_2 v/v$ and so

$$ds_1 = d\theta \left| \frac{dz}{d\zeta} \right|_c \{1 + \delta R(1 + \alpha)\} \tag{4.1}$$

is the length of the element of the perturbed arc. Then $\delta I = \int F'(v)\delta v\, ds + \int F(v)\, \delta(ds)$ becomes, after using (3.4), (3.5) and (4.1),

$$\delta I = \int F'(v)\{\delta_1 v/v - \delta R(1 + \alpha)\} 2 \sin \theta\, d\theta + \int F(v)\, \delta R(1 + \alpha) \frac{2 \sin \theta\, d\theta}{v}. \tag{4.2}$$

Again

$$\delta A = \int \delta R \left| \frac{dz}{d\zeta} \right|^2 d\theta = 2\mu \int \eta(\theta) \frac{\sin \theta\, d\theta}{v^2}, \tag{4.3}$$

so that if $\delta J = 0$, which implies

$$\delta I = K \delta A \qquad \text{with} \qquad K = \frac{1}{2} \frac{I}{A},$$

the condition for a minimal solution may be written

$$\int_0^\pi H'(\theta)\eta(\theta)\, d\theta = \int_0^\pi F'(v)\, \delta_1 v_\theta\, d\theta, \tag{4.4}$$

where

$$H = \int^\theta \{[1 + \alpha][F'(v) - F(v)/v] + 2k \sin \theta/v^2\}\, d\theta. \tag{4.5}$$

Then $\delta_1 v_\theta$, the change in the plane of the circle is with $\nu(\theta) = \eta'(\theta)$ given by

$$\delta_1 v_\theta = \frac{\mu^2}{\pi} \int_0^\pi \frac{\sin \theta}{\cos t - \cos \theta} \nu(t)\, dt \tag{4.6}$$

in which to make the operation on ν possible it is supposed for example that $\nu(t)$ possesses a derivative.

Since η vanishes at the stagnation points at the ends of the aerofoil 4.4 becomes after an integration by parts

$$\int_0^\pi F'(v)\left[\frac{1}{\pi}\int_0^\pi \frac{\sin\theta}{\cos t - \cos\theta}\nu(t)\,dt\right]d\theta + \int_0^\pi H(t)\nu(t)\,dt = 0. \tag{4.7}$$

If now $\nu(t)$ vanishes except in a small interval (4.7) gives

$$\frac{1}{\pi}\int_0^\pi \frac{F'\{v(\theta)\}\sin\theta}{\cos t - \cos\theta}\,d\theta + H(t) = 0, \tag{4.8}$$

where H is given by (4.5).

In the case $F = kv$ H simplifies greatly and (4.8) gives

$$\frac{2k \sin t}{v^2(t)} = \frac{1}{\pi}\frac{d}{dt}\left\{\log\left(\frac{1-\cos t}{1+\cos t}\right)\right\} = \frac{2}{\pi \sin t}.$$

Hence $v \propto \sin t$ and $|dz/d\zeta| =$ constant showing that the solution must be a circle. The following approximations are suggested in other cases.

(*a*) If the body is nearly circular α is small and the relation Eq. (4.8) is satisfied approximately by

$$F(v) = v + a(e^{bv^2} - 1) \qquad ve^{bv^2} = 2\,a \sin\theta,$$

where b is small. This gives *one* family of solutions.

(*b*) For thin flat aerofoils $(1 + \alpha)$ is small except at the ends and if $F(0) = 0$ the term $F - vF' = 0(\theta^2)$ at the ends so that the equation reduces to

$$\int^t \frac{2k \sin t}{v^2}\,dt + \frac{1}{\pi}\int_0^\pi \frac{\sin\theta F'\{v(\theta)\}}{\cos t - \cos\theta}\,d\theta = 0.$$

