ON MATRIX BOUNDARY VALUE PROBLEMS*

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Introduction. In a recent publication[1] a matrix type of boundary value problem was introduced in order to simplify the description of nuclear reactions. It appeared that this type of boundary value problem could find applications in other branches of mathematical physics, and the purpose of the present note is to illustrate them.

When we deal with vibrations of continuous media, with problems of heat flow etc., we usually describe the state of the system in terms of a single function which depends on position as well as on the time. As an example, we may mention the lateral displacement of a vibrating string, or the temperature function in case of problems of heat flow.

In many problems of vibration and heat conduction, of which examples will be given below, the description of the state by a single function leads to boundary value problems of great difficulty. It is possible though, in some cases, to divide the continuous medium into several regions, and with each region we can associate a function describing its state. These functions can be grouped together in the form of a column matrix or vector, which will then represent the state of the whole system. The mathematical problem we encounter then, is a matrix boundary value problem, which is, in general, much simpler than the one we would have to deal with in the usual formulation.

In the present note, we shall discuss two examples of matrix boundary value problems. The first one describing the flow of heat in a cross, illustrates the case where the interactions between the different regions appear through boundary conditions. The second one, dealing with the vibration of systems of plates with intermediate elastic media, illustrates the case where the interactions take place through the equations of motion. We shall obtain the eigenvalues and eigenmatrix functions corresponding to this type of problems, and with the help of them, give a formal solution for any initial conditions.

For the discussion of the self-adjoint properties of this type of boundary value problem, and the rigorous derivation of the series expansion theorems, we refer to other publications.[2,3,4]

1. Flow of heat in a cross. We shall consider the problem of flow of heat in a cross (Fig. 1a) whose four arms are of the same length l, and of square cross section of area a^2, where $a \ll l$. The material of the cross will have a density ρ, conductivity κ and specific heat c. The lateral sides of the cross will be coated in such a way that the outer conductivity[5] can be taken as zero, i.e. there is no radiation.

If we tried to deal with this problem as a three-dimensional heat conduction problem in a region bounded by the surface of the cross, we would have a difficult boundary value problem which would not admit a simple solution. Taking into account though, that the smallness of the cross section permits us to assume that the temperature at all points in it is the same, we can describe the state of temperature in the cross in the following fashion: with each bar of the cross we associate its temperature function $\theta_i(x, t)$, where $i = 1, 2, 3, 4$ indicates the bar in question, x represents the position of the point on the bar with $0 \leq x \leq l$ as indicated in Fig. 1a., and t is the time. The temperature state of the whole cross is then described by the column matrix:

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$$\boldsymbol{\theta}(x, t) = \begin{bmatrix} \theta_1(x, t) \\ \\ \theta_2(x, t) \\ \\ \theta_3(x, t) \\ \\ \theta_4(x, t) \end{bmatrix}. \tag{1}$$

The equation for the temperature in each bar will be the well known equation for heat flow in rods,[5] and so the matrix representing the temperature in the cross satisfies the equation:

$$\rho c(\partial \boldsymbol{\theta}/\partial t) = \kappa(\partial^2 \boldsymbol{\theta}/\partial x^2). \tag{2}$$

The boundary conditions on the temperature at the free end points $x = 0$, can have

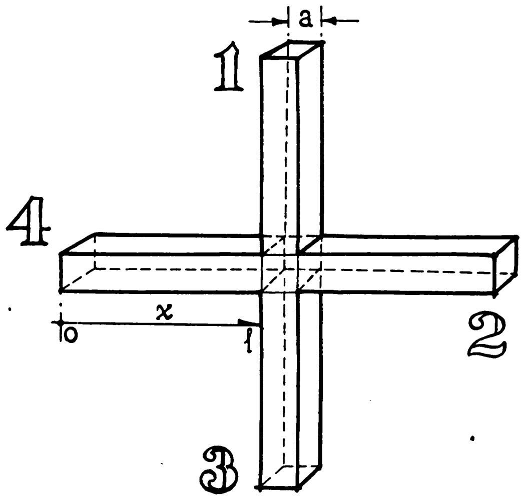

Fig. 1-a.

