WALL EFFECTS IN CAVITY FLOW—II*

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1. Introduction. In Part I of the present study,** the problems of flow about a cavitating body symmetrically placed in a channel or in a free jet have been solved in the case where the cavity extends to infinity downstream. The infinitely long cavity occurs, in each configuration, at one particular cavitation number which is a function of blockage ratio in the first case and is zero in the second. At greater values of the

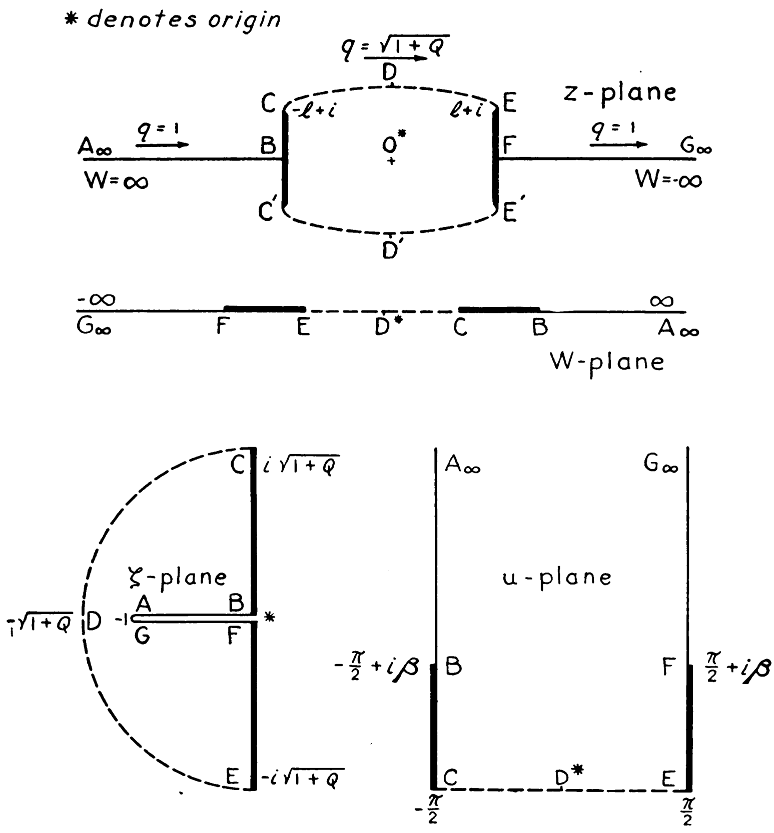

FIG. 1 - CASE A.

cavitation number, the cavity is of finite extent and a different analysis is necessary. The solutions of the corresponding problems with finite cavities are given in the present Part.

The configurations examined are again two-dimensional, this permitting the employment of conformal transformation technique. The body is taken in the form of a finite

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lamina perpendicular to the stream, so that the physical features of the flow may not be obscured by mathematical difficulties. Explicitly, the cases treated are

A. The cavitating lamina in an infinite stream;

B. The same in a channel of finite width;

C. The same in a free jet of finite width.

2. Case A. The lamina in an infinite stream. This case, where the liquid has no outer boundaries, is taken first to provide a standard for the other two. Solutions of this case have been given previously by Riabouchinsky [1] and by Fisher in an unpublished British Admiralty report, but the present treatment is much simpler than either.

Take the density of the liquid as unity, the velocity at infinity as unity, and the width of the strip forming the body as 2 units, so as to avoid unnecessary symbols. This strip is disposed between the points $(-1 \pm i)$ in the z-plane (Fig. 1). The free boundaries, there shown in broken line, start from the edges of the strip and re-form downstream on a similar, conventional, solid strip, extending between the points $(1 \pm i)$. This device, which is due to Riabouchinsky, avoids the closure jets and turbulence that would otherwise have to be taken into account. In this way, the mathematical advantages of a symmetrical problem are obtained merely by modification of the downstream conditions, to which the flow around the cavitating body is known to be insensitive. That this is so, has been clearly demonstrated by Gilbarg, Rock and Zarantonello,† who, in an as yet unpublished analysis of the similar problem with downstream closure by a re-entrant jet, find for low and moderate cavitation numbers, cavity boundaries and drag coefficients virtually indistinguishable from those that result below.