5. Problems with restrictions. In this section a brief account will be given of the way in which the method can be modified to take account of certain restrictions peculiar to flow problems. For example, let A be the area of a symmetrical profile and D the strength of the equivalent doublet which represents the flow at large distances. As in §4

$$\delta A = \frac{1}{2}\mu^2 \int_0^\pi \frac{\eta(\theta)}{\sin\theta}\left|\frac{dz}{d\zeta}\right|^2 d\theta, \tag{5.1}$$

and since δD is just the coefficient of ζ^{-1} in the perturbing potential 3.2

$$\delta D = \frac{2\mu^2}{\pi}\int_0^\pi \sin\theta\eta(\theta)\,d\theta.$$

If no restriction is made the ratio D/A is stationary only if

$$\left|\frac{dz}{d\zeta}\right|^2 = \lambda \sin^2\theta \qquad \text{or if} \qquad \left|\frac{dw}{dz}\right|$$

is constant over the whole profile. This is possible only for a strip and to avoid this trivial result, where a finite area would have to be extended indefinitely along the stream direction, it is necessary to limit the aerofoil in this direction. The simplest restriction seems to be that the aerofoil is not to lie outside of two lines drawn perpendicular to the stream. The ends then must lie on these lines and the method of variation is not applicable to such parts of the profile. Over the rest of the profile dw/dz is constant as before and the complete solution is that the aerofoils belong to the Riabouchinsky constant velocity series [2]. In such cases the area must be regarded as fixed and the theory of isoperimetrical problems used to establish the constant velocity relation. For given straight lines, i.e. for a given maximum chord, the area completely determines the solution.

In the last case the restriction was geometrical but it is also possible to limit velocity instead of the chord. In the typical case there are constant velocity portions over the middle of the aerofoil and the problem is to find perturbations of the whole boundary which leave the velocity unchanged on these arcs, say the intervals $(\pi/2) \pm \lambda$ and $(3\pi/2) \pm \lambda$ for a symmetrical aerofoil. A simple method is to take a small bulge on the free arc and then a general perturbation of the constant velocity arc. In the case quoted this is achieved by way of the transformation.

$$t = \zeta + \frac{1}{\zeta} = \sin \lambda\left(s + \frac{1}{s}\right), \tag{5.4}$$

and a suitable disturbed potential is

$$w + \delta w = u\left[\zeta + \frac{1}{\zeta} + \mu^2\left\{\frac{1}{\zeta + 1/\zeta - 2\cos\chi} + \sum_1^\infty \frac{a_n s^{-n}}{\sin\lambda}\right\}\right], \tag{5.5}$$

where the a_n are to be determined by the integral equation for

$$\nu(u) \equiv \sum n a_n \sin nu$$

$$\nu(u) - \frac{\sin\lambda \sin u}{\sqrt{1 - \sin^2\lambda \cos^2 u}}\{1 + \alpha(\theta)\}\int_0^\pi K(u, t)\nu(t)\,dt = \frac{2\beta}{1-\beta^2}\sum_1^\infty n\beta^n \sin nu \equiv G(u, \chi),$$

where

$$K \equiv \frac{2}{\pi}\sum_1^\infty \frac{\sin nu \sin nt}{n}$$

and

$$\beta = \sin\lambda/\{\cos\chi + \sqrt{\cos^2\chi - \sin^2\lambda}\}.$$

In general successive approximation would be required on account of the presence of $\alpha(\theta)$ in (5.6) but if the aerofoil is flattish or if λ is small (5.6) gives $\nu(u)$ directly to a good approximation.

The author has carried out calculations for the problem of §3 with $F \equiv v$ but requiring in addition that the maximum velocity is slightly less than twice the free stream value. In this case small constant velocity arcs appear and the method of variation gives the velocity over the rest of the profile. These solutions found by taking $\nu(u) = G(u, \chi)$ in (5.6) tend smoothly into the circle as λ vanishes. It may also be noted that the result of imposing restrictions may be to give mixed boundary conditions. For example in the preceding problem if a curve of given area π is required to lie between $y = \pm C$ where $C < 1$ the circle is replaced by a solution flattened so that $y = \pm C$ over the middle whilst the method of variation gives the velocity over the ends. This means that over one part of the aerofoil the magnitude, and over the other, the inclination of the velocity is given.

6. Minimal problems in potential theory. In dealing with aerofoils it is natural to use the conformal mapping of the region onto a circular one but this is only a device. As the following shows the true significance of the method is that it gives a functional relation between quantities on the boundary of the field. Moreover the linear transformations which arise in the method may be more compactly explained in terms of potential theory using Green's Theorem for the original and perturbing potential c.f. [3].