any of the usual forms;[5] for simplicity we will assume that the end points will be maintained at the constant temperature zero. We then have:

$$\boldsymbol{\theta}(0, t) = 0. \tag{3}$$

We now consider the boundary conditions at the end points $x = 1$. The intersection of the arms of the cross at $x = l$ gives rise to a cube of linear dimensions a illustrated in Fig. 1b. The smallness of the linear dimensions of the cube, allows us to assume that the temperature at all points of the cube can be taken as the same. The temperature at the end points of all the four arms will then be equal, and we have:

$$\theta_1(l, t) = \theta_2(l, t) = \theta_3(l, t) = \theta_4(l, t), \tag{4}$$

which gives rise to three linearly independent boundary conditions.

Finally, to obtain our last boundary condition we need to consider the total flow of heat into the cube. The flow of heat per unit time through the end sections of each bar into the cube, is given by $-\kappa a^2(\partial\theta_i/\partial x)_{x=l}$. The net inflow of heat into the cube must be equal to the increment per unit time of the quantity of heat in the cube which is $\rho c a^3(\partial\theta_1/\partial t)_{x=l}$. This quantity depends on a^3 and due to the smallness of the linear dimensions of the cube, it can be taken as of higher order than the net inflow of heat. The net flow of heat into the cube may then be assumed as ~ 0 and, as there is no radiation through the lateral sides, this leads to the boundary condition:

$$\sum_{L=1}^{4} (\partial\theta_i/\partial x)_{x=l} = 0 \tag{5}$$

The problem of flow of heat in a cross has now a complete mathematical description in terms of the equation (2) and the boundary conditions (3), (4), (5). Due to the sym-

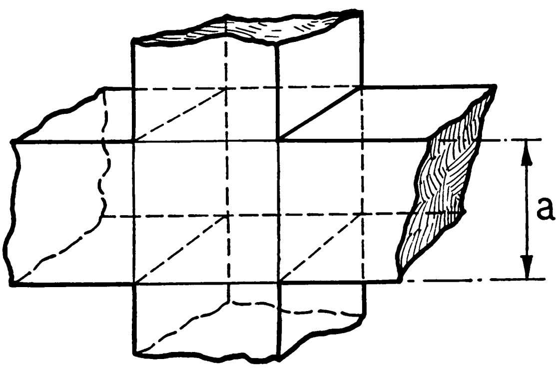

FIG. 1*b*.

metry of this particular problem, a simple linear transformation of the matrix $\boldsymbol{\theta}(x, t)$ can be found, which reduces (2)-(5) to four independent scalar problems. We shall discuss it though, as a matrix problem to illustrate the general procedure.

If we now introduce, as usual, a solution of the form $\boldsymbol{\theta}(x, t) = \boldsymbol{\theta}(x) \exp(-\lambda t)$ where λ is an arbitrary real positive constant, we are led to a matrix boundary value problem in which $\boldsymbol{\theta}(x)$ satisfies the ordinary linear equation:

$$(d^2\boldsymbol{\theta}/dx^2) + (\lambda\rho c/\kappa)\boldsymbol{\theta} = 0 \tag{6}$$

as well as the boundary conditions (3), (4), (5).

To determine the eigenvalues and eigenmatrix functions of this problem, we first notice that from (6) and (3), $\boldsymbol{\theta}(x)$ must have the form:

$$\boldsymbol{\theta}(x) = \mathbf{A} \sin (\lambda\rho c/\kappa)^{1/2}x, \tag{7}$$

where $\mathbf{A}$ is a constant column matrix of components A_i , $i = 1, 2, 3, 4$.