The flow being symmetrical about the x axis, consideration is restricted to the upper half z-plane. The corresponding regions in the W and ζ-planes are shown in Fig. 1, together with the auxiliary plane of u. Symbolism is as in Part I.

Proceeding by Kirchhoff's method for discontinuous flows, the transformation relations are found:

$$\zeta = -(1 + Q)^{1/2}\left[\frac{1 + i \tanh \beta \tan u}{1 - i \tanh \beta \tan u}\right]^{1/2}, \tag{1}$$

$$\frac{dz}{du} = S_1(\beta) \cos u\left[\frac{1 - i \tanh \beta \tan u}{1 + i \tanh \beta \tan u}\right]^{1/2}, \tag{2}$$

where

$$\beta = \tfrac{1}{2} \log (1 + Q) \tag{3}$$

and $S_1(\beta)$ can be evaluated in terms of standard Jacobian elliptic functions of modulus $k = \operatorname{sech} \beta$ as

$$S_1(\beta) = \frac{k^2}{k'^2 + E' - k^2 K'}. \tag{4}$$

The integration of (2) between appropriate limits then yields the cavity half-length and half-width a:

†Partial results were detailed previously by D. Gilbarg and H. H. Rock. Nav. Ord. Lab. Memo. 8718.

$$
\begin{aligned}
1 &= S_2(\beta) = \frac{E - k'^2 K}{k'^2 + E' - k^2 K'}, \\
a - 1 &= S_3(\beta) = \frac{k'(1 - k')}{k'^2 + E' - k^2 K'}.
\end{aligned} \tag{5}
$$

The intrinsic equation of the cavity boundary, referred to its point of departure C as origin is found to be

$$
s = S_1(\beta)\left[1 - \frac{\tan\theta}{(\tanh^2\beta + \tan^2\theta)^{1/2}}\right]. \tag{6}
$$

In Cartesian parametric form, this is equivalent to

$$
\begin{aligned}
x &= S_1(\beta)\Bigg[\cosh^2\beta E\left(k, \frac{1}{2}\pi - \theta\right) - \sinh^2\beta \cdot F\left(k, \frac{1}{2}\pi - \theta\right) \\
&\qquad\qquad - \frac{\sin\theta}{(\tanh^2\beta + \tan^2\theta)^{1/2}}\Bigg], \\
y &= S_1(\beta)\sinh^2\beta\left[\frac{\sec\theta}{(\tanh^2\beta + \tan^2\theta)^{1/2}} - 1\right],
\end{aligned} \tag{7}
$$

where E, F are the standard elliptic integrals of the second and first kinds respectively.

The drag coefficient of the lamina, based on unit velocity, is found after some reduction to be

$$
C_D = (1 + Q)S_4(\beta), \tag{8}
$$

where

$$
S_4(\beta) = \frac{2(E' - k^2 K')}{k'^2 + E' - k^2 K'}. \tag{9}
$$

Referred to the velocity on the cavity boundary, $(1 + Q)^{1/2}$, the drag coefficient is

$$
C_1 = S_4(\beta). \tag{10}
$$

The functions S_1, S_2, S_3, S_4 are all easily calculable and there is no difficulty in applying the foregoing solution in any numerically given case. The sub-class of cases in which Q is small is of especial interest: the functions then degenerate, giving the following simple results:

$$
\begin{aligned}
1 &= \frac{16}{\pi + 4}\left(\frac{1}{Q^2} + \frac{1}{Q}\right) + 0(1), \\
a - 1 &= \frac{8}{(\pi + 4)Q} + 0(Q), \\
C_D &= \frac{2\pi}{\pi + 4}(1 + Q) + 0(Q^2), \\
C_1 &= \frac{2\pi}{\pi + 4} + 0(Q^2).
\end{aligned} \tag{11}
$$

The cavity contour becomes

$$x = \frac{2}{\pi + 4}\left[\operatorname{cosec}\theta\cot\theta - \operatorname{gd}^{-1}\left(\frac{1}{2}\pi - \theta\right)\right] + 0(Q^2),$$
$$y = \frac{4}{\pi + 4}(\operatorname{cosec}\theta - 1) + 0(Q^2). \tag{12}$$

In the limit, as $Q \to 0$, (11) and (12) become the classical results for the lamina in an infinite stream.