Let

$$I = \int_\Sigma F\left(Q, \varphi, \frac{\partial \varphi}{\partial n}\right) dS_Q' , \tag{6.1}$$

where Σ is an equipotential: $\varphi = F$ and for simplicity it will be supposed that $\varphi = O(1/r)$ at great distances whilst Q expresses the dependance of F on the space-coordinates. If φ is varied the new equipotential Σ' is given by drawing from Σ a normal of length

$$\delta n = -\left[\delta\varphi \Big/ \frac{\partial \varphi}{\partial n}\right]_\Sigma. \tag{6.2}$$

Now $\delta(dS) = K\delta n\, dS$ where K is the total curvature of Σ at Q and the first variation is

$$\delta I = \int_\Sigma FK\ \delta n\, dS + \int_\Sigma F_n\ \delta n\, dS + \int_\Sigma F_{\varphi_n}\varphi_{nn}\ \delta n\, dS + \int_\Sigma F_{\varphi_n}(\delta\varphi)_n\, dS \tag{6.3}$$

suffixes denoting partial derivatives.

The last term involves $(\delta\varphi)_n$ but this may be expressed in terms of S itself by introducing a new potential Φ which takes the value of F_{φ_n} on Σ.

Then, by Green's identity,

$$\int_\Sigma F_{\varphi_n}(\delta\varphi)_n\, dS = \int \frac{\partial}{\partial n}(\Phi)\ \delta\varphi\, dS = -\int \Phi_n\varphi_n\ \delta n\, dS, \tag{6.4}$$

according to (6.2).

Applying the fundamental lemma of the calculus of variations to (6.3) the condition $\delta I = 0$ leads to

$$KF + F_n + F_{\varphi_n}\varphi_{nn} - \Phi_n\varphi_n = 0. \tag{6.5}$$

As a simple illustration let C be the capacity of an isolated conductor having a charge which gives it a potential E above that at infinity. If V is the volume $I \equiv C/V^{1/3}$ is non dimensional.

Then

$$\delta V = \int_\Sigma \delta n\, dS', \tag{6.6}$$

and for the variations just considered

$$\delta C = \frac{\delta M}{E},$$

where

$$\delta M = -\frac{1}{4\pi}\int_\Sigma \frac{\partial}{\partial n}(\delta\varphi)\, dS$$

is the change of charge on the conductor.

Hence

$$\delta C = -\frac{1}{4\pi E^2}\int_\Sigma \varphi \frac{\partial}{\partial n}(\delta\varphi)\,dS$$

$$= -\frac{1}{4\pi E^2}\int_\Sigma \frac{\partial\varphi}{\partial n}\,\delta\varphi\,dS$$

$$\delta C = +\frac{1}{4\pi E^2}\int_\Sigma \left(\frac{\partial\varphi}{\partial n}\right)^2 \delta n\,dS \tag{6.7}$$

From (6.6) (6.7) since δn is arbitrary the condition $\delta I = 0$ is

$$\left(\frac{\partial\varphi}{\partial n}\right)^2 = \text{const.}$$

so that the surface must be such that electricity is distributed uniformly over its surface. Thus if the solution were not known in advance this result would lead directly to the sphere as the solution.

Conclusion. The method of variation of [1] has now been presented in a form in which the author hopes it may be of practical utility. It is a direct method for aerofoils and the numerical work in solving the equations which arise is not greater than in finding approximate flows for given boundaries: in some cases much less. On the other hand it is to be expected that for smooth changes from true minimal shapes no large changes will appear. This is indeed an essential feature of all minimal solutions.

The author is indebted to Dr. W. H. J. Fuchs for explaining to him something of the beautiful work of M. Schiffer [4] on minimal problems in conformal mapping. It would appear that such problems as arise in analysis are far deeper than those needed in aerofoil theory. In the latter the condition of simple boundaries does not enter except trivially and since it is the essential condition in analysis the present method may not be of any use. It is immediately evident that the *proof* of the main lemma of the paper excludes cut regions where *inward* variations would require a second Riemann sheet.

The author also thanks Dr. P. M. Davidson for suggesting a variety of minimal problems in potential theory and in particular the capacity problem of §6 to which he originally gave this solution using instead of Green's identity for the potentials a physical interpretation of the problem together with the method of virtual work.

References

1. A. R. Manwell, *A method of variation for flow problems—I*, Q.J.M. (Oxford) **20**, 166-189 (1949).
2. A. R. Manwell, *Aerofoils of maximum thickness ratio*, Q.J.M.A.M. **1**, 365 (1948).
3. J. Hadamard, *Leçons sur le calcul des variations*, Tome 1, Livre 11 Ch. Vii p. 303.
4. M. Schiffer, *A method of variation within the family of simple functions*, Proc. L.M.S. (2) **44**, 432 (1938).