We now apply the boundary conditions (4), (5) to the solution (7) and we obtain the homogeneous system of linear equations:

$$\begin{gathered} A_1 \sin (\lambda\rho c/\kappa)^{1/2}l - A_i \sin (\lambda\rho c/\kappa)^{1/2}l = 0; \qquad i = 2, 3, 4 \\ \sum_{i=1}^{4} (\lambda\rho c/\kappa)^{1/2} A_i \cos (\lambda\rho c/\kappa)^{1/2}l = 0. \end{gathered} \tag{8}$$

The determinant of this system is: $4(\lambda\rho c/\kappa)^{1/2}\sin^3(\lambda\rho c/\kappa)^{1/2}l\cos(\lambda\rho c/\kappa)^{1/2}l$, and the characteristic values for which this determinant vanishes are:

$$\lambda_n = \frac{n^2\pi^2\kappa}{4\rho c l^2}, \qquad n = 0, 1, 2, \cdots . \tag{9}$$

We see from the determinant that when n is odd, the eigenvalue is non-degenerate, while when n is even, it is triply degenerate. The corresponding eigenmatrix functions are:

for odd n:

$$\boldsymbol{\theta}_n(x) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} (2l)^{-1/2}\sin(n\pi x/2l), \tag{10a}$$

for even n:

$$\boldsymbol{\theta}_n^{(1)}(x) = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} l^{-1/2}\sin(n\pi x/2l); \qquad \boldsymbol{\theta}_n^{(2)}(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} l^{-1/2}\sin(n\pi x/2l) \tag{10b}$$

$$\boldsymbol{\theta}_n^{(3)}(x) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} (2l)^{-1/2}\sin(n\pi x/2l).$$

We define the scalar product of two matrix functions $\boldsymbol{\phi}(x)$, $\boldsymbol{\psi}(x)$ as:

$$(\boldsymbol{\phi}, \boldsymbol{\psi}) = \int_0^l \boldsymbol{\phi}'(x)\boldsymbol{\psi}(x)\,dx = \int_0^l \left[\sum_{i=1}^4 \varphi_i(x)\psi_i(x)\right] dx \tag{11}$$

where $\boldsymbol{\phi}'(x)$ is the transposed form of the matrix $\boldsymbol{\phi}(x)$. It is then seen, that for even n, the eigenmatrix functions $\boldsymbol{\theta}_n^{(\alpha)}(x)$, $\alpha = 1, 2, 3$ were chosen so as to be mutually orthogonal, i.e. $(\boldsymbol{\theta}_n^{(1)}, \boldsymbol{\theta}_n^{(2)}) = 0$ etc. All the eigenmatrix functions are normalized $(\boldsymbol{\theta}_n, \boldsymbol{\theta}_n) = 1$. Finally, the eigenmatrix functions corresponding to different eigenvalues, are orthogonal, as can be seen directly from (10), and also from general considerations of self adjointness given in another publication.[2]

We assume now that the initial temperature distribution is given by a matrix function $\boldsymbol{\tau}(x)$ of class $C^{(1)}$, with sectionally continuous second derivative, which satisfies the boundary conditions (3, 4, 5). The variation of temperature with time will then be represented[3,4] by a matrix function

$$\boldsymbol{\theta}(x, t) = \sum_{n=0}^{\infty} a_{2n+1}\boldsymbol{\theta}_{2n+1}(x) \exp\left[-(2n+1)^2\pi^2\kappa t/4\rho c l^2\right] + \sum_{n=1}^{\infty}\sum_{\alpha=1}^{3} a_{2n}^{(\alpha)}\boldsymbol{\theta}_{2n}^{(\alpha)}(x) \exp\left[-n^2\pi^2\kappa t/\rho c l^2\right], \tag{12}$$

where:

$$a_{2n+1} = (\boldsymbol{\tau}, \boldsymbol{\theta}_{2n+1}), \qquad a_{2n}^{(\alpha)} = (\boldsymbol{\tau}, \boldsymbol{\theta}_{2n}^{(\alpha)}).$$

We see that in the present formulation, the problem of flow of heat in a cross admits a complete solution.