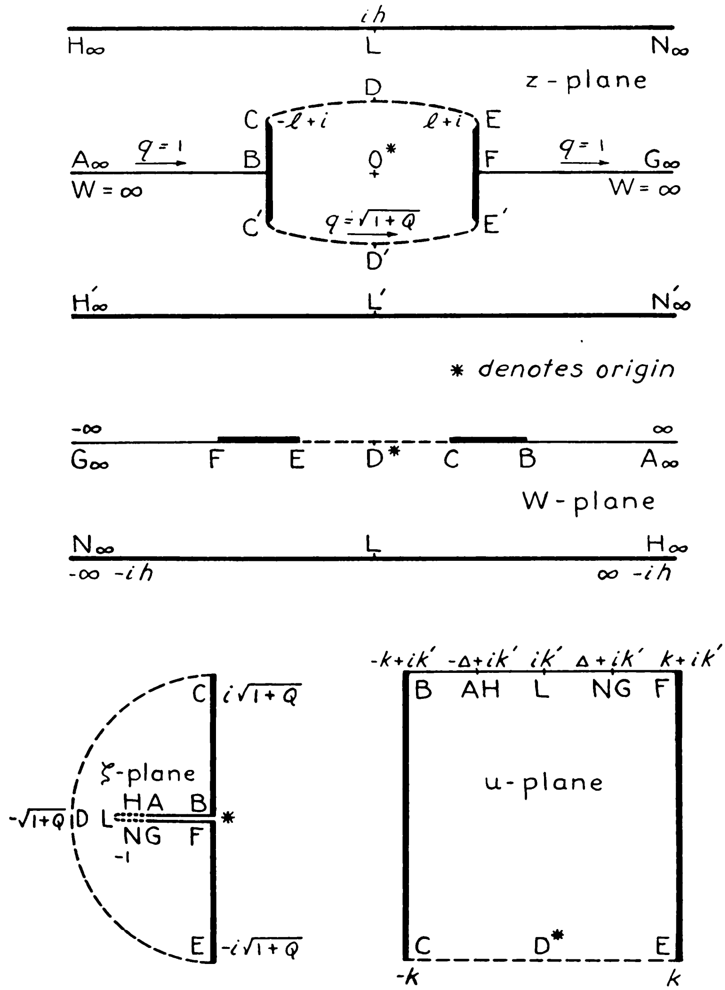

FIG. 2 - CASE B.

3. Case B. The lamina in a channel of finite width. Take the same arrangement as in Case A, but with the liquid confined between two parallel rigid walls, distant $2h$ apart, with respect to which the body is symmetrically placed (Fig. 2). Again restricting consideration to the upper half z-plane, the region in the W-plane is now an infinite strip: this, together with the ζ and u-planes, is shown in Fig. 2.

Proceeding as before, but with greater complexity due to the additional singularities, one finds the following transformation equations:

$$\zeta = -(1 + Q)^{1/2} \frac{\operatorname{cn} u + ik' \operatorname{sn} u}{\operatorname{dn} u}, \tag{13}$$

$$\frac{dz}{du} = \frac{2hk \operatorname{sn} \Delta}{\pi(1 + Q)^{1/2}} \frac{\operatorname{cn}^2 u - ik' \operatorname{sn} u \operatorname{cn} u}{1 - k^2 \operatorname{sn}^2 \Delta \operatorname{sn}^2 u}, \tag{14}$$

where

$$\operatorname{dn} \Delta = \frac{2 + Q}{Q} k'. \tag{15}$$

The constant k is not known *ab initio*, but must be determined at the end to conform with the given h.

The integration of (14), between appropriate limits, then yields after reduction the following expressions for the geometrical characteristics:

$$\frac{\pi(1 + Q)^{1/2}}{2h} = \frac{dn\, \Delta}{k \operatorname{cn} \Delta} [K'E(\Delta) + (E' - K')\Delta] + \frac{k'}{k \operatorname{cn} \Delta} \cos^{-1} (\operatorname{cd} \Delta) - kK' \operatorname{sn} \Delta, \tag{16}$$

$$1 = \frac{h}{\pi(1 + Q)^{1/2}} [k^2 \operatorname{sn} \Delta - Z(\Delta) \operatorname{dc} \Delta] \frac{2K}{k}, \tag{17}$$

$$a - 1 = \frac{2h}{\pi(1 + Q)^{1/2}} \frac{k'}{k \operatorname{cn} \Delta} [\operatorname{am} \Delta - \tan^{-1} (k' \operatorname{sc} \Delta)]. \tag{18}$$

The cavity shape is obtained in Cartesian parametric form as

$$\begin{aligned} x &= \frac{h}{\pi} \frac{2 + Q}{1 + Q} \left[\{k^2 \operatorname{sn} \Delta \operatorname{cd} \Delta - Z(\Delta)\} v + \frac{1}{2} \log \frac{\Theta_1(\Delta + v)}{\Theta_1(\Delta - v)} \right], \\ y &= \frac{h}{\pi} \frac{Q}{1 + Q} [\tan^{-1} (k' \operatorname{sc} \Delta \operatorname{nd} v) - \tan^{-1} (k' \operatorname{sc} \Delta)], \end{aligned} \tag{19}$$

where the parameter v runs from 0 to $2K$.

Again, after considerable reduction, one finds for the drag coefficients

$$\begin{aligned} C_D &= 2\left[1 + Q - \frac{hQ}{\pi} \tan^{-1} (k' \operatorname{sc} \Delta)\right], \\ C_L &= 2\left[1 - \frac{h}{\pi} \frac{Q}{1 + Q} \tan^{-1} (k' \operatorname{sc} \Delta)\right]. \end{aligned} \tag{20}$$

With these relations, the solution is formally complete in its barest essentials: if the complete flow pattern is desired, the z, u relation can be found without difficulty by integration of (14) and the velocity vector at any point is then given by (13).

The numerical solution for any given case involves some complication. Given Q and h, the value of k must be found from (16); successive approximation is the indicated method. At the same time am Δ is found from (15); the remaining results can then be evaluated.

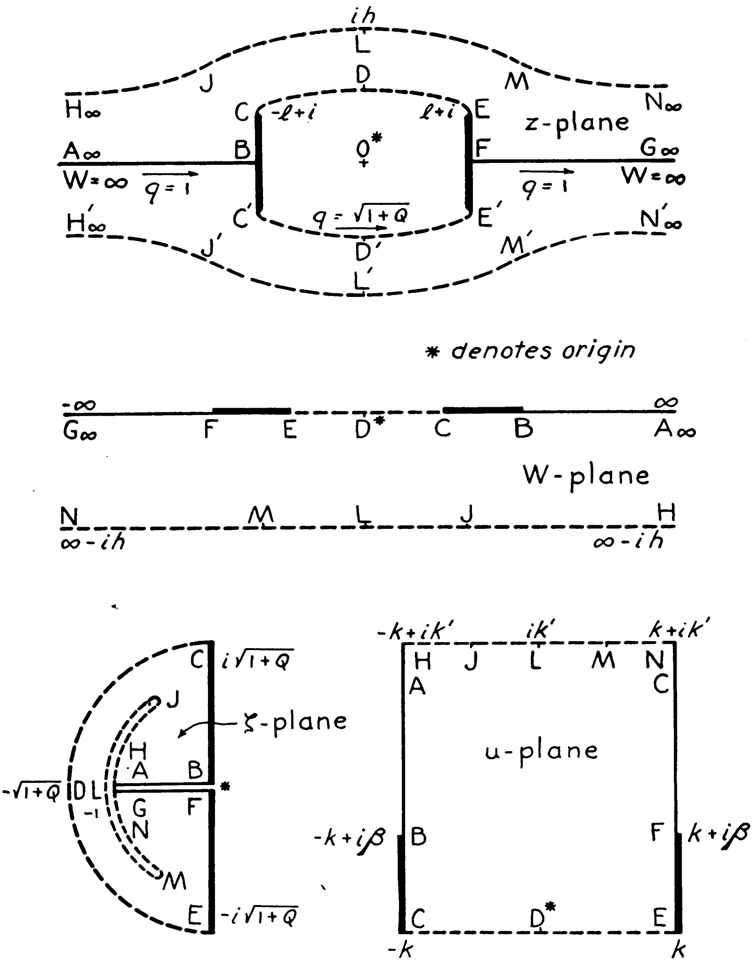

FIG. 3 - CASE C.