2. Vibration of two circular plates with an intermediate elastic medium. Let us consider a system of two circular plates of radius R, clamped at the edges, and with an intermediate elastic medium. We shall designate by ρ_1 the density, D_1 the flexural rigidity and a_1, the thickness of the first plate, and ρ_2, D_2, a_2 will have the same meaning for the second plate. Finally, we denote by k the load per unit area of the plates necessary to produce a unit compression in the elastic medium.

The state of the vibrating system can then be described in terms of the normal displacements of the two plates $u_1(r, \varphi, t)$ and $u_2(r, \varphi, t)$, in which r, φ stand for polar coordinates in the plane. As the force per unit area that the plates exert on each other, is proportional to the compression $u_1 - u_2$ of the elastic medium, we have that the equations of motion[6,7] for the vibrating system are:

$$\begin{aligned} \rho_1 a_1(\partial^2 u_1/\partial t^2) + D_1\nabla^4 u_1 + k(u_1 - u_2) &= 0, \\ \rho_2 a_2(\partial^2 u_2/\partial t^2) + D_2\nabla^4 u_2 + k(u_2 - u_1) &= 0. \end{aligned} \tag{13}$$

We introduce, as usual, a solution of this system of differential equations, of the form:

$$\mathbf{u}(r, \varphi, t) = \mathbf{u}(r) \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix} \exp(i\omega t) \tag{14}$$

in which, for simplicity in notation, we abstain from associating explicitly an index m with the two components column matrix $\mathbf{u}(r)$. We are then led to the matrix boundary value problem:

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \left(\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} - \frac{m^2}{r^2}\right)^2 \begin{bmatrix} u_1(r) \\ u_2(r) \end{bmatrix} - \begin{bmatrix} \omega^2 a_1\rho_1 - k & k \\ k & \omega^2 a_2 \rho_2 - k \end{bmatrix} \begin{bmatrix} u_1(r) \\ u_2(r) \end{bmatrix} = 0 \tag{15a}$$

$$\mathbf{u}(R) = 0 \qquad (d\mathbf{u}/dr)_{r=R} = 0 \tag{15b}$$

This type of matrix boundary value problem differs from the previous one in so far as the interactions between the components take place through the equations of motion, and not through the boundary conditions.

To find the characteristic frequencies and eigenmatrix functions of (15), we first consider the scalar boundary value problem:

$$(\mathcal{L}^2 - \lambda^2)v(r) = 0; \qquad v(R) = 0, (dv/dr)_{r=R} = 0, \tag{16}$$

where $\mathcal{L}$ is the operator $[(1/r)(d/dr)r(d/dr) - (m^2/r^2)]$, and λ is a parameter. The solution of (16) is well known, as it is the boundary value problem of a single circular plate.[7] The function $v(r)$ is given by $AJ_m(\lambda^{1/2}r) + BI_m(\lambda^{1/2}r)$ and the eigenvalues λ_n, $n = 1, 2, 3$ are givèn by the transcendental equation:

$$\left[J_m(\lambda^{1/2}r)\frac{d}{dr}I_m(\lambda^{1/2}r) - I_m(\lambda^{1/2}r)\frac{d}{dr}J_m(\lambda^{1/2}r)\right]_{r=R} = 0 \tag{17a}$$

The roots $\beta_n = \lambda_n^{1/2}R/\pi$ of this equation have been evaluated[7] for several values of m. The corresponding eigenfunctions are:

$$v_n(r) = A_n\left[J_m\left(\frac{\pi\beta_n r}{R}\right) - \frac{J_m(\pi\beta_n)}{I_m(\pi\beta_n)}I_m\left(\frac{\pi\beta_n r}{R}\right)\right], \tag{17b}$$

where A_n is an arbitrary constant.