It is soon found, on trial, that solutions do not exist for all combinations of Q and h. For each value of Q, there is a limiting value of the blockage ratio $1/h$ that cannot be exceeded: this limiting value is given by

$$\left(\frac{1}{h}\right)_{\max} = \frac{1 + \frac{1}{2}Q - (1+Q)^{1/2}}{1+Q} + \frac{1}{\pi}\frac{Q}{1+Q}\tan^{-1}\frac{Q}{2(1+Q)^{1/2}}. \tag{21}$$

It is easily verified that, in the limiting condition, the length of the cavity is infinite, so that the solution degenerates to that of Part I. Moreover, the liquid at infinity downstream is on the point of cavitating. Hence the limitation is an inherent physical one. It bears some analogy with the choking phenomenon in a transonic wind-tunnel.

The limitation, at low cavitation numbers, is extremely harsh, e.g. at $Q = 0.05$, the blockage ratio cannot exceed about 1/1500 (cf. Part I, Sect. 2). Alternatively, for a blockage ratio of 0.05, the minimum cavitation number obtainable is 0.6.

When numerical values are considered, it is found that for admissible solutions, the drag coefficients for any given cavitation number are virtually the same as in Case A: this is due to the very low blockage. The cavity tends to be larger than with infinite fluid, i.e., in effect, the cavitation number is decreased by the fixed boundaries, especially when conditions are nearly critical, but comparison of calculated cavity contours shows that this effect becomes appreciable only at points substantially downstream.

4. Case C. The lamina in a free jet of finite width. Take the same arrangement as in Case A, but with the body symmetrically placed in a free jet whose width at infinity is $2h$ units. Still restricting consideration to the upper half z-plane, the region in the W-plane is again an infinite strip and the transformation planes of ζ and u are as shown in Fig. 3. In these planes, it is necessary to take into account the points J, M at which the boundary stream-lines inflect.

Taking account of these singularities and proceeding along the same lines as in the two previous cases, one finds the transformation relations

$$\zeta = -(1 + Q)^{1/2}\left[\frac{H_1(u - i\beta)}{H_1(u + i\beta)}\right]^{1/2}, \tag{22}$$

$$\frac{dz}{du} = \frac{2hk}{\pi(1 + Q)^{1/2}}\left[\frac{H_1(u + i\beta)}{H_1(u - i\beta)}\right]^{1/2} \operatorname{cd} u, \tag{23}$$

where

$$\beta = \frac{K}{\pi} \log (1 + Q) \tag{24}$$

and H_1 is the Jacobian theta-function constructed, like *cd u*, with modulus k. k in turn must be found from the complicated integral equation

$$\frac{\pi(1 + Q)^{1/2}}{2hk} = \int_0^\beta \left[\frac{H(iu + i\beta)}{H(iu - i\beta)}\right]^{1/2} \operatorname{sn} iu\, du. \tag{25}$$

In this expression, the complex radical takes its first quadrant value.

In terms of k and β, the cavity dimensions are now found to be

$$1 = \frac{hk}{\pi(1 + Q)^{1/2}} \int_0^K \frac{H(u + i\beta) + H(u - i\beta)}{\{H(u + i\beta)H(u - i\beta)\}^{1/2}} \operatorname{sn} u\, du, \tag{26}$$

$$a - 1 = \frac{hk}{\pi(1 + Q)^{1/2}} \int_0^K \frac{H(u + i\beta) - H(u - i\beta)}{i\{H(u + i\beta)H(u - i\beta)\}^{1/2}} \operatorname{sn} u\, du. \tag{27}$$

The intrinsic equation of the cavity boundary is

$$\begin{aligned} s &= \frac{2h}{\pi(1 + Q)^{1/2}} \log \frac{1 + k}{\operatorname{dn} v + k \operatorname{cn} v}, \\ \tan\theta &= \frac{H(v + i\beta) - H(v - i\beta)}{i\{H(v + i\beta) + H(v - i\beta)\}}, \end{aligned} \tag{28}$$

where the paremeter v runs from 0 to $2K$.

The drag and lift coefficients reduce to

$$C_D = \frac{2hk}{\pi}(1+Q)^{1/2}\int_0^\beta \frac{H(i\beta+iu)-H(i\beta-iu)}{\{H(i\beta+iu)H(i\beta-iu)\}^{1/2}}\frac{\operatorname{sn} iu}{i}\,du,$$
$$C_L = \frac{2hk}{\pi(1+Q)^{1/2}}\int_0^\beta \frac{H(i\beta+iu)-H(i\beta-iu)}{\{H(i\beta+iu)H(i\beta-iu)\}^{1/2}}\frac{\operatorname{sn} iu}{i}\,du. \tag{29}$$

These relations comprise the solution of Case C. It is readily shown that, as $k \to 0$, the solution degenerates to that of Case A. This however corresponds to very great values of h and is not of practical interest. In the general case, (24) and (25) must be solved simultaneously for k and β by a method of successive approximation. The remaining results can then, with some trouble, be evaluated.

The case Q small, which is of the greatest practical interest, can be approximately solved in explicit terms. For one finds that this case corresponds to $k \to 1$, so that the elliptic and theta-functions approach degenerate forms. Thus K is logarithmically large and β/K small in comparison with unity. One develops the solution in powers of Q and retains terms of order Q. Then (24) and (25) become

$$\frac{\beta}{K} = \frac{Q}{\pi},$$
$$\frac{(1+Q)^{1/2}}{h} = S_5(\beta) + \frac{Q}{\pi^2}S_6(\beta), \tag{30}$$

where

$$S_5(\beta) = 1 - \cos\beta + \frac{1}{\pi}\sin\beta\log\frac{1+\sin\beta}{1-\sin\beta},$$
$$S_6(\beta) = \pi\sin\beta\log\sec\beta + 2[f(\tan\tfrac{1}{2}\beta) - f(-\tan\tfrac{1}{2}\beta)] - \cos\beta[f(\sin\beta) - f(-\sin\beta)], \tag{31}$$

in which

$$f(x) = \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)^2}, \tag{32}$$

a tabulated function [2]. Hence, for given values of Q and h, β is readily determined.

The simplified forms of (26), (27), (28), (29) are, respectively,

$$1 = \frac{2h}{\pi(1+Q)^{1/2}}\left[\frac{\pi}{Q}\sin\beta - \cos\beta\log\sin\beta + \log\tan\frac{1}{2}\beta\right], \tag{33}$$

$$a - 1 = \frac{2h}{\pi(1+Q)^{1/2}}\left[\frac{\pi}{Q}(1-\cos\beta) - \sin\beta\log(1+\sin\beta)\right], \tag{34}$$

$$s = \frac{2h}{\pi(1+Q)^{1/2}}\log\cosh v$$
$$(v \geq 0) \tag{35}$$
$$\theta = \cot^{-1}(\cot\beta\tanh v) - \frac{Q}{\pi}v$$

$$
\begin{aligned}
C_D &= 2h(1+Q)^{1/2}\left[1 - \cos\beta + \frac{Q}{\pi}\sin\beta\log\sec\beta\right],\\
C_1 &= \frac{2h}{(1+Q)^{1/2}}\left[1 - \cos\beta + \frac{Q}{\pi}\sin\beta\log\sec\beta\right].
\end{aligned}
\tag{36}
$$

The foregoing general solution for Q small is bound by the condition that β should not be small in comparison with Q. This merely implies an upper limit to the permissible width of jet and is no handicap in practice. Within the practical range of blockage ratios and cavitation numbers, the solution holds good.

When Q is very small, the following first approximations may be used:

$$
\begin{aligned}
\frac{1}{h} &= {}^{*}S_5(\beta),\\
1 &= \frac{2h}{Q}\sin\beta,\\
a &= \frac{2h}{Q}(1 - \cos\beta),\\
C_D &= C_1 = 2h(1 - \cos\beta).
\end{aligned}
\tag{37}
$$

When $Q \to 0$, those results become those for the infinite cavity discussed in Part I.

It is not part of the present object to give detailed numerical results for application to arbitrary configurations: these it is hoped to present elsewhere.

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A NEW VARIATIONAL PRINCIPLE FOR ISENERGETIC FLOWS*

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In a paper by Rubinov and the present author,[1] it is shown that the variational principle for irrotational flows of a compressible gas can be generalized to isenergetic flows. The functions to be varied are the stream function and the density distribution.

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[1]Lin, C. C. and Rubinov, S. I. *On the flow of curved shocks*, J. Math. and Phys. **27**, 105-129 (1948).