We propose now a solution for (15) of the form $\mathbf{u}_n(r) = \mathbf{c}v_n(r)$ where $\mathbf{c}$ is a constant column matrix of components c_1, c_2. The boundary conditions (15b) are immediately satisfied because of (16), (17).

As $\mathcal{L}^2 v_n = \lambda_n^2 v_n$, we see that the equation of motion (15a) is transformed into the algebraic linear equations

$$\left\{\begin{bmatrix} D_1\lambda_n^2 + k & -k \\ -k & D_2\lambda_n^2 + k \end{bmatrix} - \omega^2\begin{bmatrix} a_1\rho_1 & 0 \\ 0 & a_2\rho_2 \end{bmatrix}\right\}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0 \tag{18}$$

To have non-trivial solutions of this system of equations, the determinant of the matrix must vanish, and this determines the two characteristic frequencies $\omega_n^{(1)}$, $\omega_n^{(2)}$ corresponding to each eigenvalue λ_n, which take the form

$$\left.\begin{matrix}\omega_n^{(1)} \\ \\ \omega_n^{(2)}\end{matrix}\right\} = \left\{\frac{1}{2}\left(\frac{k + D_1\lambda_n^2}{a_1\rho_1} + \frac{k + D_2\lambda_n^2}{a_2\rho_2}\right) \pm \left[\alpha^2(\lambda_n^2) + \frac{k^2}{a_1\rho_1 a_2\rho_2}\right]^{1/2}\right\}^{1/2}, \tag{19}$$

where

$$\alpha(\lambda_n^2) = (2a_1\rho_1)^{-1}(k + D_1\lambda_n^2) - (2a_2\rho_2)^{-1}(k + D_2\lambda_n^2)$$

The corresponding eigenmatrices take from (18) the form

$$\mathbf{c}^{(1)} = \begin{bmatrix} k/D_1 \\ -(a_1\rho_1/D_1)\gamma(\lambda_n^2) \end{bmatrix}; \qquad \mathbf{c}^{(2)} = \begin{bmatrix} (a_2\rho_2/D_2)\gamma(\lambda_n^2) \\ k/D_2 \end{bmatrix}, \tag{20}$$

where

$$\gamma(\lambda_n^2) = -\alpha(\lambda_n^2) + [\alpha^2(\lambda_n^2) + (a_1\rho_1 a_2\rho_2)^{-1}k^2]^{1/2}.$$

The matrix boundary value problem (15) has now been solved, with the characteristic frequencies being $\omega_n^{(1)}$, $\omega_n^{(2)}$ and the corresponding eigenmatrix functions having the form: $\mathbf{u}_n^{(1)}(r) = \mathbf{c}^{(1)}v_n(r)$, $\mathbf{u}_n^{(2)}(r) = \mathbf{c}^{(2)}v_n(r)$.

It is easily established, from the general equations (13) and the boundary conditions (15b), that two eigenmatrix functions $\mathbf{u}^*$, $\mathbf{u}$ corresponding to different characteristic frequencies, are orthogonal in the sense that: $\int_0^R \mathbf{u}^{*\prime}\mathbf{W}\mathbf{u}\, rdr = 0$ where $\mathbf{u}^{*\prime}$ is the transposed of $\mathbf{u}^*$ and $\mathbf{w}$ is the matrix

$$\begin{bmatrix} a_1\rho_1 & 0 \\ 0 & a_2\rho_2 \end{bmatrix}.$$

We can normalize the eigenmatrix function $\mathbf{u}$ in the sense that $\int_0^R \mathbf{u}'\mathbf{W}\mathbf{u}\, rdr = 1$ by choosing the constant in (17b) appropriately. We would then have

$$(\mathbf{u}_n^{(\alpha)}, \mathbf{u}_l^{(\beta)}) \equiv \int_0^R \mathbf{u}_n^{(\alpha)\prime}\mathbf{W}\mathbf{u}_l^{(\beta)} r\, dr = \delta_{\alpha\beta}\delta_{nl}\ , \tag{21}$$

where α, $\beta = 1, 2$, and n, $l = 1, 2, 3, \cdots$.

We have obtained an orthonormalized set of eigenmatrix functions corresponding to this vibration problem. With their help, we could represent the state of vibration for the two plates corresponding to any initial conditions. For example, let us assume that we had at $t = 0$ a given displacement for our two plates, and that the initial velocity was zero. The initial displacement of the two plates could be represented as sum of terms of the form

$$\boldsymbol{\tau}(r)\begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix} \qquad \text{for} \qquad m = 0, 1, 2, \cdots .$$

For each term of this type, the solution of the vibration problem would be

$$\mathbf{u}(r,\ t)\begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix}$$

where $\mathbf{u}(r, t)$ has the form:

$$\mathbf{u}(r,\ t) = \sum_{n=1}^{\infty} \{(\boldsymbol{\tau}, \mathbf{u}_n^{(1)})\mathbf{u}_n^{(1)}(r) \cos \omega_n^{(1)} t + (\boldsymbol{\tau}, \mathbf{u}_n^{(2)})\mathbf{u}_n^{(2)}(r) \cos \omega_n^{(2)} t\} \tag{22}$$

and

$$(\boldsymbol{\tau}, \mathbf{u}_n^{(\alpha)}) \equiv \int_0^R \boldsymbol{\tau}'(r)\mathbf{W}\mathbf{u}_n^{(\alpha)}(r)r\, dr.$$

The generalization of the present developments to systems of more than two plates, as well as to other types of boundary conditions, and other forms for the plates, is straightforward.

3. Conclusion. A general type of matrix boundary value problem, which includes both of the preceding cases, would be the one in which there are interactions between the components of the column matrix, both, in the differential equations and in the boundary conditions. If we introduce additional components into the column matrix, so as to reduce the system of differential equations to one of the first order, the matrix boundary value problems introduced in the present note, reduce to a form which has been extensively discussed by Birkhoff and Langer.[3] While, in their formulation, the self-adjoint nature of the present problems is obscured, their proofs concerning the existence and properties of eigenvalues and eigenmatrix functions, as well as of expansion theorems, apply to the problems discussed above. We are justified then in using formal expansion theorems, such as (12), (22) in the solution of problems of the above type.

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A NOTE ON A VECTOR FORMULA*

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Of some vector formulas compiled in a recent paper[1] the one discussed in the present note seems to be of general interest in field theory.

1. Derivation of the vector formula. Let $\mathbf{B}(\mathbf{r})$ denote a vector function of the position vector $\mathbf{r}$, satisfying sufficient continuity and differentiability conditions, and let A denote a closed surface and V the region of space bounded by this surface. Using conventional vector notation we may then state Gauss' theorem in the following way

$$\int_A d\mathbf{a}\cdot\mathbf{B} = \int_V dv\ \nabla\cdot\mathbf{B}. \tag{1}$$

Letting $\varphi(\mathbf{r})$ denote a scalar function and $\mathbf{\Phi}(\mathbf{r})$ a dyade function, both possessing sufficient continuity and differentiability properties, we may derive the following equations from Gauss' theorem

$$\int_A d\mathbf{a}\ \varphi = \int_V dv\ \nabla\varphi, \tag{2}$$

$$\int_A d\mathbf{a}\cdot\mathbf{\Phi} = \int_V dv\ \nabla\cdot\mathbf{\Phi}.^{2} \tag{3}$$

Substituting in equations (2) and (3)

$$\varphi = \mathbf{r}\cdot\mathbf{B}, \tag{4}$$

$$\mathbf{\Phi} = \mathbf{r}\mathbf{B}, \tag{5}$$

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